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SOME RESULTS ON CHARACTERIZATION OF FINITE GROUPS BY NON-COMMUTING GRAPH

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ABSTRACT. The non commuting graph $\nabla(G)$ of a non-abelian finite group G is defined as follows: its vertex set is $G - Z(G)$ and two distinct vertices x and y are joined by an edge if and only if the commutator of x and y is not the identity. In this paper we prove some new results about this graph. In particular we will give a new proof of Theorem 3.24 of [A. Abdollahi, S. Akbari, H. R, Maimani, Non-commuting graph of a group, *J. Algebra*, **298** (2006) 468-492.]. We also prove that if $G_1, G_2, \ldots, G_n$ are finite groups such that $Z(G_i) = 1$ for $i = 1, 2, \ldots, n$ and they are characterizable by non commuting graph, then $G_1 \times G_2 \times \cdots \times G_n$ is characterizable by non-commuting graph.

1. Introduction

Let G be a finite group. The non-commuting graph$\nabla(G)$ of G is defined as follows: the set of vertices of $\nabla(G)$ is $G - Z(G)$, where $Z(G)$ is the center of G and two vertices x and y are connected whenever $[x, y] \neq 1$, where $[x, y]$ is the commutator of x and y. In [1] the authors put forward a conjecture as follows:

Conjecture 1. *Let G be a finite non-abelian nilpotent group and H be a group such that $\nabla(G) \cong \nabla(H)$. Then H is nilpotent.*

In this paper we prove this conjecture in the case of $|G| = |H|$. In fact this is proved in [1], but our proof is different. We say G is factorizable if G is isomorphic to a direct product of its proper subgroups. We will show that if G and H are two centerless groups and $\nabla(G) \cong \nabla(H)$, then G is factorizable if and only if H is factorizable. Moreover if $G \cong G_1 \times G_2 \times \cdots \times G_n$, then there are

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subgroups of H say $H_1, H_2, \ldots, H_n$ such that $H \cong H_1 \times H_2 \times \cdots \times H_n$. G is called characterizable by non-commuting graph if, when H is an arbitrary group with $\nabla(G) \cong \nabla(H)$, then $G \cong H$. We prove that if $G_1, G_2, \ldots, G_n$ are finite groups such that $Z(G_i) = 1$ for $i = 1, 2, \ldots, n$ and G_i is characterizable by non-commuting graph, then $G_1 \times G_2 \times \cdots \times G_n$ is characterizable by non-commuting graph. In [3] Ron Solomon and Andrew Woldar proved that all finite non-abelian simple groups are characterizable by non-commuting graph.

2. **Preliminaries**

Lemma 2.1. *Let G and H be two finite non-abelian groups. If $\nabla(G) \cong \nabla(H)$, then $\nabla(C_G(A)) \cong \nabla(C_H(\varphi(A)))$ for all $\varnothing \neq A \subseteq G - Z(G)$, where φ is the isomorphism from $\nabla(G)$ to $\nabla(H)$ and $C_G(A)$ is non-abelian.*

Proof. It is sufficient to show that $\varphi\mid_{V(C_G(A))}: V(C_G(A)) \longrightarrow V(C_H(\varphi(A)))$ is onto, where $\varphi\mid_{V(C_G(A))}$is the restriction of φ to $V(C_G(A))$ and

$$V(C_G(A)) := C_G(A) - Z(C_G(A)),$$
$$V(C_H(\varphi(A))) := C_H(\varphi(A)) - Z(C_H(\varphi(A)))$$

Assume that d is an element of $V(C_H(\varphi(A)))$. Then $d \in H - Z(H)$ and so there exists an element c of $G - Z(G)$ such that $\varphi(c) = d$. From $d = \varphi(c) \in C_H(\varphi(A))$, it follows that $[\varphi(c), \varphi(g)] = 1$ for all $g \in A$ and since φ is an isomorphism from $\nabla(G)$ to $\nabla(H)$, $[c, g] = 1$ for all $g \in A$. Therefore $c \in C_G(A)$. But $d \notin Z(C_H(\varphi(A)))$, so for an element $x \in C_H(\varphi(A))$ we have $[x, d] \neq 1$. Hence x is an element of H that does not commute with $d \in H$. This implies that $x \in H - Z(H)$. Thus there exists $x' \in G - Z(G)$, such that $\varphi(x') = x$. It is easy to see that $[x', c] \neq 1$ and therefore $c \notin Z(C_G(A))$. Therefore $c \in C_G(A) - Z(C_G(A)) = V(C_G(A))$. Hence $\varphi(c) = d$. □

We denote by I_G the set of all bijections $\phi : G \longrightarrow G$ such that $[x, y] = 1$ if and only if $[\phi(x), \phi(y)] = 1$ for all x, $y \in G$. It is easy to see that I_G is a subgroup of S_G, where S_G is the symmetric group on G.

Lemma 2.2. *Let G be a finite non-abelian group. Then $Aut(G) \leq I_G$, where $Aut(G)$ is the automorphism group of G.*

Proof. Suppose that $\psi \in Aut(G)$. If $x, y \in G$ are two arbitrary elements of G, then $[x, y] = 1$ if and only if $([x, y])\psi = 1$ and $[x\psi, y\psi] = 1$ and the proof is complete. □

Lemma 2.3. *Let G and H be two finite non-abelian groups with $\nabla(G) \cong \nabla(H)$ and $|G| = |H|$. Then $I_G \cong I_H$.*

Proof. Since $\nabla(G) \cong \nabla(H)$, $|G - Z(G)| = |H - Z(H)|$. But $|G| = |H|$ and so $|Z(G)| = |Z(H)|$. Thus there is a bijection α from $Z(G)$ to $Z(H)$. Moreover since $\nabla(G) \cong \nabla(H)$, there is a graph isomorphism

φ from $G - Z(G)$ to $H - Z(H)$. We define $\psi : I_G \longrightarrow I_H$ by

$$\psi(\phi)(x) = \varphi \circ \phi\mid_{G-Z(G)} \circ \varphi^{-1}(x)$$

if $x \notin Z(H)$ and

$$\psi(\phi)(x) = \alpha \circ \phi\mid_{Z(G)} \circ \alpha^{-1}(x)$$

if $x \in Z(H)$, for all $\phi \in I_G$, where $\circ$ denote the composition of functions. Routine checking shows that ψ is an isomorphism from I_G to I_H and so $I_G \cong I_H$. □

3. **Results and Properties**

Proposition 3.1. *Let G be a finite non-abelian nilpotent group and H be a group such that $\nabla(G) \cong \nabla(H)$ and $|G| = |H|$. Then H is nilpotent.*

Proof. We use induction on $|G| = n$. Clearly if $|G| = 1$, then the assertion holds. Suppose the result is valid for all groups K, with $|K| < n$. We will prove Proposition 3.1 when $|G| = n$. Since G is nilpotent, we can write $G \cong P_1 \times P_2 \times \cdots \times P_k$, where P_i is the p_i-Sylow subgroup of G say of order ${p_i}^{\alpha_i}$ for $i = 1, 2, \ldots, k$.

If G is a p-group for some prime number p, then since $|G| = |H|$, H is a p-group too and so H is nilpotent. If $G = P \times A$, where P is a p-group and A is an abelian group, then $\frac{G}{Z(G)}$ is a p-group and since $|G| = |H|$ and $|Z(G)| = |Z(H)|$, we conclude that $\frac{H}{Z(H)}$ is a p-group and so H is nilpotent in this case.

Let φ be an isomorphism from $\nabla(G)$ to $\nabla(H)$. We extend φ to H by defining $\varphi(z) = \psi(z)$, where ψ is an arbitrary bijective map from $Z(G)$ to $Z(H)$.

By above argument we may assume that $k > 1$ and G is not product of a p-group and an abelian group.

If $C_G(P_i) = G$, for all $i = 1, 2, \ldots, k$, then $P_i \leq Z(G)$ for $i = 1, 2, \ldots, k$ and so $G = Z(G)$, a contradiction. Hence there is a Sylow-subgroup P_i of G such that $C_G(P_i) \neq G$. But $C_G(P_i)$ is nilpotent and $\nabla(C_G(P_i)) \cong \nabla(C_H(\varphi(P_i)))$ by Lemma 2.1, where φ is an isomorphism from $\nabla(G)$ to $\nabla(H)$ and so $C_H(\varphi(P_i))$ is nilpotent by inductive hypothesis. Without loss of generality we assume that

$$G = P_1 \times P_2 \times \cdots \times P_k, k > 1$$

Let

$$K = C_G(P_1 \times \cdots \times P_{i-1} \times P_{i+1} \times \cdots \times P_k)$$

Thus

$$K = Z(P_1) \times \cdots \times Z(P_{i-1}) \times P_i \times Z(P_{i+1}) \times \cdots \times Z(P_k)$$

Therefore $\frac{K}{Z(G)}$ is a p_i-group and so

$$\frac{C_H(\varphi(P_1 \times \cdots \times P_{i-1} \times P_{i+1} \times \cdots \times P_k))}{Z(H)}$$

is a p_i-group too, because $|K| = |C_H(\varphi(P_1 \times \cdots \times P_{i-1} \times P_{i+1} \times \cdots \times P_k))|$. On the other hand

$$Z(G) = Z(C_G(P_1 \times \cdots \times P_{i-1} \times P_{i+1} \times \cdots \times P_k))$$

This implies that

$$Z(H) = Z(C_H(\varphi(P_1 \times \cdots \times P_{i-1} \times P_{i+1} \times \cdots \times P_k))),$$

because φ is an isomorphism from $\nabla(G)$ to $\nabla(H)$. Thus

$$\frac{C_H(\varphi(P_1 \times \cdots \times P_{i-1} \times P_{i+1} \times \cdots \times P_k))}{Z(C_H(\varphi(P_1 \times \cdots \times P_{i-1} \times P_{i+1} \times \cdots P_k)))}$$

is a nilpotent group and so $C_H(\varphi(P_1 \times \cdots \times P_{i-1} \times P_{i+1} \times \cdots \times P_k))$ is nilpotent. Moreover since

$$C_G(P_i) = P_1 \times \cdots \times P_{i-1} \times Z(P_i) \times P_{i+1} \times \cdots \times P_k,$$

we have ${p_1}^{\alpha_1} \cdots {p_{i-1}}^{\alpha_{i-1}} {p_{i+1}}^{\alpha_{i+1}} \cdots {p_k}^{\alpha_k} \big| |C_G(P_i)|$. Now if

$${p_1}^{\alpha_1} \cdots {p_{i-1}}^{\alpha_{i-1}} {p_{i+1}}^{\alpha_{i+1}} \cdots {p_k}^{\alpha_k} \big| |C_G(A)|$$

for an arbitrary subset A of G , then we have

$$P_1 \times \cdots \times P_{i-1} \times P_{i+1} \times \cdots \times P_k \leq C_G(A)$$

and since $Z(G) \leq C_G(A)$, we conclude that

$$P_1 \times \cdots \times P_{i-1} \times Z(P_i) \times P_{i+1} \times \cdots \times P_k = C_G(P_i) \leq C_G(A)$$

Therefore if $|C_G(A)| = |C_G(P_i)|$, then $C_G(A) = C_G(P_i)$ for all $A \subseteq G$. We know that

$$|C_H(\varphi(P_i))| = |C_H(h^{-1}\varphi(P_i)h)| = |h^{-1}C_H(\varphi(P_i))h|$$

for all $h \in H$. Thus

$$|C_G(P_i)| = |C_G(\varphi^{-1}(h^{-1}\varphi(P_i)h))|.$$

Hence

$$C_G(P_i) = C_G(\varphi^{-1}(h^{-1}\varphi(P_i)h)),$$

which implies that

$$C_H(\varphi(P_i)) = C_H(h^{-1}\varphi(P_i)h) = h^{-1}C_H(\varphi(P_i))h,$$

where h is an arbitrary element of H. Therefore $C_H(\varphi(P_i)) \trianglelefteq H$. By a similar argument we can see that

$$C_H(\varphi(P_1 \times \cdots \times P_{i-1} \times P_{i+1} \times \cdots \times P_k)) \trianglelefteq H.$$

Obviously

$$|C_G(P_i)C_G(P_1 \times \cdots \times P_{i-1} \times P_{i+1} \times \cdots \times P_k)| =$$
$$\frac{|C_G(P_i)||C_G(P_1 \times \cdots \times P_{i-1} \times P_{i+1} \times \cdots \times P_k)|}{|C_G(P_i) \cap C_G(P_1 \times \cdots \times P_{i-1} \times P_{i+1} \times \cdots \times P_k)|} = |G|.$$

Thus

$$\frac{|C_H(\varphi(P_i))||C_H(\varphi(P_1\times\cdots\times P_{i-1}\times P_{i+1}\times\cdots\times P_k))|}{|C_H(\varphi(P_i))\cap C_H(\varphi(P_1\times\cdots\times P_{i-1}\times P_{i+1}\times\cdots P_k))|}=|H|$$

and so

$$C_H(\varphi(P_i))C_H(\varphi(P_1\times\cdots\times P_{i-1}\times P_{i+1}\times\cdots\times P_k))=H$$

and since

$$C_H(\varphi(P_i)) \quad\text{and}\quad C_H(\varphi(P_1\times\cdots\times P_{i-1}\times P_{i+1}\times\cdots\times P_k))$$

are nilpotent normal subgroups of H, we conclude that H is nilpotent. □

Proposition 3.2. *Let G and H be two finite non-abelian groups. If $\nabla(G)\cong\nabla(H)$ and $|Z(G)|=|Z(H)|=1$, then G is factorizable if and only if H is factorizable. Moreover if $G\cong G_1\times G_2\times\cdots\times G_n$, then there are subgroups H_1,H_2,...,H_n of H such that $H\cong H_1\times H_2\times\cdots\times H_n$ and $\nabla(G_i)\cong\nabla(H_i)$ for $i=1,2,\ldots,n$.*

Proof. Without loss of generality assume that $G=G_1\times G_2\times\cdots\times G_n$. Put

$$M_i=1\times\cdots\times G_i\times\cdots\times 1,$$

for $1\le i\le n$. This implies that

$$C_G(M_i)=G_1\times\cdots\times G_{i-1}\times G_{i+1}\times\cdots\times G_n,$$

for $1\le i\le n$. Thus

$$\begin{aligned}|C_G(M_i)||M_i|&=|G|,\\ C_G(C_G(M_i))&=M_i,\\ M_i\cap C_G(M_i)&=1,\end{aligned}$$

for $1\le i\le n$. On the other hand

$$C_G(M_1)\cap\ldots\cap C_G(M_{i-1})\cap C_G(M_{i+1})\cap\ldots\cap C_G(M_n)=M_i,$$

for $1\le i\le n$. Therefore we have

$$C_G\big(C_G(M_1)\cap\ldots\cap C_G(M_{i-1})\cap C_G(M_{i+1})\cap\ldots\cap C_G(M_n)\big)\cap C_G\big(C_G(M_i)\big)=1,$$

for $1\le i\le n$. Since $\nabla(G)\cong\nabla(H)$, there is an isomorphism, φ from $\nabla(G)$ to $\nabla(H)$. Hence

$$\begin{aligned}|C_H(\varphi(M_i))||\varphi(M_i)|&=|H|,\\ C_H(C_H(\varphi(M_i)))&=\varphi(M_i)\end{aligned}$$

and

$$\begin{aligned}&\varphi(M_i)\cap C_H(\varphi(M_i))=1,\\ &C_H(C_H(\varphi(M_1))\cap\ldots\cap C_H(\varphi(M_{i-1}))\cap C_H(\varphi(M_{i+1}))\cap\ldots\cap C_H(\varphi(M_n)))\cap C_H(C_H(\varphi(M_i)))=1,\end{aligned}$$

for $i = 1, 2, \dots, n$. Since

$$\varphi(M_i) \cap C_H(\varphi(M_i)) = 1,$$
$$|\varphi(M_i)||C_H(\varphi(M_i))| = |H|$$

and

$$\varphi(M_i)C_H(\varphi(M_i)) = H,$$

thus

$$C_H(C_H(\varphi(M_i))) = \varphi(M_i) \trianglelefteq \varphi(M_i)C_H(\varphi(M_i)) = H.$$

Therefore $\varphi(M_i)$ for $i = 1, 2, \dots, n$ is a normal subgroup of H. Moreover we have

$$\begin{aligned}&\varphi(M_1)\dots\varphi(M_{i-1})\varphi(M_{i+1})\dots\varphi(M_n) \subseteq\\ &C_H(C_H(\varphi(M_1)) \cap \dots \cap C_H(\varphi(M_{i-1})) \cap C_H(\varphi(M_{i+1})) \cap \dots \cap C_H(\varphi(M_n)))\end{aligned}$$

and so

$$\begin{aligned}&\varphi(M_1)\dots\varphi(M_{i-1})\varphi(M_{i+1})\dots\varphi(M_n) \cap \varphi(M_i) \subseteq\\ &C_H(C_H(\varphi(M_1)) \cap \dots \cap C_H(\varphi(M_{i-1})) \cap C_H(\varphi(M_{i+1})) \cap \dots \cap C_H(\varphi(M_n)))\cap\\ &C_H(C_H(\varphi(M_i))) = 1,\end{aligned}$$

which implies that

$$\varphi(M_1)\dots\varphi(M_{i-1})\varphi(M_{i+1})\dots\varphi(M_n) \cap \varphi(M_i) = 1,$$

for $i = 1, 2, \dots, n$. Hence

$$\varphi(M_1) \cap \varphi(M_2)\dots\varphi(M_n) = 1,$$
$$\varphi(M_2) \cap \varphi(M_3)\dots\varphi(M_n) = 1,$$
$$\dots, \varphi(M_{n-1}) \cap \varphi(M_n) = 1.$$

On the other hand $|M_1|\dots|M_n| = |G|$. Now it is easy to see that

$$\varphi(M_1)\varphi(M_2)\dots\varphi(M_n) = H.$$

Put $\varphi(M_i) = H_i$, $i = 1, 2, \dots, n$. Therefore we have proved

$$H \cong H_1 \times \dots \times H_n.$$

We know that

$$G_i \cong M_i = C_G(C_G(M_i)).$$

Thus $\nabla(G_i) \cong \nabla(C_H(C_H(\varphi(M_i)))$, because

$$\nabla(C_G(C_G(M_i))) \cong \nabla(C_H(C_H(\varphi(M_i))))$$

and since

$$C_H(C_H(\varphi(M_i))) = \varphi(M_i) = H_i,$$

we conclude that $\nabla(G_i) \cong \nabla(H_i)$ for $i = 1, 2, \ldots, n$. $\square$

Corollary 3.3. *Let $G_1, G_2, \ldots, G_n$ be finite non-abelian groups. If $G_1, G_2, \ldots, G_n$ are characterizable by non-commuting graph and $Z(G_i) = 1$ for $i = 1, 2, \ldots, n$, then $G_1 \times G_2 \times \cdots \times G_n$ is characterizable by non-commuting graph.*

Proof. Assume that $\nabla(H) \cong \nabla(G_1 \times G_2 \times \cdots \times G_n)$. Thus

$$\nabla(C_H(\varphi(G_2 \times \cdots \times G_n)) \cong \nabla(C_{G_1\times\cdots\times G_n}(G_2 \times \cdots \times G_n)) = \nabla(G_1).$$

But G_1 is characterizable by non-commuting graph and so

$$G_1 \cong C_H(\varphi(G_2 \times \cdots \times G_n))$$

and since

$$Z(C_{G_1\times G_2\times\cdots\times G_n}(G_2 \times \cdots \times G_n)) = Z(G_1) = 1,$$

we have

$$Z(C_H(\varphi(G_2 \times \cdots \times G_n))) = 1.$$

It follows that $Z(H) = 1$. By Proposition 3.2, there are subgroups $H_1, H_2, \ldots, H_n$ of H such that

$$H \cong H_1 \times H_2 \times \cdots \times H_n$$

and $\nabla(G_i) \cong \nabla(H_i)$ for $i = 1, 2, \ldots, n$. But since G_i is characterizable by non-commuting graph, we have $G_i \cong H_i$, $i = 1, 2, \ldots, n$ and so

$$H_1 \times H_2 \times \cdots \times H_n \cong G_1 \times G_2 \times \cdots \times G_n.$$

Therefore $H \cong G_1 \times G_2 \times \cdots \times G_n$. $\square$

Corollary 3.4. *If $S_1, S_2, \ldots, S_m$ are finite non-abelian simple groups, then $S_1 \times S_2 \times \cdots \times S_m$ is characterizable by non-commuting graph.*

Proof. In [3] the authors prove that all simple groups are characterizable by non-commuting graph. Thus by Corollary 3.3, direct product of simple groups are characterizable by non-commuting graph. $\square$

Proposition 3.5. *Let G be a finite non-abelian group such that $I_G = Inn(G)$ and $Z(G) = 1$, where $Inn(G)$ is the group of inner automorphisms of G. If H is a group with $\nabla(G) \cong \nabla(H)$ and $|G| = |H|$, then $G \cong H$.*

Proof. By Lemma 2.3 we have $I_G \cong I_H$. But $Z(G) = 1$, $Inn(G) \cong I_G$ and so we have $G \cong I_G$. Moreover $Z(H) = 1$ and by Lemma 2.2 we can write

$$H \cong Inn(H) \leq Aut(H) \leq I_H \cong I_G \cong G.$$

Therefore H is embedded in G and since $|H| = |G|$, we have $G \cong H$.

□

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