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# ON SPECTRA OF UNITARY CAYLEY MIXED GRAPH

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ABSTRACT. In this paper we introduce mixed unitary Cayley graph $M_n$ $(n > 1)$ and compute its eigenvalues. We also compute the energy of $M_n$ for some $n$.

## 1. Introduction

Let $G(V, E)$ be a mixed graph with vertex set $V = V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E = E(G) = \{e_1, e_2, \ldots, e_m\}$, which is obtained from an undirected graph by orienting some of its edges. Mixed graphs are appropriate models of networks where both one way and reversible connection co-exist. These are called simply arcs and edges respectively. Mixed graphs include both possibilities of all edges oriented and all edges undirected as extreme cases. Thus the notion of a mixed graph generalizes both the classical approach of orienting all edges and unoriented approach. The adjacency matrix of a mixed graph G is a $n \times n$ matrix $M(G) = (a_{ij})_{n\times n}$, where

$$a_{ij} = \begin{cases} 1, & \text{if } v_iv_j \text{ is an edge,} \\ -1, & \text{if either} (v_i, v_j) \text{ or } (v_j, v_i) \text{ is an arc,} \\ 0, & \text{if } v_i \text{ and } v_j \text{ are not adjacent.} \end{cases}$$

One may note that the mixed graph is not uniquely determined by its adjacency matrix. Since $M(G)$ is a real symmetric matrix, all its eigenvalues $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ are real and the energy of the mixed

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graph $G$, $E_M(G)$ is the sum of the absolute values of its eigenvalues i.e., $E_M(G) = \sum_{i=1}^{n} |\lambda_i|$. For properties of mixed graphs one may refer to [3, 4, 7, 9, 11]. The concept of the energy of an undirected graph was introduced by Ivan Gutman [8] and was recently generalized to oriented graphs as skew energy by Adiga, Balakrishnan and Wasin So in [1].

Let $n > 1$ be a positive integer. The unitary Cayley graph $X_n = Cay(\mathbb{Z}_n, U_n)$ is a graph with vertex set $V(X_n) = \mathbb{Z}_n = \{0, 1, \ldots, n-1\}$ and edge set $E(X_n) = \{(i, j) : i, j \in \mathbb{Z}_n \text{ and } (j-i, n) = 1\}$. The graph $X_n$ is regular of degree $\phi(n)$, where $\phi(n)$ denotes the Euler totient function, and its eigenvalues are $\lambda_r$, $r = 0, 1, \ldots, n-1$, where $\lambda_r = c(r, n) := \sum_{\substack{m=1 \\ (m,n)=1}}^{n} e^{2\pi i rm/n}$.

$c(r, n)$ is the well-known Ramanujan's sum. Ramaswamy and Veena [10] have shown that the energy of $X_n := \sum_{i=1}^{n} |\lambda_i| = 2^k \phi(n)$, where $k$ is the number of distinct prime divisors of $n$. More informations about the unitary Cayley graphs can be found in [6, 10].

In order to define the mixed unitary Cayley graph $M_n$, we need the symbol $\left(\frac{a}{n}\right)$ which is defined as follows:
If $n = p$ is an odd prime, then $\left(\frac{a}{p}\right)$ is defined as follows:

$$\left(\frac{a}{p}\right) = \begin{cases} 1 & \text{if } \quad a \quad \text{is a quadratic residue mod } p, \\ -1 & \text{if } \quad a \quad \text{is a quadratic non residue mod } p, \\ 0 & \text{if } \quad p \mid a. \end{cases}$$

And

$$\left(\frac{a}{2}\right) = \begin{cases} 1 & \text{if } \quad a \quad \text{is odd}, \\ 0 & \text{if } \quad a \quad \text{is even}. \end{cases}$$

If $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k}$ is the prime decomposition of $n$, then

$$\left(\frac{a}{n}\right) = \left(\frac{a}{p_1}\right)^{\alpha_1} \left(\frac{a}{p_2}\right)^{\alpha_2} \cdots \left(\frac{a}{p_k}\right)^{\alpha_k}.$$

It may be noted that if n is an odd integer then $\left(\frac{a}{n}\right)$ is the Jacobi symbol.

We now define the mixed unitary Cayley graph $M_n$ $(n > 1)$. The vertex set of $M_n$ is $V(M_n) = \{0, 1, \ldots, n-1\}$ and the edge set of $M_n$ is $E(M_n)$ and is defined as follows : for $i, j \in V(M_n)$ with $i < j$, $(j-i, n) = 1$, $ij$ is an edge if $\left(\frac{j-i}{n}\right) = 1$ and $(i, j)$ is an arc if $\left(\frac{j-i}{n}\right) = -1$.

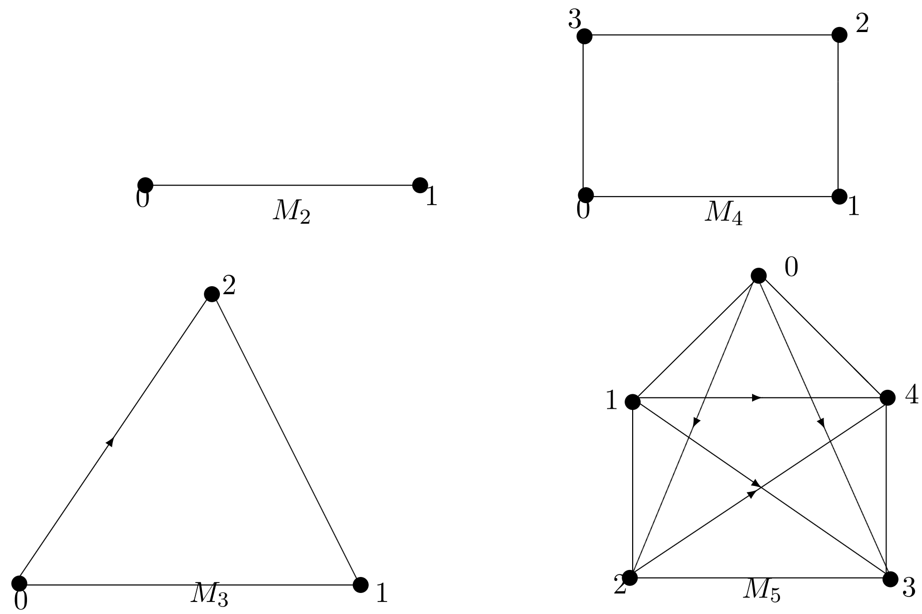


Fig. 1

Mixed unitary Cayley graphs of order 2, 3, 4 and 5 are given in Fig. 1.

The main purpose of this paper is to compute the eigenvalues of a mixed unitary Cayley graph $M_n$ and also to compute the energy of $M_n$ for some $n$.

## 2. Preliminaries

In this section we state certain basic definitions and results which will be used to prove the main results .

**Definition 2.1.** *Let $G$ be an arbitrary group. A complex valued function $f$ defined on $G$ is called a character of $G$ if $f$ is multiplicative, i.e, $f(ab) = f(a)f(b)$ for all $a$, $b$ in $G$, and if $f(c) \neq 0$ for some $c$ in $G$.*

Let $G$ be the group of reduced residue classes modulo $n$. For each character $f$ of $G$ we define an arithmetical function $\chi_f$ as follows:

$$\chi_f(m) = f(\hat{m}), \text{ if } (m, n) = 1,$$

and

$$\chi_f(m) = 0, \text{ if } (m, n) > 1,$$

where $\hat{m}$ is the set of all integers congruent to $m$ modulo $n$. The function $\chi_f$ is called a Dirichlet character modulo $n$. Clearly every group $G$ has atleast one character, namely $f = 1$, i.e, $f(a) = 1$ for all $a \in G$ and is called as principal character and the associated arithmetical function $\chi_1$ is called Dirichlet principal character mod $n$. For more details about Dirichlet character modulo $n$, the reader may refer to [2].

**Definition 2.2.** *For any Dirichlet character $\chi_f$ mod $n$, the sum*

$$G(r,\chi_f)=\sum_{m=1}^{n}\chi_f(m)e^{2\pi irm/n}$$

*is called the Gauss sum associated with $\chi_f$.*

If $\chi_f=\chi_1$, then $G(r,\chi_f)=c(r,n)$, the Ramanujan's sum.

**Theorem 2.3** ([2]). *If $\chi_f$ is any Dirichlet character mod $n$, then $G(r,\chi_f)=\bar{\chi}_f(r)G(1,\chi_f)$, whenever $(r,n)=1$.*

**Definition 2.4.** *A matrix of the form*

$$A_n=\begin{pmatrix} a_0 & a_1 & \cdots & a_{n-2} & a_{n-1} \\ a_{n-1} & a_0 & \cdots & a_{n-3} & a_{n-2} \\ \vdots & \vdots & & \vdots & \vdots \\ a_1 & a_2 & \cdots & a_{n-1} & a_0 \end{pmatrix}$$

*is called a circulant matrix.*

There is an explicit formula [5] for the eigenvalues $\lambda_r,\ 0\le r\le n-1$, of the circulant matrix $A_n$, given by $\lambda_r=\sum_{j=0}^{n-1}a_jw^{rj}$, $w=e^{2i\pi/n}$.

**Definition 2.5.** *A matrix of the form*

$$S_n=\begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-2} & a_{n-1} \\ -a_{n-1} & a_0 & a_1 & \cdots & a_{n-3} & a_{n-2} \\ -a_{n-2} & -a_{n-1} & a_0 & \cdots & a_{n-4} & a_{n-3} \\ \vdots & \vdots & & \vdots & \vdots & \\ -a_2 & -a_3 & -a_4 & \cdots & a_0 & a_1 \\ -a_1 & -a_2 & -a_3 & \cdots & -a_{n-1} & a_0 \end{pmatrix}$$

*is called a skew circulant.*

The eigenvalues of $S_n$ [5] are $\lambda_r,\ 0\le r\le (n-1)$, where $\lambda_r=\sum_{j=0}^{n-1}a_jw^{(2r+1)j/2}$.

**Definition 2.6.** *Let $A=(a_{ij})$ be a $n\times m$ matrix. $B=(b_{ij})$ be a $p\times q$ matrix then the Kronecker product of $A$ and $B$, $A\otimes B$ is the $np$ by $mq$ matrix obtained by replacing each entry $a_{ij}$ by $a_{ij}B$.*

It is well-known that the eigenvalues of $A\otimes B$ are given by $\lambda_i\mu_j$, where $\lambda_i$ and $\mu_j$ are the eigenvalues of $A$ and $B$ respectively.

**Definition 2.7.** *Let $G_1$ and $G_2$ be two graphs with disjoint vertex sets. The direct product of $G_1$ and $G_2$, denoted by $G_1 \times G_2$, is the graph with vertex set $V(G_1) \times V(G_2)$ such that two vertices $(x_1, y_1)$ and $(x_2, y_2) \in V(G_1 \times G_2)$ are adjacent if and only if $x_1x_2$ and $y_1y_2$ are edges in $G_1$ and $G_2$ respectively.*

We now define the direct product of mixed graphs as follows: Let $G_1$ and $G_2$ be two mixed graphs with disjoint vertex sets $V(G_1) = \{x_1, x_2, \ldots, x_n\}$ and $V(G_2) = \{y_1, y_2, \ldots, y_m\}$ having the order $x_i < x_{i+1}$ and $y_i < y_{i+1}$. The direct product of $G_1$ and $G_2$, denoted by $G_1 \times G_2$, is the mixed graph with vertex set $V(G_1 \times G_2) = V(G_1) \times V(G_2)$. If $u_1 = (x_1, y_1)$ and $u_2 = (x_2, y_2) \in V(G_1 \times G_2)$ with $u_1 < u_2$ (dictionary order), then $u_1u_2$ is an edge in $G_1 \times G_2$, if both $x_1x_2$ and $y_1y_2$ are either oriented edges or unoriented edges in $G_1$ and $G_2$ respectively, and $(u_1, u_2)$ is an oriented edge in $G_1 \times G_2$ if either $x_1x_2$ is an edge and $(y_1, y_2)/(y_2, y_1)$ is an oriented edge or vice versa.

Let $M(G_1)$ and $M(G_2)$ be the adjacency matrices of mixed graphs $G_1$ and $G_2$ respectively. One can note that $M(G_1 \times G_2) = M(G_1) \otimes M(G_2)$, the Kronecker product of $M(G_1)$ and $M(G_2)$. And hence $E_M(G_1 \times G_2) = E_M(G_1)E_M(G_2)$.

**Theorem 2.8** ([10])**.** *If $(m, n) = 1$, then the direct product of the unitary Cayley graphs $X_m$ and $X_n$ is isomorphic to $X_{mn}$.*

## 3. Main Results

In this section we obtain the eigenvalues of a mixed unitary Cayley graph $M_n$ and also we compute the energy of $M_n$, when $n = 2^j p_1^{\alpha_1}, \ldots, p_k^{\alpha_k} m$, where $p_i$'s are distinct primes with $p_i \equiv 1(mod\ 4)$ for $1 \leq i \leq k$, $\alpha_i$'s are odd positive integer and $m$ is a perfect square with $(2p_1, \ldots, p_k,\ m) = 1$.

**Theorem 3.1.** *If $n$ has even number of prime factors of the form $4k + 3$, then the eigenvalues of the mixed unitary Cayley graph $M_n$ are the Gauss sums $G(r, \chi_f)$, $r = 0, 1, \ldots, (n - 1)$, where $\chi_f = \left(\frac{m}{n}\right)$ is the Dirichlet character mod $n$.*

*Proof.* The adjacency matrix of $M_n$ with respect to the natural order of the vertices $0, 1, \ldots, (n - 1)$ is

$$\left( \begin{array}{cccccc} \left(\frac{0}{n}\right) & \left(\frac{1}{n}\right) & \left(\frac{2}{n}\right) & \cdots\left(\frac{i-1}{n}\right) & \cdots & \left(\frac{n-1}{n}\right) \\ \left(\frac{1}{n}\right) & \left(\frac{0}{n}\right) & \left(\frac{1}{n}\right) & \cdots\left(\frac{i-2}{n}\right) & \cdots & \left(\frac{n-2}{n}\right) \\ \vdots & & \ddots & & & \\ \left(\frac{i-1}{n}\right) & \left(\frac{i-2}{n}\right) & \left(\frac{i-3}{n}\right) & \cdots\left(\frac{0}{n}\right) & \cdots & \left(\frac{n-i}{n}\right) \\ \vdots & & & & \ddots & \\ \left(\frac{n-1}{n}\right) & \left(\frac{n-2}{n}\right) & \left(\frac{n-3}{n}\right) & \cdots\left(\frac{n-i}{n}\right) & \cdots & \left(\frac{0}{n}\right) \end{array} \right). \tag{3.1}$$

Since $n$ has even number of prime factors of the form $4k + 3$ and for an odd prime $p$,

$$\left(\frac{-1}{p}\right) = \begin{cases} 1 & \text{if } p \equiv 1(mod\ 4), \\ -1 & \text{if } p \equiv 3(mod\ 4), \end{cases} \tag{3.2}$$

we have $\left(\frac{-1}{n}\right) = 1$. So,

$$\left(\frac{n-a}{n}\right) = \left(\frac{-a}{n}\right) = \left(\frac{-1}{n}\right)\left(\frac{a}{n}\right) = \left(\frac{a}{n}\right).$$

Hence the adjacency matrix of $M_n$ is circulant. Consequently the eigenvalues are given by

$$\begin{aligned} \lambda_r &= \sum_{m=0}^{n-1} \left(\frac{m}{n}\right) \omega^{rm},\ r = 0, 1, \ldots, n-1, \\ &= \sum_{m=1}^{n} \left(\frac{m}{n}\right) \omega^{rm},\ \omega = e^{2\pi i/n}, \\ &= G(r, \chi_f), \end{aligned}$$

where $\chi_f = \left(\frac{m}{n}\right)$ is the Dirichlet character mod $n$. $\square$

**Theorem 3.2.** *If $n$ has odd number of prime factors of the form $4k+3$, then the eigenvalues of the mixed unitary Cayley graph $M_n$ are $\lambda_r,\ 0 \le r \le n-1$, where $\lambda_r = \sum_{m=0}^{n-1} \left(\frac{m}{n}\right) w^{(2r+1)m/2}$.*

*Proof.* The adjacency matrix of $M_n$ with respect to the natural order of the vertices $0, 1, \ldots, (n-1)$ is same as (3.1). Since $n$ has odd number of prime factors of the form $4k+3$, using (3.2), we have

$\left(\dfrac{-1}{n}\right) = -1$. So,

$$\left(\frac{n-a}{n}\right) = \left(\frac{-a}{n}\right) = \left(\frac{-1}{n}\right)\left(\frac{a}{n}\right) = -\left(\frac{a}{n}\right).$$

Thus one can easily see that the adjacency matrix of $M_n$ is a skew circulant. Hence its eigenvalues are given by $\lambda_r = \sum_{m=0}^{n-1}\left(\frac{m}{n}\right) w^{(2r+1)m/2}$, $r = 0, 1, \ldots, n-1$. □

**Theorem 3.3.** *Let $m$ and $n$ be positive integers having even number of prime factors of the form $4k+3$ and let $(m,n) = 1$. Then the direct product of the mixed unitary Cayley graphs $M_m$ and $M_n$ is isomorphic to $M_{mn}$.*

*Proof.* Set $(i,j) = k_{ij}$, where $k_{ij} \equiv i(mod\ m)$ and $k_{ij} \equiv j(mod\ n)$, for all $(i,j) \in V(M_m \times M_n)$. Since $(m,n) = 1$, by the Chinese Remainder theorem, it follows that the map $(i,j) \to k_{ij}$ is an isomorphism from $\mathbb{Z}_m \times \mathbb{Z}_n$ onto $\mathbb{Z}_{mn}$. By Theorem 2.8, it follows that the underlying graph $X_m \times X_n$ of $M_m \times M_n$ is isomorphic to $X_{mn}$, the underlying graph of $M_{mn}$. We shall now show that $((i_1,j_1),(i_2,j_2))$ is an oriented edge in $M_m \times M_n$ if and only if $(k_{i_1j_1}, k_{i_2j_2})$ is an oriented edge in $M_{mn}$.
Let $(k_{i_1j_1} = (i_1,j_1), k_{i_2j_2} = (i_2,j_2))$ be an oriented edge in $M_m \times M_n$. Then $\left(\dfrac{i_2-i_1}{m}\right)\left(\dfrac{j_2-j_1}{n}\right) = -1 \iff \left(\dfrac{k_{i_2j_2}-k_{i_1j_1}}{m}\right)\left(\dfrac{k_{i_2j_2}-k_{i_1j_1}}{n}\right) = -1$, as $m$ and $n$ are positive integers having even number of prime factors of the form $4k+3$ and $\left(\dfrac{a}{c}\right) = \left(\dfrac{b}{c}\right)$, for $a \equiv b(mod\ c)$. Similarly one can show that $(i_1,j_1)(i_2,j_2)$ is an unoriented edge in $M_m \times M_n$ if and only if $k_{i_1j_1}k_{i_2j_2}$ is an unoriented edge in $M_{mn}$. This completes the proof. □

**Theorem 3.4.** *If $p$ is a prime and $p \equiv 1(mod\ 4)$, then the energy of $M_p$ is*

$$E_M(M_p) = (p-1)\sqrt{p}.$$

*Proof.* By Theorem 3.1. the eigenvalues of $M_p$ are

$$\lambda_r = G(r, \chi_f),\ 0 \le r \le p-1,$$

where $f$ is the quadratic character mod $p$.
Hence the energy of $M_p$, is given by

$$\begin{aligned} E_M(M_p) &= \sum_{r=0}^{p-1} |\lambda_r| \\ &= \sum_{r=0}^{p-1} |G(r,\chi_f)| \\ &= \sum_{r=1}^{p-1} |\bar{\chi}_f(r)||G(1,\chi_f)| \text{ (by Theorem 2.3)} \\ &= |G(1,\chi_f)|\phi(p). \end{aligned}$$

Now consider,

$$\begin{aligned}
|G(1,\chi_f)|^2 &= G(1,\chi_f)\overline{G(1,\chi_f)} \\
&= G(1,\chi_f)\sum_{m=1}^{p}\bar{\chi}_f(m)e^{-2\pi im/p} \\
&= \sum_{m=1}^{p} G(m,\chi_f)e^{-2\pi im/p} \text{ (by Theorem 2.3)} \\
&= \sum_{m=1}^{p}\sum_{j=1}^{p}\left(\frac{j}{p}\right)e^{2\pi ijm/p}e^{-2\pi im/p} \\
&= \sum_{j=1}^{p}\left(\frac{j}{p}\right)\sum_{m=1}^{p}\omega^{m(j-1)}, \text{ where } \omega = e^{2\pi i/p} \\
&= \left(\frac{1}{p}\right)\sum_{m=1}^{p}1, \text{since} \sum_{m=1}^{p}\omega^{m(j-1)} = 0, \text{ if } j > 1 \\
&= p.
\end{aligned}$$

Thus

$$E_M(M_p) = |G(1,\chi_f)|\phi(p) = \sqrt{p}\ (p-1).$$

□

**Theorem 3.5.** *If $p$ is a prime and $p \equiv 1 (mod\ 4)$ then*

$$E_M(M_{p^\alpha}) = \begin{cases} 2\ \phi(p^\alpha) & \text{if } \alpha \text{ is even,} \\ \sqrt{p}\ \phi(p^\alpha) & \text{if } \alpha \text{ is odd.} \end{cases}$$

*Proof.* If $\alpha$ is even, then $M_{p^\alpha} \cong X_{p^\alpha}$. Since the energy of $X_{p^\alpha}$ is $2\ \phi(p^\alpha)$, $E_M(M_{p^\alpha}) = 2\ \phi(p^\alpha)$. Now suppose that $\alpha$ is odd, then $\left(\frac{a}{p^\alpha}\right) = \left(\frac{a}{p}\right)^\alpha = \left(\frac{a}{p}\right)$ and also $\left(\frac{p+a}{p}\right) = \left(\frac{a}{p}\right)$. Hence the adjacency matrix of $M_{p^\alpha}$ is of the form $A \otimes M_p$, where $A$ is square matrix of order $p^{\alpha-1}$ with all its entries as 1. Thus $E_M(M_{p^\alpha}) = p^{\alpha-1}E_M(M_p) = \sqrt{p}\ \phi(p^\alpha)$. □

**Theorem 3.6.** *Let $n = 2^j p_1^{\alpha_1}, \ldots, p_k^{\alpha_k} m$, where $p_i$'s are distinct primes with $p_i \equiv 1 (mod\ 4)$ for $1 \le i \le k$, $\alpha_i$'s are odd positive integer and $m$ is a perfect square. If $(2p_1, \ldots, p_k,\ m) = 1$, then $E_M(M_n) = 2^{l+1}(p_1, p_2, \ldots, p_k)^{1/2}\phi(n)$, where $l$ is the number of distinct prime factors of $m$.*

*Proof.* The proof follows from Theorem 3.3 and Theorem 3.5 and the fact that the energy of $X_t = 2^s\phi(t)$, where $s$ is the number of distinct prime factors of $t$. □

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