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A NEW CONSTRUCTION FOR VERTEX DECOMPOSABLE GRAPHS

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ABSTRACT. Let G be a finite simple graph on the vertex set $V(G)$ and let $S \subseteq V(G)$. Adding a whisker to G at x means adding a new vertex y and edge xy to G where $x \in V(G)$. The graph $G \cup W(S)$ is obtained from G by adding a whisker to every vertex of S. We prove that if $G \setminus S$ is either a graph with no chordless cycle of length other than 3 or 5, chordal graph or C_5, then $G \cup W(S)$ is a vertex decomposable graph.

1. **Introduction**

Let G be a finite simple graph on the vertex set $V(G)$ and edge set $E(G)$. Adding a *whisker* to G at x means adding a new vertex y and edge xy to G where $x \in V(G)$. Villarreal showed that the graph obtained from G by adding a whisker to every vertex of G is Cohen-Macaulay [7, Proposition 2.2]. Moreover, it was shown that this graph is also pure and vertex decomposable [3, Theorem 4.4]. Suppose $S \subseteq V(G)$. The graph $G \cup W(S)$ is obtained from G by adding a whisker to every vertex of S. Francisco and H'a proved that if $G \setminus S$ is a chordal graph or C_5, then $G \cup W(S)$ is a sequentially Cohen-Macaulay graph [4, Theorem 3.3]. For any graph G the following hierarchy is known:

vertex decomposable $\Rightarrow$ shellable $\Rightarrow$ sequentially Cohen-Macaulay.

It is known that the implications are strict. We extend the above result as follows:

Let G be a finite simple graph on the vertex set $V(G)$ and let $S \subseteq V(G)$. If $G \setminus S$ is a chordal graph or C_5, then $G \cup W(S)$ is a vertex decomposable graph, see Theorem 3.1.

As a consequence of Theorem 3.1 we give bound $|S| \geq |V(G)| - 3$ on the number of vertices of S is sharp, see Corollary 3.6 and Example 3.8.

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Assume that $|S| \geq |V(G)| - 3$. Then $G \cup W(S)$ is a vertex decomposable graph.

As the second main result we prove the following result:

If $G \setminus S$ is a graph with no chordless cycles of length other than 3 or 5, then $G \cup W(S)$ is a vertex decomposable graph, see Theorem 3.9.

2. Preliminaries

In this section we recall some definitions and properties that will be used in this article.

Let k be a field. To any finite simple graph G with vertex set $V(G) = \{x_1, \ldots, x_n\}$ and edge set $E(G)$ one associates *edge ideal* $I(G) \subset k[x_1, \ldots, x_n]$ whose generators are all monomials $x_i x_j$ such that $\{x_i, x_j\} \in E(G)$. The graph G is called *sequentially Cohen-Macaulay or Cohen-Macaulay* over k if $k[x_1, \ldots, x_n]/I(G)$ has this property.

Let G be a graph with the vertex set $V(G) = \{x_1, \ldots, x_n\}$ and the edge set $E(G)$. The *induced subgraph* $G|_W$ for $W \subseteq V(G)$ is defined by

$$G|_W := (W, \{e \in E(G); e \subset W\}).$$

For $W \subseteq V(G)$ we denote $G|_{V(G)\setminus W}$ by $G \setminus W$. If $W = \{x\}$, we write $G \setminus x$ instead of $G \setminus \{x\}$. For any $x \in V(G)$ we denote the *open neighbor* set of x in G by $N_G(x)$, i.e, $N_G(x) := \{y \in V(G) \mid \{x, y\} \in E(G)\}$, $N_G[x] := N_G(x) \cup \{x\}$.

A *simplicial complex* Δ on a finite set V is a collection of subsets of V which is closed under inclusion. Members of Δ are called *faces*. The maximum faces of Δ with respect to inclusion are called the *facets*. The *dimension* of a face F is $|F| - 1$ and the dimension of a complex Δ is the maximum of the dimensions of its facets. If all the facets of Δ have the same dimension we say that Δ is *pure* and a complex with a unique facet is called a *simplex*.

A simplicial complex Δ is called *shellable* if its facets can be ordered $F_1, \ldots, F_m$ such that for all $1 \leq i < j \leq m$, there exists some $v \in F_j \setminus F_i$ and some $l \in \{1, \ldots, j-1\}$ with $F_j \setminus F_l = \{v\}$.

Let F be a face of Δ. The *link* and *deletion* of F in Δ are the simplicial complexes defined by

$$\operatorname{link}_\Delta F := \{G \in \Delta \mid G \cap F = \emptyset, G \cup F \in \Delta\} \quad \text{and}$$

$$\operatorname{del}_\Delta F := \{G \in \Delta \mid F \nsubseteq G\}.$$

If $F = \{x\}$, we write $\operatorname{link}_\Delta x$ (resp. $\operatorname{del}_\Delta x$) instead of $\operatorname{link}_\Delta \{x\}$ (resp. $\operatorname{del}_\Delta \{x\}$). See [6] for detailed information.

Let $G = (V(G), E(G))$ be a graph. A subset F of $V(G)$ is called an *independent set* if no two vertices of F are adjacent. The independence complex of G, denoted by $\operatorname{Ind}(G)$, is the simplicial complex on the vertex set $V(G)$, defined by

$$\operatorname{Ind}(G) := \{F \subseteq V(G) \mid \text{F is an independent set of G}\}.$$

A simplicial complex Δ is called *vertex decomposable* if it is a simplex or else has some vertex x such that

(1) both $\operatorname{del}_\Delta x$ and $\operatorname{link}_\Delta x$ are vertex decomposable, and

(2) there is no face of $\mathrm{link}_\Delta\, x$ which is also a facet of $\mathrm{del}_\Delta\, x$.

A *shedding vertex* is the vertex x which satisfies condition (2). Vertex decomposability were introduced in the pure case by Billera and Provan [1] and extended to non-pure complexes by Björner and Wachs [2]. A graph G is called *vertex decomposable or shellable* if $\mathrm{Ind}(G)$ has this property. In [8] Woodroofe translated the notion of vertex decomposable for graphs as follows:

A graph G is *vertex decomposable* if it is totally disconnected (with no edges) or else has some vertex x such that

(1) both $G \setminus N_G[x]$ and $G \setminus x$ are vertex decomposable, and
(2) for every independent set S in $G \setminus N_G[x]$, there exists some $y \in N_G(x)$ such that $S \cup \{y\}$ is independent in $G \setminus x$.

A vertex x which satisfies condition (2) is called a *shedding vertex* for G.

A graph G is called *chordal* if every induced cycle of G of length ≥ 4 has a chord. In [8] Woodroofe proved that every chordal graph is vertex decomposable and so shellable. The cycle of length n is denoted by C_n.

3. vertex decomposable graphs

In the following, we prove the first main result of this article.

Theorem 3.1. *Let G be a finite simple graph on the vertex set $V(G)$ and let $S \subseteq V(G)$. If $G \setminus S$ is a chordal graph or C_5, then $G \cup W(S)$ is a vertex decomposable graph.*

Proof. Suppose $S = \{x_1, \dots, x_n\}$. We prove the assertion by induction on $n = |S|$. Assume for $1 \leq i \leq n$, y_i is a whisker associated to x_i. Set $G' = G \cup W(S)$. Since $N_{G'}[y_i] \subseteq N_{G'}[x_i]$, x_i is a shedding vertex for G'. It is enough to show that $G' \setminus x_1$ and $G' \setminus N_{G'}[x_1]$ are vertex decomposable. If $n = 1$, then $G' \setminus x_1 = (G \setminus x_1) \cup \{y_1\}$ is a chordal graph or $C_5 \cup \{y_1\}$. Note that y_1 is a isolated vertex in $G' \setminus x_1$ and hence has no effect on vertex decomposablity. It follows from [5, Proposition 4.1] and [8, Corollary 7] that $G \setminus x_1$ is vertex decomposable, and so $G' \setminus x_1$ is vertex decomposable. If $G' \setminus x_1$ is a chordal graph, then $G' \setminus N_{G'}[x_1]$ is an induced subgraph of $G' \setminus x_1$, and so is a chordal graph. Therefore $G' \setminus N_{G'}[x_1]$ is vertex decomposable by [8, Corollary 7]. So suppose $G' \setminus N_{G'}[x_1]$ is an induced subgraph of C_5 then it is path or C_5. Since path and C_5 [5, Proposition 4.1] are vertex decomposable so $G' \setminus N_{G'}[x_1]$ is vertex decomposable. Assume that $n > 1$ and the assertion holds for any graph G such that $|S| \leq n-1$ and $G \setminus S$ is a chordal graph or C_5. We have $G' \setminus x_1 = (G \setminus x_1) \cup W(S_1) \cup \{y_1\}$ where $S_1 = \{x_2, \dots, x_n\}$ and so $(G \setminus x_1) \cup W(S_1)$ is vertex decomposable by induction hypothesis. Therefore $G' \setminus x_1$ is vertex decomposable. If $S \subseteq N_{G'}[x_1]$, then $G' \setminus N_{G'}[x_1] = (G \setminus N_G[x_1]) \cup \{y_i\}_{i=2}^n$ and so it is vertex decomposable by the similar arguments to case $n = 1$. Assume that $S \not\subset N_{G'}[x_1]$ we have $G' \setminus N_{G'}[x_1] = G_1 \cup W(S_1) \cup \{y_j | x_j \in N_G[x_1], j \neq 1\}$ where $G_1 = G \setminus N_G[x_1]$ and $S_1 = S \setminus N_G[x_1]$. Since $G_1 \setminus S_1$ is an induced subgraph of $G \setminus S$ and so is chordal graph, path or C_5. Therefore $G_1 \cup W(S_1)$ is vertex decomposable by induction. Since $\{y_j | x_j \in N_G[x_1], j \neq 1\}$ are isolated vertices in $G' \setminus N_{G'}[x_1]$, we know that $G' \setminus N_{G'}[x_1]$ is vertex decomposable. Thus G' is vertex decomposable. $\square$

Suppose $G \setminus S$ satisfies the condition in Theorem 3.1. Then $G \cup W(S)$ is vertex decomposable and so shellable and sequentially Cohen-Macaulay. Therefore the following result holds.

Corollary 3.2. [4, Theorem 3.3] *Let G be a finite simple graph on the vertex set $V(G)$ and let $S \subseteq V(G)$. Suppose $G \setminus S$ is a chordal graph or C_5, then $G \cup W(S)$ is a sequentially Cohen-Macaulay graph.*

In the following example it is shown that if $G \setminus S$ is not chordal and C_5, then $G \cup W(S)$ may not (in general) be vertex decomposable.

Example 3.3. *Let G be a graph shown in Figure 1, and let $S = \{x_1\}$. The graph $G \cup W(S)$ is obtained by adding a whisker to the graph G at vertex x_1. Note that the only shedding vertex of $G \cup W(S)$ is x_1. Since $(G \cup W(S)) \setminus N[x_1] = C_4$ is not vertex decomposable, $G \cup W(S)$ is not vertex decomposable. Although $G \setminus S$ is vertex decomposable.*

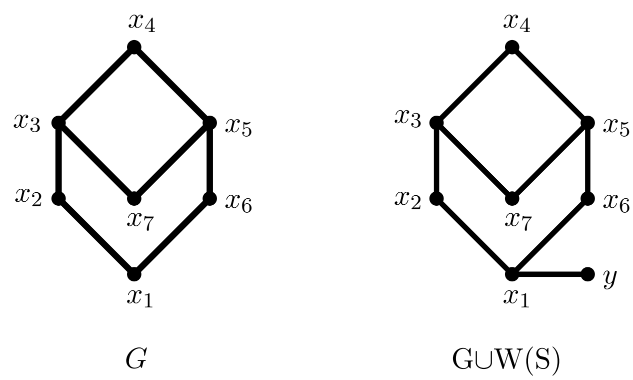

FIGURE 1.

Corollary 3.4. *Let G be a simple graph, and let $S \subset V(G)$.*

(i) *If S is a vertex cover of G, then $G \cup W(S)$ is vertex decomposable.*
(ii) *If $G \setminus S$ is a forest, then $G \cup W(S)$ is vertex decomposable.*
(iii) *If $G = C_n$ is a cycle and x is a vertex of C_n , then $G \cup W(x)$ is vertex decomposable.*

Proof. For (i), since S is a vertex cover of G, $G \setminus S$ is a edge less graph and so it is a chordal graph. It follows from Theorem 3.1. For (ii), every forest is a chordal graph. The assertion holds by Theorem 3.1. Finally, the resulting graph of removing a vertex from C_n is a tree and hence (iii) follows from (ii). □

For any graph G the following hierarchy is known:
vertex decomposable ⇒ shellable ⇒ sequentially Cohen-Macaulay.
From Corollary 3.4, we have the following result:

Corollary 3.5. [4, Corollary 3.4] *Let G be a simple graph, and let $S \subset V(G)$.*

(i) *If S is a vertex cover of G, then $G \cup W(S)$ is sequentially Cohen-Macaulay.*
(ii) *If $G \setminus S$ is a forest, then $G \cup W(S)$ is sequentially Cohen-Macaulay.*
(iii) *If $G = C_n$ is a cycle and x is a vertex of C_n , then $G \cup W(x)$ is sequentially Cohen-Macaulay.*

Corollary 3.6. *Let G be a simple graph, and let $S \subset V(G)$. Assume that $|S| \geq |V(G)| - 3$. Then $G \cup W(S)$ is a vertex decomposable graph.*

Proof. Since $|S| \geq |V(G)| - 3$, $G \setminus S$ is a graph on at most 3 vertices. So $G \setminus S$ is a tree, edge less graph, $K_1 \cup K_2$ or C_3. Thus $G \setminus S$ is chordal and the assertion holds by Theorem 3.1. $\square$

The same arguments before Corollary 3.5, we have the following result:

Corollary 3.7. [4, Corollary 3.5] *Let G be a simple graph, and let $S \subset V(G)$. Assume that $|S| \geq |V(G)| - 3$. Then $G \cup W(S)$ is a sequentially Cohen-Macaulay graph.*

In the following example it is shown that the bound $|V(G)| - 3$ in Corollary 3.6 is sharp.

Example 3.8. *Let G be a graph shown in Figure 2, and let $S = \{x_1, x_2\}$. Note that the only shedding vertex of $G \cup W(S)$ are x_1 and x_2. Then $G \cup W(S)$ is not vertex decomposable , because $((G \cup W(S)) \setminus x_1) \setminus x_2 = \{C_4\} \cup \{y, z\}$ is not vertex decomposable.*

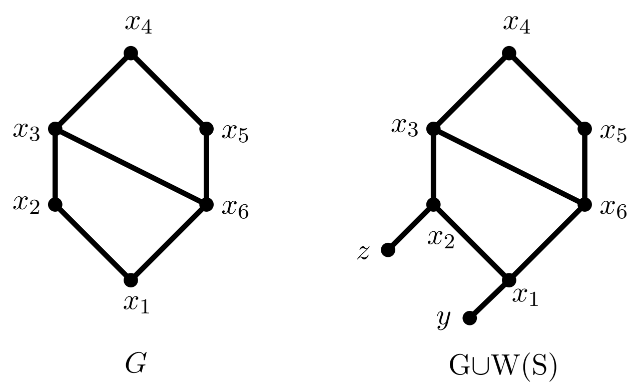

FIGURE 2.

The induced cycle C_n of G is called *chordless cycle* of length n. In [8, Theorem 1] Woodroofe proved that If G is a graph with no chordless cycles of length other than 3 or 5, then G is vertex decomposable. Now, we prove the second main result of this article.

Theorem 3.9. *Let G be a finite simple graph on the vertex set $V(G)$ and let $S \subseteq V(G)$. If $G \setminus S$ is a graph with no chordless cycles of length other than 3 or 5, then $G \cup W(S)$ is a vertex decomposable graph.*

Proof. Suppose $S = \{x_1, \ldots, x_n\}$. We prove the assertion by induction on $n = |S|$. Assume for $1 \leq i \leq n$, y_i is a whisker associated to x_i. Set $G' = G \cup W(S)$. Since $N_{G'}[y_i] \subseteq N_{G'}[x_i]$, x_i is a shedding vertex for G'. It is enough to show that $G' \setminus x_1$ and $G' \setminus N_{G'}[x_1]$ are vertex decomposable.

For $n = 1$, we have $G' \setminus x_1 = (G \setminus x_1) \cup \{y_1\}$. Since $G \setminus x_1$ has no chordless cycles of length other than 3 or 5, it is vertex decomposable by [8, Theorem 1]. Therefore $G' \setminus x_1$ is vertex decomposable. Also, we have $G' \setminus N_{G'}[x_1] = G \setminus N_G[x_1]$. Since $G \setminus N_G[x_1]$ is an induced subgraph of $G \setminus x_1$, it has no chordless cycles of length other than 3 or 5, and so $G' \setminus N_{G'}[x_1]$ is vertex decomposable by [8, Theorem 1]. Assume that $n > 1$ and the assertion holds for any graph G such that $|S| \leq n - 1$ and $G \setminus S$ is a graph with no chordless cycles of length other than 3 or 5. We have $G' \setminus x_1 = (G \setminus x_1) \cup W(S_1) \cup \{y_1\}$, where $S_1 = \{x_2, \ldots, x_n\}$. Thus it is vertex decomposable by induction. If $S \subseteq N_{G'}[x_1]$, then $G' \setminus N_{G'}[x_1] = (G \setminus N_G[x_1]) \cup \{y_i\}_{i=2}^{n}$ and so it is vertex decomposable by the similar arguments to case $n = 1$. So suppose $S \not\subset N_{G'}[x_1]$ we have $G' \setminus N_{G'}[x_1] = G_1 \cup W(S_1) \cup \{y_j | x_j \in N_G[x_1], j \neq 1\}$ where $G_1 = G \setminus N_G[x_1]$ and $S_1 = S \setminus N_G[x_1]$. Note that $\{y_j | x_j \in N_G[x_1], j \neq 1\}$ are isolated vertex in $G' \setminus N_{G'}[x_1]$ and hence have no effect on vertex decomposablity. Since $G_1 \setminus S_1$ is an induced subgraph $G \setminus S$, it has no chordless cycles of length other than 3 or 5, and hence $G' \setminus N_{G'}[x_1]$ is vertex decomposable by induction. Since $G' \setminus x_1$ and $G' \setminus N_{G'}[x_1]$ are vertex decomposable, G' is vertex decomposable.

$\square$

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