www.combinatorics.ir

Transactions on Combinatorics
ISSN (print): 2251-8657, ISSN (on-line): 2251-8665
Vol. 5 No. 3 (2016), pp. 1-9.

www.ui.ac.ir

TOTAL k-DISTANCE DOMINATION CRITICAL GRAPHS

D. A. MOJDEH, A. SAYED-KHALKHALI, H. ABDOLLAHZADEH AHANGAR* AND Y. ZHAO

Communicated by Bijan Taeri

Abstract. A set S of vertices in a graph $G = (V, E)$ is called a total k-distance dominating set if every vertex in V is within distance k of a vertex in S. A graph G is total k-distance domination-critical if $\gamma_t^k(G - x) < \gamma_t^k(G)$ for any vertex $x \in V(G)$. In this paper, we investigate some results on total k-distance domination-critical of graphs.

1. Introduction

The terminology and notation in [3] will be used throughout. The distance $d_G(u, v)$ between two vertices u and v of G is the length of the shortest u-v path if such path exists, otherwise $d_G(u, v) = \infty$. The open k-neighborhood $N_k(X)$ of a subset $X \subseteq V(G)$ is the set of vertices in $V(G) - X$ of distance at most k from each element of X and the closed k-neighborhood is defined by $N_k[X] = N_k(X) \cup X$. If $X = \{v\}$ is a single vertex, then we denote the (closed) k-neighborhood of v by $N_k(v)$ ($N_k[v]$, respectively). The (closed) 1-neighborhood of a vertex v or a set X of vertices is usually denoted by $N(v)$ or $N(X)$, respectively ($N[v]$ or $N[X]$, respectively). The minimum k-degree $\delta_k(G)$ equals $\min\{|N_k(v)| : v \in V\}$, while the maximum k-degree $\Delta_k(G)$ equals $\max\{|N_k(v)| : v \in V\}$. For a set $S \subseteq V(G)$, we denote the subgraph of G induced by S by $\langle S \rangle$. The k-th power of a graph G is the graph G^k with vertex set $V(G^k) = V(G)$ and edge set $E(G^k) = \{xy : 1 \leq d_G(x, y) \leq k\}$.

Given $k \leq n$, place n vertices around a circle, equally spaced. If k is even, form the Harary graph $H_{k,n}$ by making each vertex adjacent to the nearest $k/2$ vertices in each direction around the circle, [12]. It is well known that every m-th power of a cycle C_n with n vertices C_n^m is $H_{2m,n}$.

MSC(2010): Primary: 05C69.

Keywords: Distance domination, total k-distance dominating set, total k-distance domination-critical.

Received: 11 July 2015, Accepted: 15 December 2015.

*Corresponding author.

The circulant graph $C(n; M)$ is the graph with the vertex set $V(C(n; M)) = \{v_i | 0 \leq i \leq n-1\}$ and the edge set $E(C(n; M)) = \{v_i v_j | 0 \leq i \leq n-1, 0 \leq j \leq n-1, (i-j)(mod\ n) \in M\}$, $M \subseteq \{1, 2, \ldots, \lfloor n/2 \rfloor\}$, [1, 12].

The dominating set (total dominating set) D of a graph G is a set of vertices of G such that every vertex of $V(G) - D$ (respectively,$V(G)$) is adjacent to some vertex of D. The domination number $\gamma(G)$ (total domination number $\gamma_t(G)$) of G is the minimum cardinality of a dominating set (total dominating set) of G.

A subset $S \subseteq V(G)$ is a k-distance dominating set if every vertex in $V - S$ is within distance k of at least one vertex in S for integers $k > 1$. That is $N_k[S] = V(G)$. A subset $S \subseteq V(G)$ is a total k-distance dominating set (TkDDS for short) if every vertex $u \in V(G)$ is within distance k from at least one vertex in S other than itself. The minimum cardinality of a (total) distance k-dominating set in G is the (total) distance k-domination number of G, denoted by $\gamma^k(G)$ ($\gamma_t^k(G)$, respectively). Any TkDDS of cardinality $\gamma_t^k(G)$ is called a γ_t^k-set of G, [4].

2. **Preliminary results**

By the definitions of TkDDS and G^k it immediately follows:

Observation 2.1. *Let G be a nontrivial connected graph. Then a set $D \subseteq V(G)$ is a TkDDS of G if and only if D is a total dominating set of G^k.*

Corollary 2.2. *Each γ_t-set of G^k is a γ_t^k-set of G and vice versa, that is, $\gamma_t^k(G) = \gamma_t(G^k)$.*

An end-vertex is a vertex of degree one and a support vertex is one that is adjacent to an end-vertex. Let $S(G)$ be the set of support vertices of G. We say that a vertex $v \in V(G) - S(G)$ is γ_t^k-critical if $\gamma_t^k(G - v) < \gamma_t^k(G)$.

Observation 2.3. *Let v be a γ_t^k-critical vertex of a graph G. Then:*

(i) *Each vertex of $N_k(v)$ is not in any γ_t^k-set of $G - v$;*
(ii) *if T is a γ_t^k-set of $G - v$, then $T \cup \{u\}$ is a γ_t^k-set of G for every $u \in N_k(v)$;*
(iii) *$\gamma_t^k(G) = \gamma_t^k(G - v) + 1$.*

Proof. (i) If each vertex of $N_k(v)$ is in any γ_t^k-set of $G - v$, then $\gamma_t^k(G - v) \geq \gamma_t^k(G)$, a contradiction. (ii) and (iii) If T is a γ_t^k-set of $G - v$ and $u \in N_k(v)$, then $T \cup \{u\}$ is a TkDDS of G and $|T \cup \{u\}| = \gamma_t^k(G - v) + 1 \leq \gamma_t^k(G)$. Thus $\gamma_t^k(G) = \gamma_t^k(G - v) + 1$ and $T \cup \{u\}$ is a γ_t^k-set of G. $\square$

Since total domination may not be defined for a graph with isolated vertices, we say that a graph G is total k-distance domination vertex critical, or just γ_t^k-critical, if every vertex of $V(G) - S(G)$ is a γ_t^k-critical vertex. If G is γ_t^k-critical, and $\gamma_t^k(G) = r$, then we say that G is r-γ_t^k-critical. Note that a graph is vertex γ_t^k-critical if and only if each of its component is γ_t^k-critical.

Corollary 2.4. *If G is a connected γ_t^k-critical graph, then $\gamma_t^k(G - v) = \gamma_t^k(G) - 1$ for any $v \in V(G) - S(G)$.*

Proposition 2.5. *Let G be a graph.*

(i) *If v is a γ_t^k-critical vertex of G, then v is a γ_t-critical vertex of G^k.*

(ii) *If v is a γ_t-critical vertex of G^k, then v is a γ_t^k-critical vertex of G.*

Proof. The case of $k = 1$ is trivial, so we assume that $k \geq 2$. (i) Since v is γ_t^k-critical, $\gamma_t^k(G - v) = \gamma_t^k(G) - 1$. By Corollary 2.2, $\gamma_t^k(G - v) = \gamma_t((G - v)^k)$ and $\gamma_t^k(G) = \gamma_t(G^k)$. Since the total domination number does not increase when edges are added to a graph and since $(G - v)^k$ is a spanning subgraph of $G^k - v$, it follows that $\gamma_t((G - v)^k) \geq \gamma_t(G^k - v)$. Thus $\gamma_t(G^k - v) \leq \gamma_t^k(G - v) = \gamma_t^k(G) - 1 = \gamma_t(G^k) - 1$. (ii) Since v is a γ_t-critical vertex of G^k, $\gamma_t(G^k - v) = \gamma_t(G^k) - 1 = \gamma_t^k(G) - 1$ and no neighbor of v in G^k belongs to some γ_t-set of $G^k - v$. Hence each γ_t-set of $G^k - v$ is a total dominating set of $(G - v)^k$ which implies $\gamma_t(G^k - v) \geq \gamma_t((G - v)^k)$. Since always $\gamma_t(G^k - v) \leq \gamma_t((G - v)^k)$, the equality $\gamma_t(G^k - v) = \gamma_t((G - v)^k)$ holds. Thus $\gamma_t^k(G) - 1 = \gamma_t^k(G - v)$ as required. $\square$

Corollary 2.6. *Let G be a graph with $\delta(G) \geq 2$. Then G is γ_t^k-critical if and only if G^k is γ_t-critical.*

3. Total k-distance domination

We start this section with an important result from [2].

Theorem 3.1. [2] *Let G be a γ_t-critical graph of order n. Then $n \leq \Delta(G)(\gamma_t(G) - 1) + 1$.*

In what follows, for any vertex v in G, S_v denotes a total k-distance dominating set of the subgraph $G_v = G - v$ with minimum size, and S_v^u denotes the set $S_v \cup \{u\}$ for $u \in V(G)$.

Theorem 3.2. *Let G be a γ_t^k-critical graph of order n. Then $n \leq \Delta_k(G)(\gamma_t^k(G) - 1) + 1$.*

Proof. Let $v \in V(G) - S(G)$. Total criticality of G implies that there exists a S_v with $|S_v| = \gamma_t^k(G) - 1$ for G_v. Since each vertex of S_v can k-distance dominate at most $\Delta_k(G)$ vertices, S_v can k-distance dominate at most $\Delta_k(G)(\gamma_t^k(G) - 1)$ vertices, which implies that $n = |V(G_v)| + 1 \leq \Delta_k(G)(\gamma_t^k(G) - 1) + 1$. $\square$

In [7] it has been given the result:

Theorem 3.3. [7, Theorem 1] *Any γ_t-critical graph G of order $n = \Delta(G)(\gamma_t(G) - 1) + 1$ is regular.*

One can have the following which shows that Theorem 3.3 cannot be generalized for total k-distance domination of G.

Theorem 3.4. *There is no γ_t^k-critical graph of order $n = \Delta_k(G)(\gamma_t^k(G) - 1) + 1$.*

Proof. Let G be a $k - \gamma_t^k$-critical graph of order $\Delta_k(G)(\gamma_t^k(G) - 1) + 1$. So G^k is a $k - \gamma_t$-critical graph of order $\Delta_{G^k}(k - 1) + 1$. Let $v \in G^k$ and S_v is a $\gamma_t(G^k - v)$-set. Then S_v is an efficient total dominating set for $G^k - v$ and no neighbor of v is in S_v. Hence in G^k each edge belongs to triangles, there is at least one vertex $u \in G^k$ such that $| N_{G^k}(u) \cap S_v | \neq 1$. Therefore by above assumptions, the bound $\Delta(G^k)(\gamma_t(G^k) - 1) + 1$ is not attainable. $\square$

Combining of Theorems 3.2 and 3.4, we have the following corollary:

Corollary 3.5. If G is a γ_t^k-critical graph of order n, then $n \leq \Delta_k(G)(\gamma_t^k(G) - 1)$.

We need the results from [10] and [2].

Lemma 3.6. [10] *For each $k \geq 1$, if the vertices x and y are two vertices in G such that $\rho_G(x, y) = d(G)$, then $d_{G^k}(x, y) = d(G^k)$. Furthermore, $d(G^k) = \lceil \frac{d(G)}{k} \rceil$.*

Theorem 3.7. [2] *For $m \leq 8$, the diameter of a m-γ_t-critical graph is at most the value given by the following table:*

m	3	4	5	6	7	8
diam	3	4	6	7	9	11

A generalization of Lemma 3.6 and Theorem 3.7 is the next result.

Proposition 3.8. *If G is an m-γ_t^k-critical graph for $m \leq 8$, then $d(G) \leq 11k$.*

Proof. Using Lemma 3.6 and Theorem 3.7, we have $\frac{d(G)}{k} \leq d(G^k) \leq 11$. □

Example 3.9. *For $n \leq 7$, $\gamma_t^2(C_n) = \gamma_t^2(P_n) = 2$, and for $n \geq 8$, $\gamma_t^2(C_n) = \gamma_t^2(P_n) = \lceil \frac{n}{7} \rceil + \lceil \frac{n-1}{7} \rceil - 1$ if $n \equiv 2 \pmod 7$ and $\gamma_t^2(C_n) = \gamma_t^2(P_n) = \lceil \frac{n}{7} \rceil + \lceil \frac{n-1}{7} \rceil$ otherwise.*

Example 3.10. *$\gamma_t^3(C_n) = \gamma_t^3(P_n) = \lceil \frac{n}{5} \rceil + 1$ if $n = 10k + 4$, and $\gamma_t^3(C_n) = \gamma_t^3(P_n) = \lceil \frac{n}{5} \rceil$ otherwise.*

We have the result from [6] that is useful for other results.

Theorem 3.11. [6, Theorem 13] Let $H_{2m,n}$ be a Harary graph with n vertices and $n = (3m+1)l + r$, where $0 \leq r \leq 3m$. Then

$$\gamma_t(H_{2m,n}) = \begin{cases} 2l & \text{if} \quad r = 0 \\ 2l+1 & \text{if} \quad 1 \leq r \leq m \\ 2l+2 & \text{if} \quad m+1 \leq r \leq 3m. \end{cases}$$

Now using Theorem 3.11 to follow the following result.

Theorem 3.12. *$H_{2k,n}$ is γ_t-critical if and only if $3k+1 \mid n-1$ or $3k+1 \mid n-(k+1)$.*

Proof. Corollary 2.2 and Theorem 3.11 imply that $\gamma_t^k(C_n) = \gamma_t(C_n^k) = \gamma_t(H_{2k,n})$. Now we show that $H_{2k,n}$ is γ_t-critical if and only if $3k+1 \mid n-1$ or $3k+1 \mid n-(k+1)$.
It is easy to see that, two adjacent vertices dominate at most $3k+1$ vertices and three adjacent vertices dominate at most $4k+1$ vertices. Let $n = (3k+1)l + r$ where $0 \leq r \leq 3k$. If $r = 0$, then $\gamma_t(H_{2k,n}) = 2l$. Now consider v as any vertex and $H_{2k,n} - v$ then the size of $H_{2k,n} - v$ is $(3k+1)(l-1) + 3k$. This shows $\gamma_t(H_{2k,n} - v) = 2l$, it is not critical, for $n = 3k+1)l$.

If $r = 1$, then $\gamma_t(H_{2k,n}) = 2l + 1$. Now consider $H_{2k,n} - v_n$ (note that all vertices play same role), then the set

$$S = \{v_{k+1}, v_{2k+1}; v_{4k+2}, v_{5k+2}; v_{7k+3}, v_{8k+3}; \dots ; v_{(3l-5)k+l}, v_{(3l-4)k+l}; v_{(3l-2)k+l}, v_{(3l-1)k+l}\}$$

is a total dominating set of $H_{2k,n} - v_n$ with size $|S| = 2l$. Thus $H_{2k,n}$ for $n = (3k + 1)l + 1$ is total critical.
Let $2 \le r \le k$. Then $\gamma_t(H_{2k,n}) = 2l + 1$. Since $2l$ vertices totaly dominate at most $(3k + 1)l$ vertices and for $2 \le r \le k$ the graph $H_{2k,n} - v$ has at least $(3k + 1)l + 1$ vertices, then $\gamma_t(H_{2k,n} - v) = 2l + 1$. Therefore $H_{2k,n}$ is not total critical, where $n = (3k + 1)l + r$ and $2 \le r \le k$.

Let $r = k + 1$. Then $\gamma_t(H_{2k,n}) = 2l + 2$. Same as above consider $H_{2k,n} - v_n$, then the set vertex

$$S = \{v_{k+1}, v_{2k+1}; v_{4k+2}, v_{5k+2}; \dots ; v_{(3l-5)k+l}, v_{(3l-4)k+l}; v_{(3l-2)k+l}, v_{(3l-1)k+l}, v_{(3l)k+l}\}$$

is a total dominating set of $H_{2k,n} - v_n$ with size $|S| = 2l + 1$. Thus $H_{2k,n}$ for $n = (3k + 1)l + k + 1$ is total critical.

Let $k + 2 \le r \le 3k$. Then $\gamma_t(H_{2k,n}) = 2l + 2$. Since $2l + 1$ vertices totaly dominate at most $(3k + 1)l + k$ vertices and for each v, $H_{2k,n} - v$ has at least $(3k + 1)l + k + 1$ vertices. And so $\gamma_t(H_{2k,n} - v) = 2l + 2$. Thus $H_{2k,n}$ for $n = (3k + 1)l + r$ for $k + 2 \le r \le 3k$ is not total critical. Therefore the proof is complete. $\square$

From Theorem 3.12, for $H_{2k,n}$ where $3k+1 \mid n-1$ or $3k+1 \mid n-(k+1)$ we have $n < \Delta(G)(\gamma_t(G) - 1) + 1$. This result shows that the converse of Theorem 3.1(ii) is not true.
As an immediate result from Theorems 3.11 and 3.12 we have:

Proposition 3.13. Let C_n be a cycle with n vertices and $n = (3k + 1)l + r$, where $0 \le r \le 3k$. Then

(i)

$$\gamma_t^k(C_n) = \begin{cases} 2l & \text{if } \ r = 0 \\ 2l + 1 & \text{if } \ 1 \le r \le k \\ 2l + 2 & \text{if } \ k + 1 \le r \le 3k. \end{cases}$$

(ii) C_n is γ_t^k-critical if and only if $3k + 1 \mid n - 1$ or $3k + 1 \mid n - (k + 1)$.

Proof. By Corollary 2.2, $\gamma_t^k(C_n) = \gamma_t(C_n^k)$. Since $C_n^k = H_{2k,n}$, (i) holds by Theorem 3.11 and (ii) holds by Theorem 3.12. $\square$

Observation 3.14. *There is no γ_t^k-critical graph with $\Delta_k = 2$.*

Proof. Let G be a γ_t^k-critical graph with $\Delta_k = 2$ and $x \in V(G)$. Since $\Delta_k = 2$, $deg(x) \le 2$. Therefore G is P_2, P_3 or C_3 and none of them is γ_t^k-critical graph. $\square$

Proposition 3.15. *Let G be a corona of G' with $\delta(G') \ge 2$. Then G is γ_t-critical, and for $k \ge 2$, G is not necessary γ_t^k-critical.*

Proof. It is clear that $\gamma_t(G) = |V(G')|$. If v is a vertex in $V(G) \setminus S(G)$, then $\gamma_t(G - v) = |V(G')| - 1$. For $k \ge 2$, let $G' = K_n$ $(n \ge 3)$ and G be corona of K_n. Then $\gamma_t^k(G) = \gamma_t^k(G - v)$ for $k \ge 2$. $\square$

Theorem 3.16. *If G has a cut vertex, and G is not corona of a graph G' with $\delta(G') \geq 2$, then G is not a γ_t^k-critical graph with $k \geq 1$.*

Proof. Let u be a cut vertex of G such that $u \notin S(G)$, and $G - u$ has two components, say G_1 and G_2. Suppose on the contrary that G is a γ_t^k-critical graph and S is a γ_t^k-set for G. Let S_1 and S_2 be the γ_t^k-set for G_1 and G_2. Let $n_1 = |S \cap G_1|$ and $n_2 = |S \cap G_2|$. Since G is a γ_t^k-critical graph, $\gamma_t^k(G-u) = \gamma_t^k(G_1) + \gamma_t^k(G_2)$, and then we will have $\gamma_t^k(G) = \gamma_t^k(G_1) + \gamma_t^k(G_2) + 1$.

If $u \in S$, one of the followings holds:

(i) $n_1 = \gamma_t^k(G_1)$, $n_2 = \gamma_t^k(G_2)$; (ii) $n_1 > \gamma_t^k(G_1)$, $n_2 < \gamma_t^k(G_2)$; (iii) $n_1 < \gamma_t^k(G_1)$, $n_2 > \gamma_t^k(G_2)$.

Remove the vertex u from γ_t^k-set. It is easy to see that G can be total k-distance dominated by $S_1 \cup S_2 \cup \{x\}$, where $x \in N_k(u)$, a contradiction.

Let now that $u \notin S$. We have one of the followings:

(a): $n_1 = \gamma_t^k(G_1)$, $n_2 > \gamma_t^k(G_2)$;
(b): $n_1 > \gamma_t^k(G_1)$, $n_2 = \gamma_t^k(G_2)$;
(c): $n_1 < \gamma_t^k(G_1)$, $n_2 > \gamma_t^k(G_2)$;
(d): $n_1 > \gamma_t^k(G_1)$, $n_2 < \gamma_t^k(G_2)$.

Case (a): It is obvious that $n_2 = \gamma_t^k(G_2) + 1$. Suppose G_1 and G_2 are γ_t^k-critical graphs. Let $x \in N_k(u) \cap G_1$ and $y \in N_k(u) \cap G_2$. Let S_x, S_y be the γ_t^k-set for $G_1 - x$ and $G_2 - y$. Let $S' = S_x \cup S_y \cup \{u, v\}$, for which $v \in N_u$. It is easy to check that S' is a γ_t^k-set for G with $\gamma_t^k(G_1) + \gamma_t^k(G_2)$ elements, a contradiction. Thus at most one component is γ_t^k-critical graph. Without lose of generality assume that G_1 is γ_t^k-critical graph. For $k = 1$, let $x \in G_1$ be a vertex with distance two from u. We can total dominate G with $S_x \cup S_2 \cup \{y\}$ where $y \in N(x) \cap N(u)$ and S_x is total dominating set for $G_1 - x$, a contradiction. If $k \geq 2$ let $x \in N_k(u)$ and $y \in N_k(u) \cap N_k(x)$, we can total k-distance dominate G with $S_x \cup S_2 \cup \{y\}$ where S_x is γ_t^k-set for $G_1 - x$ with $\gamma_t^k(G_1) + \gamma_t^k(G_2)$ elements, a contradiction. So suppose both of G_1 and G_2 are not γ_t^k-critical graphs. Let $x \in G_2$. Since G_2 is not γ_t^k-critical graph, one of the following holds:

(a-1) $\gamma_t^k(G_2 - x) = \gamma_t^k(G_2)$;
(a-2) $\gamma_t^k(G_2 - x) > \gamma_t^k(G_2)$;
(a-3) $\gamma_t^k(G_2 - x) < \gamma_t^k(G_2)$.

In Case (a-1), if $\gamma_t^k(G_2 - x) = \gamma_t^k(G_2)$, then $\gamma_t^k(G - x) = \gamma_t^k(G_1) + \gamma_t^k(G_2)$, a contradiction.

In Case (a-2), if $\gamma_t^k(G_2 - x) > \gamma_t^k(G_2)$, then $\gamma_t^k(G - x) \geq \gamma_t^k(G_1) + \gamma_t^k(G_2) + 1$, a contradiction.

In Case (a-3), if $\gamma_t^k(G_2 - x) < \gamma_t^k(G_2)$, then $\gamma_t^k(G_2 - x) = \gamma_t^k(G_2) - 1$. Let S_x be a γ_t^k-set for $G_2 - x$ and $y \in N_k(x) \cap N_k(u)$. It is easy to see that $S_x \cup S_1 \cup \{y\}$ is a γ_t^k-set for G with $\gamma_t^k(G_1) + \gamma_t^k(G_2)$ elements, a contradiction.

Case (b): The proof is similar to Case (a).

Case (c): $n_1 < \gamma_t^k(G_1)$, $n_2 > \gamma_t^k(G_2)$. If G_2 is γ_t^k-critical graph. Since $n_1 < \gamma_t^k(G_1)$, so some vertices of G_1 are k-distance dominated by elements of G_2. There is at least one vertex of G_2 which k-distance dominates some vertices of G_1. Let $s_1 \in S \cap G_2$ that k-distance dominate some vertices of G_1. Let S_{s_1} be a γ_t^k-set for $G_2 - s_1$ with $\gamma_t^k(G_2) - 1$ elements and $S_1 := S \cap G_1$ with n_1 elements.

It is clear that $S_{s_1} \cup S_1 \cup \{u, s_1\}$ is a γ_t^k-set for G with $\gamma_t^k(G_2) + n_1 + 1$ elements which is less than $\gamma_t^k(G_1) + \gamma_t^k(G_2) + 1$, a contradiction. If G_2 is not γ_t^k-critical graph, then there is a vertex $x \in G_2$ such that $\gamma_t^k(G_2 - x) \geq \gamma_t^k(G_2)$. Therefore $\gamma_t^k(G - x) \geq \gamma_t^k(G)$, a contradiction.

Case (d): It is similar to Case (c). $\square$

From Theorem 3.16 we have:

Corollary 3.17. *If T is a tree with size $n \geq 3$, then T is not γ_t^k-critical graph.*

4. **On the equivalence of a conjecture**

Mojdeh et al., in [7] conjectured: for $r \geq 6$, there is no 3-γ_t-critical r-regular graph of order $2r + 1$, that has been disproved by J. Rad et al., in [5]. Afterward it was also studied more by Sohn et al., in [9] and Wang et al, in [11]. Authors of [5] and [11] showed:

Theorem 4.1. [5, 11, Theorems 2.2, 2.1] *For any even $r \geq 6$, if $M_r = \{2k - 1 : 1 \leq k \leq r/2\}$. The circulant graphs $G_r = C(2r + 1; M_r)$ are 3-γ_t-critical graphs of order $2r + 1$.*

In this section we study a similar result on power k of G, that is, we wish to study this problem when $\Delta_k(G) \geq 6$:

Theorem 4.2. *For any even $k \geq 2$, there is a 3-γ_t^k-critical graph of order $3\frac{\Delta_k(G)}{2} + 2$.*

Proof. Let C_n be a cycle with size $n = 3k + 2$. Then C_n^k is a Harary graph $G = H_{2k,n}$. Since $n = 3k + 2 = (3k + 1) + 1$, from Theorem 3.12 $H_{2k,n}$ is a 3-γ_t-critical graph of order $n = 3k + 2$. Therefore from Proposition 3.13, C_n is a 3-γ_t^k-critical graph of order $3\frac{\Delta_k(G)}{2} + 2$. $\square$

Also in [5] we have

Theorem 4.3. [5, Theorem 2.1] *A graph G of order 9 is 3-γ_t-critical if and only if $G = F$, where F be the graph with vertex set $\{x, y, z, v_1, v_2, v_3, v_4, v_5, v_6\}$ and edge set*
$\{xv_i : i = 1, 2, 3, 4\} \cup \{v_2v_3, v_1v_5, v_1y, v_1v_6, v_2y, v_2v_6, v_3v_5, v_3z, v_4v_5, v_4z, v_4v_6, v_5y, yz, zv_6\}$.

We can have corresponding result here.

Theorem 4.4. *There is no 3-γ_t^k-critical graph with $\delta_k = \Delta_k = 4$ of order 9.*

Proof. Let G be a 3-γ_t^k-critical graph of order 9 with $\delta_k = \Delta_k = 4$. Then G^k should be a 3-γ_t-critical 4-regular graph of order 9. Since with Theorem 4.3, the only 3-γ_t-critical graph of order 9 is F, so $G^k = F$. In F two vertices z and y are adjoint, therefore in G we should have $N_k(y) \cap N_k(z) \neq \emptyset$, but in F we have $N(y) \cap N(z) = \emptyset$, a contradiction. $\square$

Now we can pose a conjecture as follows:
Conjecture. For $\Delta_k(G) \geq 6$, there is no 3-γ_t^k-critical graph of order $2\Delta_k(G) + 1$.
we have followings from [8] and [9] respectively.

Theorem 4.5. [8, Theorem 3.6] *There is no* 3-γ_t*-critical graph of order* $\Delta(G)+3$ *with* $\Delta(G) = 3, 5$ *and* $\delta(G) \geq 2$.

Theorem 4.6. [8, Theorem 12] *There is no* 4-γ_t*-critical graph* G *of order* $\Delta(G)+4$ *with* $\Delta(G)) = 3, 5, 7$ *and* $\delta(G) \geq 2$.

Now we have corresponding results for power graph.

Theorem 4.7. (i) *There is no* 3-γ_t^k*-critical graph of order* $\Delta_k(G)+3$ *with* $\Delta_k(G) = 3, 5$ *and* $\delta_k(G) \geq 2$.
(ii) *There is no* 4-γ_t^k*-critical graph of order* $\Delta_k(G)+4$ *with* $\Delta_k(G) = 3, 5, 7$ *and* $\delta_k(G) \geq 2$.

Proof. (i) Suppose on the contrary, there is a graph G which is 3-γ_t^k-critical with the given properties. Then G^k is a 3-γ_t-critical graph of order $\Delta(G^k)+3$ with $\Delta(G^k) = 3, 5$ and $\delta(G^k) \geq 2$, a contradiction.
(ii). This item has similar proof and it is left. □

To close of this paper, we pose the following problem:

problem 4.8. *How do we can generalize the properties of* γ_t*-criticality of a graph* G *to the* γ_t^k*-criticality of* G*?*

Acknowledgments

The authors thank the referee for his/her helpful comments and suggestions to improve the exposition and readability of the paper.

References

[1] J. A. Bondy and U. S. R. Murty, *Graph Theory*, Graduate Texts in Mathematics, **244**, Springer, New York, 2008.

[2] W. Goddard, T. W. Haynes, M. A. Henning and L. C. van der Merwe, The diameter of total domination vertex critical graphs, *Discrete Math.*, **286** (2004) 255–261.

[3] T. W. Haynes, S. T. Hedetniemi and P. J. Slater, *Fundamentals of Domination in Graphs*, Monographs and Textbooks in Pure and Applied Mathematics, **208**, Marcel Dekker, Inc., New York, 1998.

[4] M. A. Henning, O. R. Oellermann and H. C. Swart. Bounds on distance domination parameters, *J. Combin. Inform. System Sci.*, **16** (1991) 11–18.

[5] N. Jafari Rad and S. Rahimi Sharebaf, A Remark on Total Domination Critical Graphs, *Appl. Math. Sci. (Ruse)*, **2** (2008) 2025–2028.

[6] A. Khodkar, D. A. Mojdeh and A. P. Kazemi, Domination in Harary graphs, *Bull. Inst. Combin. Appl.*, **49** (2007) 61–78.

[7] D. A. Mojdeh and N. Jafari Rad, On an open problem concerning total domination critical graphs, *Expo. Math.*, **25** (2007) 175–179.

[8] D. A. Mojdeh and N. Jafari Rad, On the total domination critical graphs, Electronic Notes in Discrete Mathematics, **24**, Elsevier Sci. B. V., Amsterdam, 2006 89–92.

[9] M. Y. Sohn, D. Kim, Y. S. Kwon and J. Lee, On the existence problem of the total domination vertex critical graphs, *Discrete Appl. Math.*, **159** (2011) 46–52.

[10] F. Tian and J.-Ming Xu, Distance domination-critical graphs, *Appl. Math. Lett.*, **21** (2008) 416–420.

[11] H. Wang and G. Wang, A Note on Total Domination Critical Graphs, ResearchGate, 2014,`http://www.researchgate.net/publication/265845738`.

[12] D. B. West, *Introduction to gragh theory*, (2nd edition), Prentice Hall, U.S.A., 2001.

Doost Ali Mojdeh

Department of Mathematics, University of Mazandaran Babolsar, Iran, and

Department of Mathematics, University of Tafresh Tafresh, Iran

Email: damojdeh@umz.ac.ir

A. Sayed-Khalkhali

Department of Mathematics, University of Tafresh Tafresh, Iran

Email: a.s.khalkhali@gmail.com

Hossein Abdollahzadeh Ahangar

Department of of Basic Science, Babol University of Technology Babol, Iran

Email: ha.ahangar@nit.ac.ir

Yancai Zhao

Department of Basic Science, Wuxi City College of Vocational Technology Jiangsu 214153, China

Email: zhaoyc69@126.com