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EXTREME EDGE-FRIENDLY INDICES OF COMPLETE BIPARTITE GRAPHS

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Abstract. Let $G = (V, E)$ be a simple graph. An edge labeling $f : E \to \{0, 1\}$ induces a vertex labeling $f^+ : V \to \mathbb{Z}_2$ defined by $f^+(v) \equiv \sum\limits_{uv\in E} f(uv) \pmod 2$ for each $v \in V$, where $\mathbb{Z}_2 = \{0, 1\}$ is the additive group of order 2. For $i \in \{0, 1\}$, let $e_f(i) = |f^{-1}(i)|$ and $v_f(i) = |(f^+)^{-1}(i)|$. A labeling f is called edge-friendly if $|e_f(1) - e_f(0)| \le 1$. $I_f(G) = v_f(1) - v_f(0)$ is called the edge-friendly index of G under an edge-friendly labeling f. Extreme values of edge-friendly index of complete bipartite graphs will be determined.

1. Introduction

Let $G = (V, E)$ be a simple graph. An edge labeling $f : E \to \{0, 1\} \subset \mathbb{N}$ induces a vertex labeling $f^+ : V \to \mathbb{Z}_2$ defined by $f^+(v) \equiv \sum\limits_{uv\in E} f(uv) \pmod 2$ for each $v \in V$, where $\mathbb{Z}_2 = \{0, 1\}$ is the additive group of order 2. We sometimes view the value of $f^+(v)$ as an integer. For $i \in \{0, 1\}$, let $e_f(i) = |f^{-1}(i)|$ and $v_f(i) = |(f^+)^{-1}(i)|$. Let $I_f(G) = v_f(1)-v_f(0)$. An edge labeling f is *edge-friendly* if $|e_f(1) - e_f(0)| \le 1$. The concept of edge-friendly index maybe first introduced by Lee and Ng [4] on considering edge cordial labeling. Unfortunately, we cannot find this paper even through we have asked the authors. Readers are referred to [1] for detail about edge cordial.

The number $I_f(G)$ is called the *edge-friendly index* of G under f if f is an edge-friendly labeling of G. The set $\mathrm{FEFI}(G) = \{I_f(G) \mid f \text{ is edge-friendly}\}$ is called the *full edge-friendly index set* of G. This is a dual concept of full friendly index set which was first introduced by the author and H. Kwong [9]. Readers who are interested on friendly index or friendly index set may refer to [2, 3, 5–15].

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In this paper, we shall study the extreme values of edge-friendly index of complete bipartite graphs $K_{m,n}$.

2. Some Basic Properties

In the following, all graphs are simple and connected. The codomain of any edge labeling is $\mathbb{Z}_2$. Suppose f is an edge labeling. A vertex (resp. an edge) is called an *i-vertex* (resp. *i-edge*) under f if it is labeled by $i \in \{0, 1\}$. Notation and concepts not defined here are referred to [16].

Suppose G is a graph of order p. Since $v_f(1) + v_f(0) = p$ for any edge labeling f of G, $I_f(G) = 2v_f(1) - p$. Thus, it suffices to study the number of 1-vertices instead of studying the edge-friendly index of G under f.

Lemma 2.1. *Let f be any edge labeling of a graph $G = (V, E)$. Then $v_f(1)$ must be even.*

Proof. Since the value of each edge contributes twice toward the sum of values of vertex,

$$v_f(1) = \sum_{u \in V} f^+(v) \equiv 2 \sum_{e \in E} f(e) \equiv 0 \pmod 2.$$

□

By means of the above lemma, we may write $v_f(1) = 2j$ for some j with $0 \le j \le \lfloor p/2 \rfloor$, where f is an edge labeling of a graph G of order p. So $I_f(G) = 4j - p$ for some j, $0 \le j \le \lfloor p/2 \rfloor$. It implies that

$$\mathrm{FEFI}(G) \subseteq \{4j - p \,|\, 0 \le j \le \lfloor p/2 \rfloor\}.$$

A *labeling matrix* $L_f(G)$ for an edge labeling f of a graph G is a matrix whose rows and columns are indexed by the vertices of G and the (u, v)-entry is $f(uv)$ if $uv \in E$, and is $*$ otherwise.

Suppose $L_f(G)$ is a labeling matrix for the edge labeling f of G. If we view the entries of $L_f(G)$ as elements in $\mathbb{Z}_2$, then $f^+(v)$ is the v-row sum (as well as v-column sum), where entries with $*$ will be treated as 0.

Let $X = \{x_1, \ldots, x_m\}$ and $Y = \{y_1, \ldots, y_n\}$ be the bipartition of the complete bipartite graph $K_{m,n}$. Under this indexing of vertices, a labeling matrix for any edge labeling f is of the form

$$\begin{pmatrix} \bigstar_m & A \\ A^T & \bigstar_n \end{pmatrix},$$

where $\bigstar_r$ is a square matrix of order r with all entries being $*$ and A is an $m \times n$ matrix whose entries are elements of $\mathbb{Z}_2$. So the multi-set of row sums and column sums of A is equal to the sequence $\{f^+(x_1), \ldots, f^+(x_m), f^+(y_1), \ldots, f^+(y_n)\}$. Thus, we shall only consider such matrix A and we shall denote it as $A_f(G)$ when there is some ambiguity. Thus, we shall use such matrix $A_f(G)$ (or A) to define an edge labeling f. Let $v_A(1)$ denote the number of 1's being row sum or column sum. Then $v_A(1) = v_f(1)$. Similarly, we may define $v_A(0)$, which will equal to $v_f(0)$. Also we may define $e_A(1)$ and $e_A(0)$ to be the number of 1 and 0 used to form the matrix A, respectively. So $e_A(i) = e_f(i)$, $i = 0, 1$.

An $m \times n$ matrix A satisfying the following conditions is called a *friendly matrix* of $K_{m,n}$:

1. Each entry of A is either 1 or 0;
2. $|e_A(1) - e_A(0)| \le 1$.

In order to find an edge-friendly labeling, it suffices to find a suitable size of friendly matrix. Following is a well-known lemma.

Lemma 2.2. *Let (X, Y) be a bipartition of a bipartite graph G. Then*
$$\sum_{x\in X}\deg(x) = \sum_{y\in Y}\deg(y) = q(G),$$
where $q(G)$ is the size of G.

3. Minimum Value of $v_f(1)$

For easy to describe some matrices, we let $J_{m,n}$ be the $m \times n$ matrix whose entries are 1 and $O_{m,n}$ be the $m \times n$ zero matrix.

For $K_{1,n}$, it is easy to obtain that

$$\mathrm{FEFI}(K_{1,n}) = \begin{cases} \{-2, 2\}, & n = 4k+1; \\ \{1\}, & n = 4k+2; \\ \{0\}, & n = 4k+3; \\ \{-1\}, & n = 4k+4, \end{cases}$$

where $k \ge 0$. For $K_{2,2} \cong C_4$, it is easy to get that

$$FEFI(K_{2,2}) = \{0, 4\}.$$

So we assume that $m, n \ge 2$ and $\max\{m, n\} \ge 3$.

Lemma 3.1. *For $k \ge 1$, there is no edge-friendly labeling f of $K_{2,4k+2}$ such that $v_f(1) = 0$. But there is an edge-friendly labeling μ of $K_{2,4k+2}$ such that $v_\mu(1) = 2$.*

Proof. Suppose there is an edge-friendly labeling f of $K_{2,4k+2}$ such that $v_f(1) = 0$. Let H be the subgraph of $K_{2,4k+2}$ induced by all 1-edges. Since f is edge-friendly, the size of H is $4k+2$. Let X and Y be the bipartition of $K_{2,4k+2}$. Also let $U \subseteq X$ and $W \subseteq Y$ be the bipartition of H. Since $v_f(1) = 0$, the degree of each vertex in H must be even and positive. Hence $\deg_H(w) = 2$ for each $w \in W$. Thus $U = X$. By means of the size of H, we have $|W| = 2k+1$. Hence $H \cong K_{2,2k+1}$ which contradicts $\deg_H(u)$ being even for $u \in U$.

Let the block matrix $A_{2,4k+2} = \begin{pmatrix} J_{2,2k+1} & O_{2,2k+1} \end{pmatrix}$. $A_{2,4k+2}$ is the required friendly matrix which induces an edge-friendly labeling μ for $K_{2,4k+2}$ with $v_\mu(1) = v_{A_{2,4k+2}}(1) = 2$. □

We define some useful friendly matrices first. Let

$$A_{2h,4}=\begin{pmatrix}J_{2h,2} & O_{2h,2}\end{pmatrix}\text{ for } h\geq 1,\quad A_{3,4}=\begin{pmatrix}1&1&0&0\\0&1&1&0\\1&0&1&0\end{pmatrix},$$

$$D_h=\begin{pmatrix}J_{h,6}\\O_{h,6}\end{pmatrix}\text{ for } h\geq 1,\qquad A_{6,6}=\begin{pmatrix}1&1&1&1&1&1\\1&1&1&1&0&0\\1&1&1&0&1&0\\1&1&1&0&0&1\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix}.$$

Lemma 3.2. *For $k\geq 1$, there is a friendly matrix A of $K_{m,4k}$ such that $v_A(1)=0$.*

Proof. For $m=2s\geq 2$, let

$$A=A_{2s,4k}=\begin{pmatrix}A_{2s,4} & A_{2s,4} & \cdots & A_{2s,4}\end{pmatrix}.$$

For $m=2s+1\geq 3$, let

$$A=A_{2s+1,4k}=\begin{pmatrix}A_{2s-2,4} & A_{2s-2,4} & \cdots & A_{2s-2,4}\\ A_{3,4} & A_{3,4} & \cdots & A_{3,4}\end{pmatrix}. \tag{3.1}$$

Note that, when $s=1$, the first row of block matrices does not appear. It is easy to check that each matrix above is friendly $v_A(1)=0$. □

Example 3.1. Let $A_{4,8}=\left(\begin{array}{cccc|cccc}1&1&0&0&1&1&0&0\\1&1&0&0&1&1&0&0\\1&1&0&0&1&1&0&0\\1&1&0&0&1&1&0&0\end{array}\right)$ and $A_{7,8}=\left(\begin{array}{cccc|cccc}1&1&0&0&1&1&0&0\\1&1&0&0&1&1&0&0\\1&1&0&0&1&1&0&0\\1&1&0&0&1&1&0&0\\\hline 1&1&0&0&1&1&0&0\\0&1&1&0&0&1&1&0\\1&0&1&0&1&0&1&0\end{array}\right)$.

They define edge-friendly labelings for $K_{4,8}$ and $K_{7,8}$ with $v_{A_{4,8}}(1)=0$ and $v_{A_{7,8}}(1)=0$, respectively.

Lemma 3.3. *For $k\geq 0$ and odd m, there is no edge-friendly labeling f of $K_{m,4k+2}$ such that $v_f(1)=0$. But there is an edge-friendly labeling μ of $K_{m,4k+2}$ such that $v_\mu(1)=2$.*

Proof. Suppose there is an edge-friendly labeling f of $K_{m,4k+2}$ such that $v_f(1)=0$. Let H be the subgraph of $K_{m,4k+2}$ induced by all 1-edges. Then $q(H)=m(2k+1)$, which is odd. Let U and W be the bipartition of H. Since $v_f(1)=0$, the degree of each vertex in H must be even and positive. By Lemma 2.2 we know that it is impossible.

For the second part of the lemma. Let

$$Z_{m,2}=\begin{pmatrix}J_{2h,2}\\1\quad 0\\O_{2h,2}\end{pmatrix}\text{ if } m=4h+1, Z_{m,2}=\begin{pmatrix}J_{2h+1,2}\\1\quad 0\\O_{2h+1,2}\end{pmatrix}\text{ if } m=4h+3.$$

It is easy to see that the matrices above are friendly. Then

$$A_{m,4k+2} = \begin{pmatrix} A_{m,4k} & Z_{m,2} \end{pmatrix}$$

is the required friendly matrix which induces an edge-friendly labeling μ of $K_{m,4k+2}$ such that $v_\mu(1) = 2$, where $A_{m,4k}$ is defined in (3.1). $\square$

Example 3.2. Let $A_\mu(K_{3,10}) = \left(\begin{array}{cccc|cccc|cc} 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \end{array}\right)$ and $A_\mu(K_{5,10}) = \left(\begin{array}{cccc|cccc|cc} 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ \hline 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \end{array}\right)$. They define edge-friendly labelings μ for $K_{3,10}$ and $K_{5,10}$ with $v_\mu(1) = 2$, respectively.

Lemma 3.4. *For $h, k \geq 1$, there is a friendly matrix A of $K_{4h+2,4k+2}$ such that $v_A(1) = 0$.*

Proof. Let

$$A = A_{4h+2,4k+2} = \left(\begin{array}{cccc|c} A_{4h-4,4} & A_{4h-4,4} & \cdots & A_{4h-4,4} & D_{2h-2} \\ \hline A_{3,4} & A_{3,4} & \cdots & A_{3,4} & \multirow{2}{*}{$A_{6,6}$} \\ A_{3,4} & A_{3,4} & \cdots & A_{3,4} & \end{array}\right)$$

It is easy to check that A is friendly and $v_A(1) = 0$. $\square$

Example 3.3. Let $A_\mu(K_{10,10}) = \left(\begin{array}{cccc|cccccc} 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$. This matrix defines an edge-friendly labeling μ for $K_{10,10}$ with $v_\mu(1) = 0$.

Lemma 3.5. *For odd m and n, there is a friendly matrix A of $K_{m,n}$ such that $v_A(1) = 0$.*

Proof. Suppose one of m and n is of the form $4k + 3$ for some $k \geq 0$. Without loss of generality, we assume that $n = 4k + 3$.

We consider $k = 0$ first. Let

$$A_{3,3} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, A_{4,3} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } A_{5,3} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Clearly, $v_{A_{3,3}}(1) = 0$, $v_{A_{4,3}}(1) = 0$ and $v_{A_{5,3}}(1) = 0$.

Suppose there is a friendly matrix $A_{2s-3,3}$ of size $(2s-3)\times 3$ such that $v_{A_{2s-3,3}}(1)=0$, where $s\geq 3$. Let $A_{2s+1,3}=\begin{pmatrix} A_{4,3} \\ A_{2s-3,3}\end{pmatrix}$. Clearly $A_{2s+1,3}$ is a friendly matrix of $K_{2s+1,3}$ such that $v_{A_{2s+1,3}}(1)=0$. By mathematical induction, we know that there is friendly matrix A of $K_{2s+1,3}$ such that $v_A(1)=0$ for all $s\geq 1$.

For the general case, i.e., $m=2s+1$ and $n=4k+3$, let

$$A_{2s+1,4k+3}=\begin{pmatrix} A_{2s+1,4k} & A_{2s+1,3}\end{pmatrix},$$

where $A_{2s+1,4k}$ is defined in (3.1) and $A_{2s+1,3}$ is defined above. Then $A_{2s+1,4k+3}$ is a required friendly matrix.

The remaining case is when $m=4h+1$ and $n=4k+1$ for some $h,k\geq 1$. When $n=5$. Let

$$A_{4,5}=\begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \text{ and } A_{5,5}=\begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Clearly, $A_{5,5}$ is a friendly matrix of $K_{5,5}$ such that $v_{A_{5,5}}(1)=0$. Suppose there is a friendly matrix $A_{4h-3,5}$ of $K_{4h-3,5}$ such that $v_{A_{4h-1,5}}(1)=0$, where $h\geq 2$. Let $A_{4h+1,5}=\begin{pmatrix} A_{4,5} \\ A_{4h-3,5}\end{pmatrix}$. This is a required friendly matrix. By mathematical induction, we know that there is a friendly matrix A of $K_{4h+1,5}$ such that $v_A(1)=0$ for all $h\geq 1$.

For the general case, similar to the construction above, we let

$$A_{4h+1,4k+1}=\begin{pmatrix} A_{4h+1,4k-4} & A_{4h+1,5}\end{pmatrix},$$

where $k\geq 2$. Hence this matrix is a required friendly matrix. □

Corollary 3.6. *Let μ be an edge-friendly labeling for $K_{m,n}$ attaining the minimum number of 1-edges. Then $v_\mu(1)=0$ except the following cases:*
(a) $n=4k+2$ and $m=2$ or m is odd, for some $k\geq 0$;
(b) $m=4h+2$ and $n=2$ or n is odd, for some $h\geq 0$.
Moreover, for these exceptional cases, $v_\mu(1)=2$.

Note that, the above exceptional cases are the same by symmetry. In the following, we shall keep the definitions of $A_{i,j}$'s defined in this section.

4. Maximum Value of $v_f(1)$

In this section, we shall define a friendly matrix $M_{m,n}$ which induces an edge-friendly labeling φ for $K_{m,n}$ such that $v_\varphi(1)=v_{M_{m,n}}(1)$ is maximum.

From Lemma 2.1 we have

Corollary 4.1. *Suppose m and n are of opposite parity. There is no edge-friendly labeling f of $K_{m,n}$ such that $v_f(1) = m + n$.*

Let

$$M_{2,4k+2} = \begin{pmatrix} J_{1,2k+1} & O_{1,2k+1} \\ O_{1,2k+1} & J_{1,2k+1} \end{pmatrix}, k \geq 0; \qquad M_{2,4k} = \begin{pmatrix} J_{1,2k+1} & O_{1,2k-1} \\ O_{1,2k+1} & J_{1,2k-1} \end{pmatrix}, k \geq 1;$$

$$N_{2s,4k+2} = \begin{pmatrix} J_{2s,2k+1} & O_{2s,2k+1} \end{pmatrix}, k \geq 0, s \geq 1; \; N_{4h,4k} = \begin{pmatrix} J_{2h,2k+1} & O_{2h,2k-1} \\ O_{2h,2k+1} & J_{2h,2k-1} \end{pmatrix}, h, k \geq 1.$$

$$N_{4h,2t+1} = \begin{pmatrix} A^T_{2t-1,4h} & N_{4h,2} \end{pmatrix}, h \geq 1, t \geq 2;$$

$$M_{4,4} = \begin{pmatrix} J_{2,2} & I_2 \\ I_2 & O_{2,2} \end{pmatrix}; \qquad M_{4,4k} = \begin{pmatrix} M_{4,4} & N^T_{4k-4,4} \end{pmatrix}, k \geq 2,$$

where I_2 is the identity matrix of order 2. Note that $v_{M_{2,4k+2}}(1) = 4k+4$, $v_{M_{2,4k}}(1) = 4k+2$, $v_{M_{4,4k}}(1) = 4k+4$, $v_{N_{2s,4k+2}}(1) = 2s$, $v_{N_{4h,4k}}(1) = 4h$, $v_{N_{4h,2t+1}}(1) = 4h$. More precisely, each row sum of $N_{2s,4k+2}$ (resp. $N_{4h,4k}$, $N_{4h,2t+1}$) is 1 and each column sum of $N_{2s,4k+2}$ (resp. $N_{4h,4k}$, $N_{4h,2t+1}$) is 0.

Lemma 4.2. *Suppose m and n are even. There is a friendly matrix M of $K_{m,n}$ such that $v_M(1) = m + n$.*

Proof. Suppose $n = 4k + 2$ for some $k \geq 0$. Let $m = 2s$ for some $s \geq 1$. When $s = 1$, the required matrix is $M_{2,4k+2}$. When $s \geq 2$. Then $M_{2s,4k+2} = \begin{pmatrix} M_{2,4k+2} \\ N_{2s-2,4k+2} \end{pmatrix}$ is a required matrix.

Suppose $n = 4k$ for some $k \geq 0$. Suppose $m = 4h + 2$ for some $h \geq 0$. When $h = 0$, the required matrix is $M_{2,4k}$. When $h \geq 1$. Then $M_{4h+2,4k} = \begin{pmatrix} M_{2,4k} \\ N_{4h,4k} \end{pmatrix}$ is a required matrix. Suppose $m = 4h$ for some $h \geq 1$. When $h = 1$. The required matrix is $M_{4,4k}$. When $h \geq 2$. Then $M_{4h,4k} = \begin{pmatrix} M_{4,4k} \\ N_{4h-4,4k} \end{pmatrix}$ is a required matrix. $\square$

Example 4.1. $M_{4,6} = \begin{pmatrix} M_{2,6} \\ N_{2,6} \end{pmatrix} = \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ \hline 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{array}\right)$,

$$M_{6,8} = \begin{pmatrix} M_{2,8} \\ N_{4,8} \end{pmatrix} = \left(\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ \hline 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{array}\right), \; M_{8,8} = \begin{pmatrix} M_{4,8} \\ N_{4,8} \end{pmatrix} = \left(\begin{array}{cccc|cccc} 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ \hline 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{array}\right).$$

Lemma 4.3. *Suppose m and n are odd. There is a friendly matrix M of $K_{m,n}$ such that $v_M(1) = m + n$.*

Proof. Suppose one of m and n is congruence 3 modulo 4. Without loss of generality, we assume that $m = 4h+3$ for some $h \geq 0$. Consider $m = 3$ first. Let $M_{3,3} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, $M_{3,5} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$ and $R_{3,4} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$. Clearly, $v_{M_{3,3}}(1) = 6$, $v_{M_{3,5}}(1) = 8$ and $v_{R_{3,4}}(1) = 4$. Moreover, the row sums of $R_{3,4}$ are 0 and columns sums are 1. Putting a suitable numbers of $R_{3,4}$ on the left hand side of $M_{3,3}$ and $M_{3,5}$ to form a large matrix will get a friendly matrix $M_{3,2t+1}$ of $K_{3,2t+1}$ such that $v_{M_{3,2t+1}}(1) = 2t+4$, for some $t \geq 1$.

For the general case, let $M_{4h+3,2t+1} = \begin{pmatrix} N_{4h,2t+1} \\ M_{3,2t+1} \end{pmatrix}$. Then $v_{M_{4h+3,2t+1}}(1) = v_{M_{3,2t+1}}(1) + 4h = 4h + 2t + 4$.

Suppose both of m and n are congruence 1 modulo 4. Let $m = 4h+1$ and $n = 4k+1$ with $h, k \geq 1$. Suppose one of m and n equal to 5. Without loss of generality, we assume $m = 5$. Let

$$M_{5,5} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad M_{5,4k+1} = \begin{pmatrix} N^T_{4k-4,5} & M_{5,5} \end{pmatrix} \text{ if } k \geq 2.$$

Clearly, these two matrices are required matrices.

The remaining case is when $h, k \geq 2$. Let

$$M_{4h+1,4k+1} = \begin{pmatrix} A_{4h-4,4k-4} & N_{4h-4,5} \\ N^T_{4k-4,5} & M_{5,5} \end{pmatrix}.$$

We may check that $v_{M_{4h+1,4k+1}}(1) = 4h + 4k + 2$. $\square$

Note that $e_{M_{4h+3,4k+3}}(1) = e_{M_{4h+3,4k+3}}(0) + 1$; $e_{M_{4h+1,4k+1}}(1) = e_{M_{4h+1,4k+1}}(0) + 1$ and $e_{M_{4h+3,4k+1}}(1) = e_{M_{4h+3,4k+1}}(0) - 1$.

Example 4.2. $M_{7,7} = \begin{pmatrix} N_{4,7} \\ M_{3,7} \end{pmatrix}$, $N_{4,7} = \begin{pmatrix} A^T_{5,4} & N_{4,2} \end{pmatrix}$, $A_{5,4} = \begin{pmatrix} A_{2,4} \\ A_{3,4} \end{pmatrix}$.

Hence $A^T_{5,4} = \left(\begin{array}{cc|ccc} 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$, $N_{4,7} = \left(\begin{array}{ccccc|cc} 1 & 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right)$,

$M_{3,7} = \begin{pmatrix} R_{3,4} & M_{3,3} \end{pmatrix} = \left(\begin{array}{cccc|ccc} 1 & 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right)$, $M_{7,7} = \left(\begin{array}{ccccccc} 1 & 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ \hline 1 & 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right)$.

Similarly, $M_{7,9} = \begin{pmatrix} N_{4,9} \\ M_{3,9} \end{pmatrix}$,

$$N_{4,9} = \begin{pmatrix} A^T_{7,4} & N_{4,2} \end{pmatrix} = \begin{pmatrix} A^T_{4,4} & A^T_{3,4} & N_{4,2} \end{pmatrix} = \left(\begin{array}{cccc|ccc|cc} 1&1&1&1&1&0&1&1&0 \\ 1&1&1&1&1&1&0&1&0 \\ 0&0&0&0&0&1&1&1&0 \\ 0&0&0&0&0&0&0&1&0 \end{array}\right) \text{ and}$$

$$M_{3,9} = \begin{pmatrix} R_{3,4} & M_{3,5} \end{pmatrix} = \left(\begin{array}{cccc|ccccc} 1&1&0&0&1&1&1&1&1 \\ 1&0&1&0&1&0&0&0&0 \\ 1&0&0&1&1&0&0&0&0 \end{array}\right). \text{ So } M_{7,9} = \left(\begin{array}{ccccccccc} 1&1&1&1&1&0&1&1&0 \\ 1&1&1&1&1&1&0&1&0 \\ 0&0&0&0&0&1&1&1&0 \\ 0&0&0&0&0&0&0&1&0 \\ \hline 1&1&0&0&1&1&1&1&1 \\ 1&0&1&0&1&0&0&0&0 \\ 1&0&0&1&1&0&0&0&0 \end{array}\right).$$

$$M_{9,13} = \begin{pmatrix} A_{4,8} & N_{4,5} \\ N^T_{8,5} & M_{5,5} \end{pmatrix}. \text{ So we have } M_{9,13} = \left(\begin{array}{cccccccc|ccccc} 1&1&0&0&1&1&0&0&1&0&1&1&0 \\ 1&1&0&0&1&1&0&0&1&1&0&1&0 \\ 1&1&0&0&1&1&0&0&0&1&1&1&0 \\ 1&1&0&0&1&1&0&0&0&0&0&1&0 \\ \hline 1&1&0&0&1&1&0&0&1&1&1&1&1 \\ 0&1&1&0&0&1&1&0&1&1&1&0&0 \\ 1&0&1&0&1&0&1&0&1&1&1&0&0 \\ 1&1&1&1&1&1&1&1&1&0&0&0&0 \\ 0&0&0&0&0&0&0&0&1&0&0&0&0 \end{array}\right).$$

Lemma 4.4. *There is a friendly matrix M of $K_{m,n}$ such that $v_M(1) = m + n - 1$, where $m \not\equiv n \pmod 2$.*

Proof. Without loss of generality, we may assume m is even.

Let $S_{3,3} = \begin{pmatrix} 1&1&1 \\ 1&0&0 \\ 0&1&0 \end{pmatrix}$; $T_{3,3} = \begin{pmatrix} 1&1&0 \\ 1&0&0 \\ 0&1&0 \end{pmatrix}$; $S_{3,5} = \begin{pmatrix} 1&1&1&0&0 \\ 1&1&1&0&0 \\ 0&0&0&1&0 \end{pmatrix}$; $T_{3,5} = \begin{pmatrix} 1&1&1&1&0 \\ 1&1&1&0&0 \\ 0&0&0&1&0 \end{pmatrix}$;

$S_{3,4k+1} = \begin{pmatrix} A_{3,4k-4} & S_{3,5} \end{pmatrix}$ and $T_{3,4k+1} = \begin{pmatrix} A_{3,4k-4} & T_{3,5} \end{pmatrix}$ if $k \ge 2$;

$S_{3,4k+3} = \begin{pmatrix} A_{3,4k} & S_{3,3} \end{pmatrix}$ and $T_{3,4k+3} = \begin{pmatrix} A_{3,4k} & T_{3,3} \end{pmatrix}$ if $k \ge 1$. Then $v_{S_{3,4k+1}}(1) = v_{S_{3,4k+3}}(1) = 4$ in which all row sums are 1 and exactly one column sum is 1. $v_{T_{3,4k+1}}(1) = v_{T_{3,4k+3}}(1) = 2$ in which all column sums are 0. Note that $e_{S_{3,4k+1}}(1) = e_{S_{3,4k+1}}(0) - 1$; $e_{T_{3,4k+1}}(1) = e_{T_{3,4k+1}}(0) + 1$; $e_{S_{3,4k+3}}(1) = e_{S_{3,4k+3}}(0) + 1$; and $e_{T_{3,4k+3}}(1) = e_{T_{3,4k+3}}(0) - 1$.

1. $m = 4h$ and $n = 4k + 1$.
   Let $M_{4h,4k+1} = \begin{pmatrix} M_{4h-3,4k+1} \\ S_{3,4k+1} \end{pmatrix}$. Since $e_{M_{4h-3,4k+1}}(1) = e_{M_{4h-3,4k+1}}(0) + 1$, $M_{4h,4k+1}$ is friendly. Moreover, $v_{M_{4h,4k+1}}(1) = (4h - 3) + (4k + 1) + 3 - 1 = 4h + 4k = m + n - 1$.
2. $m = 4h$ and $n = 4k + 3$.
   Let $M_{4h-3,4k+3} = M^T_{4k+3,4h-3}$ and $M_{4h,4k+3} = \begin{pmatrix} M_{4h-3,4k+3} \\ S_{3,4k+3} \end{pmatrix}$. Since $e_{M_{4h-3,4k+3}}(1) = e_{M_{4h-3,4k+3}}(0) - 1$, $M_{4h,4k+3}$ is friendly. Moreover, $v_{M_{4h,4k+3}}(1) = (4h - 3) + (4k + 3) + 3 - 1 = 4h + 4k + 2 = m + n - 1$.
3. $m = 2$ and n is odd. The required labeling matrix is $\begin{pmatrix} J_{1,n} \\ O_{1,n} \end{pmatrix}$.

4. $m = 4h+2$ and $n = 4k+1$, where $h \ge 1$.
 Let $M_{4h+2,4k+1} = \begin{pmatrix} M_{4h-1,4k+1} \\ T_{3,4k+1} \end{pmatrix}$. Since $e_{M_{4h-1,4k+1}}(1) = e_{M_{4h-1,4k+1}}(0) - 1$, $M_{4h+2,4k+1}$ is friendly.
 Moreover, $v_{M_{4h+2,4k+1}}(1) = (4h-1)+(4k+1)+2 = 4h+4k+2 = m+n-1$.
5. $m = 4h+2$ and $n = 4k+3$, where $h \ge 1$.
 Let $M_{4h+2,4k+3} = \begin{pmatrix} M_{4h-1,4k+3} \\ T_{3,4k+3} \end{pmatrix}$. Since $e_{M_{4h-1,4k+3}}(1) = e_{M_{4h-1,4k+3}}(0) + 1$, $M_{4h+2,4k+3}$ is friendly.
 Moreover, $v_{M_{4h+2,4k+3}}(1) = (4h-1)+(4k+3)+2 = 4h+4k+4 = m+n-1$.

This completes the proof. □

Example 4.3. $M_{8,5} = \begin{pmatrix} M_{5,5} \\ S_{3,5} \end{pmatrix} = \left(\begin{array}{ccccc} 1&1&1&1&1\\ 1&1&1&0&0\\ 1&1&1&0&0\\ 1&0&0&0&0\\ 1&0&0&0&0\\ \hline 1&1&1&0&0\\ 1&1&1&0&0\\ 0&0&0&1&0 \end{array}\right)$,

$$M_{8,7} = \begin{pmatrix} M_{5,7} \\ S_{3,7} \end{pmatrix} = \left(\begin{array}{c|c} \begin{matrix} A_{3,4} \\ N_{4,2}^T \end{matrix} & M_{3,5}^T \\ \hline A_{3,4} & S_{3,3} \end{array}\right) = \left(\begin{array}{cccc|ccc} 1&1&0&0&1&1&1\\ 0&1&1&0&1&0&0\\ 1&0&1&0&1&0&0\\ \hline 1&1&1&1&1&0&0\\ 0&0&0&0&1&0&0\\ \hline 1&1&0&0&1&1&1\\ 0&1&1&0&1&0&0\\ 1&0&1&0&0&1&0 \end{array}\right),$$

$$M_{6,5} = \begin{pmatrix} M_{3,5} \\ T_{3,5} \end{pmatrix} = \left(\begin{array}{ccccc} 1&1&1&1&1\\ 1&0&0&0&0\\ 1&0&0&0&0\\ \hline 1&1&1&1&0\\ 1&1&1&0&0\\ 0&0&0&1&0 \end{array}\right),$$

$$M_{6,7} = \begin{pmatrix} M_{3,7} \\ T_{3,7} \end{pmatrix} = \begin{pmatrix} R_{3,4} & M_{3,3} \\ A_{3,4} & T_{3,3} \end{pmatrix} = \left(\begin{array}{cccc|ccc} 1&1&0&0&1&1&1\\ 1&0&1&0&1&0&0\\ 1&0&0&1&1&0&0\\ \hline 1&1&0&0&1&1&0\\ 0&1&1&0&1&0&0\\ 1&0&1&0&0&1&0 \end{array}\right).$$

The next problem is to determine the full edge-friendly index set of $K_{m,n}$. We conjecture that

$$\mathrm{FEFI}(K_{m,n}) = \begin{cases} \{4j-(m+n) \mid 1 \le j \le \lfloor (m+n)/2 \rfloor\}, & \text{if } n \equiv 2 \pmod 4 \text{ and } m = 2 \text{ or } m \text{ is odd;} \\ \{4j-(m+n) \mid 1 \le j \le \lfloor (m+n)/2 \rfloor\}, & \text{if } m \equiv 2 \pmod 4 \text{ and } n = 2 \text{ or } n \text{ is odd;} \\ \{4j-(m+n) \mid 0 \le j \le \lfloor (m+n)/2 \rfloor\}, & \text{otherwise.} \end{cases}$$

It will take more iterations on finding each possible value of edge-friendly index. Since this paper seems too long, we will study this problem at the next paper. Following we provide two examples as the ending of this paper.

Example 4.4. Matrices $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ induce the edge-friendly labelings μ and φ for $K_{2,3}$ such that $v_\mu(1) = 2$ and $v_\varphi(1) = 4$, respectively. Hence $FEFI(K_{2,3}) = \{-1, 3\}$.

Example 4.5. Consider $K_{2,4k+2}$ for $k \geq 1$. We start from $A_0 = A_{2,4k+2}$. Note that $v_{A_0}(1) = 2$. Obtain a new matrix A_1 by shifting the second row of A_0 to right by one entry cyclically. Then $v_{A_1}(1) = 4$. Repeat shifting the second row to right of the newly matrix. We can see that the number of 1-edges increase by 2 after each shifting. For example, when $k = 1$:

$$A_0 = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{pmatrix} \to A_1 = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \end{pmatrix} \to A_2 = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix} \to \cdots$$

So, we get that

$$FEFI(K_{2,4k+2}) = \{4j - 4k - 4 \mid 1 \leq j \leq 2k + 2\}.$$

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