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SOME RESULTS ON THE COMAXIMAL IDEAL GRAPH OF A COMMUTATIVE RING

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Abstract. Let R be a commutative ring with unity. The comaximal ideal graph of R, denoted by $\mathcal{C}(R)$, is a graph whose vertices are the proper ideals of R which are not contained in the Jacobson radical of R, and two vertices I_1 and I_2 are adjacent if and only if $I_1 + I_2 = R$. In this paper, we classify all comaximal ideal graphs with finite independence number and present a formula to calculate this number. Also, the domination number of $\mathcal{C}(R)$ for a ring R is determined. In the last section, we introduce all planar and toroidal comaximal ideal graphs. Moreover, the commutative rings with isomorphic comaximal ideal graphs are characterized. In particular we show that every finite comaximal ideal graph is isomorphic to some $\mathcal{C}(\mathbb{Z}_n)$.

1. Introduction

Algebraic combinatorics is an area of mathematics which employs methods of abstract algebra in various combinatorial contexts and vice versa. Associating a graph to an algebraic structure is a research subject in this area and has attracted considerable attention. When one assigns a graph to an algebraic structure numerous interesting algebraic problems arise from the translation of some graph-theoretic parameters such as clique number, chromatic number, independence number and so on. There are a lot of papers which apply combinatorial methods to obtain algebraic results, for instance see [1], [2], [4], [6], [7], [8] and [9].

The *comaximal ideal graph* of a commutative ring R, denoted by $\mathcal{C}(R)$, is a graph whose vertices are the proper ideals of R which are not contained in the Jacobson radical of R, and two vertices I_1 and I_2 are adjacent if and only if $I_1 + I_2 = R$. This graph was first introduced and studied in [11].

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They showed that $\mathcal{C}(R)$ is simple, connected and with diameter less than or equal to three, and has girth less than or equal to four if it contains a cycle. For some recent works on comaximal ideal graphs of commutative rings see [3], [12] and [13].

In this paper, we study the domination and independence number of $\mathcal{C}(R)$. It is proved that if $\alpha(\mathcal{C}(R)) < \infty$, then $R \cong R_1 \times \cdots \times R_k$, where R_i is a local ring for $i = 1, \ldots, k$. Moreover, we present a formula for calculating $\alpha(\mathcal{C}(R))$ using intersecting set families of $\{1, \ldots, k\}$. In the last section, we derive several necessary conditions for the nonplanarity of the comaximal ideal graphs of commutative rings and determine all commutative rings whose comaximal ideal graphs are planar. Also, all toroidal comaximal ideal graphs, up to isomorphism, are introduced. Moreover, the commutative rings with isomorphic comaximal ideal graphs are characterized.

First we would like to recall some facts and notations related to this paper. For a graph G, by $V(G)$ and $E(G)$, we denote the set of all vertices and all edges, respectively. The degree of $v \in V(G)$ denoted by $d_G(v)$. For a vertex v of G, we define $N(v)$ be the set of all neighbors of v. An *r-partite* graph is one whose vertex set can be partitioned into r subsets so that an edge has both ends in no subset. A *complete r-partite* graph is an r-partite graph in which each vertex is adjacent to every vertex that is not in the same subset. The complete bipartite (i.e., 2-partite) graph with part sizes m and n is denoted by $K_{m,n}$. We use K_n to denote the complete graph with n vertices. A subset S of the vertex set of G is an *independent set* if no two vertices of S are adjacent. A *maximal independent set* is an independent set such that adding any other vertex to the set forces the set to contain an edge. The largest size of independent sets is called independence number and is denoted by $\alpha(G)$. A subset D of the vertices of G is called a *dominating set*, if every vertex of $V(G) \setminus D$ is adjacent to some vertex of D. The minimum size of such a subset is called the *domination number* of G. The genus of a graph G, denoted by $\gamma(G)$, is smallest nonnegative integer g such that the graph G can be embedded on the surface obtained by attaching g handles to a sphere such that the edges intersect only in vertices. The graphs of genus 0 and 1 are called *planar* graphs and *toroidal* graphs respectively.

Throughout this paper, all rings are assumed to be commutative with identity. The intersection of all maximal ideals of R is called the *Jacobson radical* of R and denoted by $J(R)$. For a ring R, we use $\mathbb{I}(R)$ and $Max(R)$ to denote the set of all ideals and the set of maximal ideals of R, respectively. A ring with only one maximal ideal is called a *local ring*. A ring R is called *uniserial* if for every two ideals $I, J \in \mathbb{I}(R)$, we have $I \subseteq J$ or $J \subseteq I$. An element $e \in R$ is called an idempotent if $e^2 = e$. If $e \neq 0, 1$ is an idempotent then $R \cong Re \times R(1-e)$ as rings.

2. **Domination and independence number of comaximal ideal graphs**

In this section, we study the domination and independence number of $\mathcal{C}(R)$. Before stating main results, we need the following lemma and theorems.

Lemma 2.1. *Let R be a ring and $a \in R \backslash (J(R) \cup U(R))$. Then we have the following statements:*

(1) *If $I = Ra$ is an idempotent ideal of R, then I is generated by a nontrivial idempotent. In particular, R is a direct product of two rings.*

(2) *Let A be a subset of $\mathbb{I}(R)$ such that if $I \in A$, then $I^2 \in A$ and $Ri \in A$ for some $i \in I$. If A has a minimal element J, then $J = Re$ is generated by an idempotent.*

Proof. (1) Since $Ra = Ra^2$, there is an $r \in R$ such that $a = ra^2$. Let $e = ra$. So $e^2 = r^2a^2 = ra = e$ and $a = ea$. This implies that e is a non-trivial idempotent and $I = Ra = Re$.

(2) By assumption, there is a $j \in J$ such that $Rj \in A$. Since J is minimal in A, we conclude that $Rj = J = J^2$ is an idempotent ideal of R. Thus $J = Re$ is generated by an idempotent and the proof is complete.

□

Remark 2.2. *Let I be an ideal of a ring R and $a \in R \backslash (J(R) \cup U(R))$. Then the following sets of ideals satisfy the conditions in part (2) of Lemma 2.1.*

(1) $A = \{J \in \mathbb{I}(R) : J \subseteq I, J \nsubseteq J(R)\}$.
(2) $A = \{Ra^i : i \in \mathbb{N}\}$.
(3) *Every maximal independent set of $\mathcal{C}(R)$.*
(4) $A = \{J : J \in \mathbb{I}(R),\ J + I = R\} = \{$*all the neighbors of the vertex I in* $\mathcal{C}(R)\}$.

Theorem 2.3. *Let R be a ring and $a \in R \backslash (J(R) \cup U(R))$. Then we have the following statements:*

(1) *If the chain $\cdots \subseteq Ra^2 \subseteq Ra$ is finite, then R has a nontrivial idempotent.*
(2) *If R has no nontrivial idempotent, then each edge of $\mathcal{C}(R)$ is contained in an induced subgraph isomorphic to $K_{\infty,\infty}$. In particular, every vertex has infinite degree.*

Proof. (1) Since the chain $\cdots \subseteq Ra^2 \subseteq Ra$ is finite, $A = \{Ra^i : i \in \mathbb{N}\}$ has a minimal element. Also, the set A satisfies the conditions in part (2) of Theorem 2.1 and so the result follows.

(2) Let I_0, J_0 be two comaximal ideals. Then there are $i \in I_0$ and $j \in J_0$ such that $i + j = 1$. Let $I_n = Ri^n$ and $J_n = Rj^n$. It is easy to see that $I_n + J_m = R$, for every $n, m \geq 0$. Since R has no nontrivial idempotent, the chains $\cdots \subseteq I_1 \subseteq I_0$ and $\cdots \subseteq J_1 \subseteq J_0$ are infinite, by part (1). So the induced subgraph of $\mathcal{C}(R)$ on the vertices $\{I_n\}_{n\geq 0} \cup \{J_n\}_{n\geq 0}$ is isomorphic to $K_{\infty,\infty}$ and we are done.

□

Theorem 2.4. *Let I be an ideal of the ring R such that the set $S = \{J \in \mathbb{I}(R) : I + J = R\}$ has a minimal element K. Then K is a minimum element of S and $K = Re$ generated by an idempotent. In particular, R/K is a direct factor of R.*

Proof. First note that $S = \{J \in \mathbb{I}(R) : I + J = R\}$ satisfies conditions in part (2) of Theorem 2.1. Thus K is generated by an idempotent e. Hence $R \cong Re \times R(1 - e) \cong Re \times R/K$. Now, if $J \neq K$ be an element of S then $I + K = I + J = R$ implies that $I + (K\ J) = R$. By minimality of K, we conclude that $K \subseteq J$. So K is minimum and the proof is complete. □

Theorem 2.5. *For a ring R, $\mathcal{C}(R)$ has an ideal I of finite degree n if and only if $R \cong R_1 \times R_2$, where R_1 has $n + 1$ ideals.*

Proof. Assume that $R = R_1 \times R_2$ such that R_1 has $n + 1$ ideals. So $I = R_1 \times \{0\}$ has degree n. Conversely, assume that I has degree n. So $\{J \in \mathbb{I}(R) : I + J = R\}$ has a minimum element K which is generated by an idempotent e, by Theorem 2.4. Hence $R \cong R/K \times T$. It is clear that R/K has $n + 1$ ideals. □

Corollary 2.6. *For a ring R, $\mathcal{C}(R)$ has a vertex of degree one if and only if R is a direct product of a field and another ring.*

The following remark is used frequently without reference.

Remark 2.7. *Let $R = R_1 \times \cdots \times R_n$. Then $\mathcal{C}(R_i)$ is isomorphic to an induced subgraph of $\mathcal{C}(R)$, for $1 \leq i \leq n$. Hence $\alpha(\mathcal{C}(R_i)) \leq \alpha(\mathcal{C}(R))$ and $\gamma(\mathcal{C}(R_i)) \leq \gamma(\mathcal{C}(R))$, for $1 \leq i \leq n$.*

Theorem 2.8. *If $K_{3,\infty}$ is not a subgraph of $\mathcal{C}(R)$ then $R \cong R_1 \times \cdots \times R_n$, where R_i's are local rings. Moreover, if $n \geq 3$, then R_i's have finitely many ideals.*

Proof. Since $\mathcal{C}(R)$ doesn't contain $K_{\infty,\infty}$, R has a nontrivial idempotent, by Theorem 2.3. So R is a direct product of finitely many rings. Since the induced subgraph of $\mathcal{C}(R)$ on maximal ideals of R is a complete graph and $\mathcal{C}(R)$ dose not contain K_∞, we conclude that $|Max(R)| < \infty$. Assume that n is the largest number such that $R \cong R_1 \times \cdots \times R_n$. Thus R_i has no nontrivial idempotent. Since comaximal ideal graph of each factor is a subgraph of $\mathcal{C}(R)$, we conclude that each factor must be a local ring. If $n \geq 3$ and R_i contains infinitely many ideals, then $\mathcal{C}(R)$ contains $K_{3,\infty}$ which is a contradiction and so R_i's have finitely many ideals and the proof is complete. □

Corollary 2.9. *Let R be a ring such that every vertex of $\mathcal{C}(R)$ has finite degree. Then $R \cong R_1 \times \cdots \times R_n$, where R_i's are local rings with finitely many ideals.*

Proof. By Theorem 2.8, we have $R \cong R_1 \times \cdots \times R_n$, where R_i's are local rings. Moreover, if $n \geq 3$, then R_i's have finitely many ideals. If $n = 2$, then $\mathcal{C}(R)$ is a complete bipartite graph and so R_1 and R_2 have finitely many ideals. □

In the following, we show that if $\mathcal{C}(R)$ is finite, then R is a direct product of finitely many uniserial rings and a finite ring and moreover $\mathcal{C}(R) \cong \mathcal{C}(S)$ if and only if $R \cong R_1 \times \cdots \times R_n$ and $S \cong S_1 \times \cdots \times S_n$, where R_i and S_i are local rings with the same number of ideals, for $1 \leq i \leq n$. Before stating these results, we need the following lemma.

Lemma 2.10. *Let (R, m) be a local ring with finitely many ideals. Then the following hold:*

(1) *If the residue field R/m is infinite, then m is a principal ideal and every ideal of R is a power of m. In particular, R is a uniserial ring.*
(2) *If the residue field R/m is finite, then R is a finite ring.*

Proof. Since R is an Artinian local ring, $m^n = 0, m^{n-1} \neq 0$, for a natural number n. Also, m^i/m^{i+1} is a vector space with finitely many subspaces.

(1) Since m/m^2 is a vector space with finitely many subspaces, the dimension of m is at most 1 and so $m = (x)$ is a principal ideal. Since R is an Artinian ring, it is well-known that every ideal is a power of m.

(2) Since m^i/m^{i+1} is a finite dimensional vector space over the finite field R/m, we conclude that $|m^i/m^{i+1}|$ is finite. Therefore, $|R| = |R/m||m/m^2| \cdots |m^{n-1}/m^n|$ is finite and we are done. □

Corollary 2.11. *Let R be a ring such that $\mathcal{C}(R)$ is finite. Then $R \cong R_1 \times \cdots \times R_n$, where R_i is a finite local ring or a uniserial ring, for $1 \le i \le n$.*

In the following, we characterize the ring R when the minimum degree of $\overline{\mathcal{C}(R)}$ is finite.

Theorem 2.12. *Let R be a ring such that $\delta(\overline{\mathcal{C}(R)}) < \infty$. Then $R \cong R_1 \times \cdots \times R_k$, where R_i's are local rings. Also, if $k \ge 3$, then R has finitely many ideals.*

Proof. Let $\delta(\overline{\mathcal{C}(R)}) = k < \infty$ and I be a maximal ideal such that $d_{\overline{\mathcal{C}(R)}}(I) = k$. I contains only finitely many ideals which are not contained in $J(R)$. By Lemma 2.1, there is an idempotent in I. Assume that Re is a maximal element in $\{Ra \subseteq I : a^2 = a\}$. So, $R \cong Re \times R(1-e) = S_1 \times S_2$. Since $Re \subseteq I$, we have $I = S_1 \times I'$, where $I' \in Max(S_2)$. S_2 has no nontrivial idempotent, because Re is maximal. Hence S_2 is a local ring with finitely many ideals. Also, S_1 is a local ring or S_1 has finitely many ideals which are not contained in $J(S_1)$, which means $\mathcal{C}(S_1)$ has finitely many ideals by Corollary 2.11. Thus S_1 is a local ring or S_1 has finitely many ideals. □

Corollary 2.13. *Assume that $R = S_1 \times S_2$ as in the proof of Theorem 2.12, where S_1 has finitely many ideals and (S_2, m) is a local ring such that $d_{\overline{\mathcal{C}(R)}}(S_1 \times m) = \delta(\overline{\mathcal{C}(R)}) = k$. Let $a_1 = |\{I \in \mathbb{I}(S_1) |\, I \nsubseteq J(S_1)\}|$ and $b_1 = |\{I \in \mathbb{I}(S_2) |\, I \subseteq m\}|$. Then $k = a_1 b_1 - 1$.*

Corollary 2.14. *Let R be a ring. Then we have the following statements:*

(1) $\delta(\overline{\mathcal{C}(R)}) = 0$ *if and only if R is a direct product of a field and a local ring.*

(2) $\delta(\overline{\mathcal{C}(R)}) = 1$ *if and only if R is a direct product of a ring with 3 ideals and a local ring which is not a field.*

Corollary 2.15. [3] Let R be a ring. Then $\mathcal{C}(R)$ has a vertex which is adjacent to all other vertices if and only if R is isomorphic to the direct product of a field and a local ring.

Theorem 2.16. *Let R and S be two rings such that $\mathcal{C}(R)$ is finite. Then $\mathcal{C}(R) \cong \mathcal{C}(S)$ if and only if $R \cong R_1 \times \cdots \times R_n$ and $S \cong S_1 \times \cdots \times S_n$, where R_i and S_i are local rings with the same number of ideals, for $1 \le i \le n$.*

Proof. First assume that $\mathcal{C}(R) \cong \mathcal{C}(S)$. Thus by Corollary 2.11, we have $R \cong R_1 \times \cdots \times R_m$ and $S \cong S_1 \times \cdots \times S_n$, where R_i and S_j are local rings with finitely many ideals, for $1 \le i \le m$ and $1 \le j \le n$. Let $I_i = \{(a_k) | a_k = 0 \text{ for all } k \ne i\}$ and $J_j = \{(b_k) | b_k = 0 \text{ for all } k \ne j\}$, for $1 \le i \le m$ and $1 \le j \le n$. One can see that $N(I_i)$ and $N(J_j)$ are minimal elements of the set $\{N(v) | v \in V(\mathcal{C}(R))\}$

and $\{N(v)|v \in V(\mathcal{C}(S))\}$, for $1 \leq i \leq m$ and $1 \leq j \leq n$, respectively. Clearly, $|N(I_i)| = |\mathbb{I}(R_i)| - 1$ and $|N(J_j)| = |\mathbb{I}(S_j)| - 1$. Since $\mathcal{C}(R) \cong \mathcal{C}(S)$, we have $m = n$. Also, by a suitable permutation, we conclude that R_i and S_j are local rings with the same number of ideals, for $1 \leq i \leq n$. Conversely, let $f_i : \mathbb{I}(R_i) \to \mathbb{I}(S_i)$ be a corresponding such that $f_i(R_i) = S_i$. Then it is easy to see that $f : \mathcal{C}(R) \to \mathcal{C}(S)$ such that $f(I_1 \times \cdots \times I_n) = f_1(I_1) \times \cdots \times f_n(I_n)$ is an isomorphism and so $\mathcal{C}(R) \cong \mathcal{C}(S)$. □

Corollary 2.17. *Let R be a non-local ring such that $\mathcal{C}(R)$ is a finite graph. Then $\mathcal{C}(R) \cong \mathcal{C}(\mathbb{Z}_n)$, for a natural number n.*

In [3], Akbari and et.al. show that if $\alpha(\mathcal{C}(R)) < \infty$, then $\mathcal{C}(R)$ is a finite graph. In the following, it is proved that if $\alpha(\mathcal{C}(R)) < \infty$, then $R \cong R_1 \times \cdots \times R_k$, where R_i is a local ring for $i = 1, \ldots, k$. Moreover, we present a formula for calculating $\alpha(\mathcal{C}(R))$ using intersecting proper set families of $\{1, \ldots, k\}$.

Remark 2.18. *If $R = S \times T$ where S and T are local rings, then $\mathcal{C}(R)$ is the complete bipartite graph $K_{m-1,n-1}$, where S has m ideals and T has n ideals and $\alpha(\mathcal{C}(R)) = max\{m - 1, n - 1\}$. In the following theorem we generalize this formula.*

Theorem 2.19. *Let R be a non-local ring such that $\alpha(\mathcal{C}(R)) < \infty$. Then we have the following statements:*

(1) *$R = R_1 \times \cdots \times R_k$ and each R_i is a local ring with finitely many ideals. So $\mathcal{C}(R)$ is a finite graph.*
(2) *$\alpha(\mathcal{C}(R)) = max\{\sum_{A \in T} \prod_{i \in A}(n_i - 1) : T$ is an intersecting set family of proper subsets of $\{1, \ldots, k\}$ and n_i is the number of ideals of $R_i\}$.*

Proof. (1) By Theorem 2.8, $R = R_1 \times \cdots \times R_k$ and R_i has finitely many ideals for $k \geq 3$. If $k = 2$ then $\mathcal{C}(R)$ is a complete bipartite graph. So each R_i has finitely many ideals.

(2) Let K be the set $\{1, \ldots, k\}$ and $P(K)$ be the power set of K. Define a function f on the set of all ideals of R as follows. If $I = I_1 \times \cdots \times I_k$ then $f(I) = \{i_1, \ldots, i_t\}$ where $I_{ij} \neq R_{ij}$, for $1 \leq j \leq t$. Let I and J be two non-adjacent ideals of R. So $I + J \neq R$ and there exists i, $1 \leq i \leq k$, such that $I_i, J_i \neq R_i$. Thus $f(I) \cap f(J) \neq \varnothing$. Hence, if S is an independent set of $\mathcal{C}(R)$, then $f(S)$ is an intersecting set family of proper subsets of K. Now, define a function g on $P(K)$ such that $g(\{i_1, \ldots, i_t\}) = \{I = I_1 \times \cdots \times I_k : I_{i_j} \neq R_{i_j}$, for $1 \leq j \leq t\}$. If $A, B \subseteq K$ and $A \cap B \neq \varnothing$, then $g(A) \cup g(B)$ is an independent set of $\mathcal{C}(R)$. So, if T is an intersecting set family of proper subsets of K, then $\bigcup g(T)$ is an independent set of $\mathcal{C}(R)$. It is easy to see that $S \subseteq \bigcup gof(S)$. Hence $S = \bigcup gof(S)$, where S is a maximal independent set of $\mathcal{C}(R)$. Clearly, for an intersecting set family of K, T, we have $|\bigcup g(T)| = \sum_{A \in T} \prod_{i \in A}(n_i - 1)$. Since $\alpha(\mathcal{C}(R)) = max\{|S| : S$ is a maximal independent set of $\mathcal{C}(R)\} = max\{|\bigcup gof(S)| : S$ is a maximal independent set of $\mathcal{C}(R)\}$, we are done.

□

Corollary 2.20. *Let $R = R_1 \times R_2 \times R_3$, where R_1, R_2 and R_3 are local rings. Then $\alpha(\mathcal{C}(R)) = max\{(n_1 - 1)(n_2 + n_3 - 1), (n_1 - 1)(n_2 - 1) + (n_1 - 1)(n_3 - 1) + (n_2 - 1)(n_3 - 1)\}$, where $n_1 \geq n_2 \geq n_3$.*

Example 2.21. *If* $R = \mathbb{Z}_4\times\mathbb{Z}_4\times\mathbb{Z}_4$, *then* $\alpha(\mathcal{C}(R)) = (n_1-1)(n_2-1)+(n_1-1)(n_3-1)+(n_2-1)(n_3-1) = 12$. *If* $R = \mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, *then* $\alpha(\mathcal{C}(R)) = (n_1 - 1)(n_2 + n_3 - 1)) = 6$.

In the following, we determine the dominating set and domination number of the comaximal ideal graph of a ring.

Remark 2.22. *Let* R *be a ring with two maximal ideals. Then,* $\mathcal{C}(R)$ *is a bipartite graph, by* [11, Lemma 4.1]. *So, the domination number of* $\mathcal{C}(R)$ *is equal* 1 *or* 2.

Theorem 2.23. *Let* R *be a ring,* $|Max(R)| \geq 3$ *and the domination number of* $\mathcal{C}(R)$ *be finite. Then* $Max(R)$ *is a dominating set of* $G = \mathcal{C}(R)$. *Moreover, the domination number of* G *is equal to* $|Max(R)|$.

Proof. Let k be the domination number of $G = \mathcal{C}(R)$ and $S = \{J_1, \ldots, J_k\}$ be a dominating set of G. If $|Max(R)|$ is infinite, then there are maximal ideals $M_1, \ldots, M_k$ such that $J_i \subseteq M_i$ for $1 \leq i \leq k$. Assume that $M_{k+1}, \ldots, M_{2k+1}$ are $k + 1$ distinct maximal ideals such that $M_i \neq M_j$, for every i and j, $1 \leq i \leq k$ and $k + 1 \leq j \leq 2k + 1$. Set $N_i = \cap_1^{k+i} M_t$ for $1 \leq i \leq k + 1$. These are distinct ideals of R. Also N_j is not adjacent to any J_i. Thus $N_1, \ldots, N_{k+1}$ are contained in S, which is a contradiction. Hence $|Max(R)| = n$ is finite. Note that every ideal of R is adjacent to a maximal ideal. So $Max(R)$ is a dominating set and the domination number of G is at most $|Max(R)|$. Now, we prove that the domination number of G is at least $|Max(R)|$. Let $I_j = \cap_{i\neq j} M_i \nsubseteq J(R)$, for $j = 1, \ldots, n$. It is clear that $I_i \cap I_j = J(R)$, $i \neq j$, and so $T = \{I_j : 1 \leq j \leq n\}$ is an independent set. Also, I_j and I_i have no common neighbor, for all $i \neq j$ and $j = 1, \ldots, n$. Because otherwise $I + I_i = I + I_j = R$ implies that $I + I_i \cap I_j = I + J(R) = R$, which is a contradiction. Now, define $f : T \setminus S \to S$ such that $f(I)$ is adjacent to I. Since T is independent, we conclude that $f(I) \in S \setminus T$. Moreover, f is an injective function and so $|T \setminus S| \leq |S \setminus T|$. Thus $n = |T| \leq |S| = k$ and the proof is complete. □

In the following, we show that if $\mathcal{C}(R)$ is a tree, then it is a star graph. Moreover, in this case the ring R will be characterized.

Theorem 2.24. *Let* R *be a ring and* $\mathcal{C}(R)$ *be a tree. Then* $\mathcal{C}(R)$ *is a star graph and* R *is isomorphic to the direct product of a field and a local ring.*

Proof. First assume that $|Max(R)| \geq 3$ and M_1, M_2, M_3 are maximal ideals of R. It is easy to see that M_1, M_2 and M_3 are adjacent in $\mathcal{C}(R)$. So there exists a cycle, a contradiction. Thus $|Max(R)| = 2$ and $\mathcal{C}(R)$ is a star graph. Hence $\mathcal{C}(R)$ has vertices of degree one. By Theorem 2.6, R is isomorphic to the direct product of a field and another ring. Set $R = R_1 \times R_2$. Clearly, the center of the star graph is a maximal ideal of R. With no loss of generality, we can assume that $R_1 \times M$ is the center of $\mathcal{C}(R)$. If $M \neq 0$, then $R_1 \times M$ is not adjacent to $R_1 \times 0$, a contradiction. So R_2 is a field. One can see that R_1 dose not have more than one maximal ideal. Thus R is isomorphic to the direct product of a field and a local ring. □

Corollary 2.25. *Let* R *be a ring. Then the domination number of* $\mathcal{C}(R)$ *is equal* 1 *if and only if* R *is isomorphic to the direct product of a field and a local ring.*

Proof. First note that R has only two maximal ideals, by Theorem 2.23. So $\mathcal{C}(R)$ is a bipartite graph. Hence it is a star graph. Thus the proof is complete, by Theorem 2.24. □

Corollary 2.26. *Let R be a ring. Then the domination number of $\mathcal{C}(R)$ is equal 2 if and only if R has two maximal ideals none of them is generated by an idempotent.*

Proof. First note that R has only two maximal ideals, by Theorem 2.23. If $M = Re$ is generated by an idempotent then $R \cong Re \times R(1-e)$. Hence Re and $R(1-e)$ are local rings. In the ring $R(1-e) \cong R/Re$ the zero ideal is maximal. So $R(1-e)$ is a field. Hence the domination number of $\mathcal{C}(R)$ is equal 1 by Corollary 2.25, which is a contradiction. Conversely, If R has two maximal ideals none of them is generated by an idempotent then $\mathcal{C}(R)$ is a bipartite graph. Also R is not isomorphic to the direct product of a field and a local ring. So the domination number of $\mathcal{C}(R)$ is equal 2. □

3. Genus of the comaximal ideal graph of a ring

n this section, we want to study the genus of $\mathcal{C}(R)$. In particular, all comaximal ideal graphs with genus 0 and 1 are characterized. In the following theorem we remind some well-known formulas:

Theorem 3.1. [10] *The following statements hold:*

(1) *For $n \geq 3$ we have $\gamma(K_n) = \lceil \frac{(n-3)(n-4)}{12} \rceil$.*
(2) *For $m, n \geq 2$ we have $\gamma(K_{m,n}) = \lceil \frac{(m-2)(n-2)}{4} \rceil$.*

According to Theorem 3.1 we have $\gamma(K_n) = 0$ for $1 \leq n \leq 4$ and $\gamma(K_n) = 1$ for $5 \leq n \leq 7$ and for other value of n, $\gamma(K_n) \geq 2$. Also $\gamma(K_{1,n}) = \gamma(K_{2,n}) = 0$ and $\gamma(K_{3,3}) = \gamma(K_{3,4}) = \gamma(K_{3,5}) = \gamma(K_{3,6}) = \gamma(K_{4,4}) = 1$ and for other values of $m \leq n, \gamma(K_{m,n}) \geq 2$.

Theorem 3.2. *Let R be a non-local ring such that $\gamma(\mathcal{C}(R)) < \infty$. Then $R = R_1 \times \cdots \times R_k$ where R_i's are local rings. Also, if R has infinitely many ideals, then $k = 2$ and $\gamma(\mathcal{C}(R)) = 0$.*

Proof. By Theorem 2.8, $R = R_1 \times \cdots \times R_k$ where R_i's are local rings and for $k \geq 3$ each R_i has only finitely many ideals. If $k = 2$, then $\mathcal{C}(R)$ is a complete bipartite graph. Set $|\mathbb{I}(R_1)| = m$ and $|\mathbb{I}(R_2)| = n$. If $m = \infty, n \geq 4$, then $K_{\infty,3} \subseteq \mathcal{C}(R)$ which is a contradiction. So $m = \infty$ and $n \leq 3$. Hence $\gamma(\mathcal{C}(R)) = 0$. □

Theorem 3.3. *Let R be a non-local ring. If $\mathcal{C}(R)$ is a planar graph, then one of the following statements occurs:*

(1) *$|Max(R)| = 2$ and $R \cong R_1 \times R_2$, where R_1 and R_2 are local rings and R_1 is a field or has two proper ideals.*
(2) *$|Max(R)| = 3$ and $R \cong R_1 \times R_2 \times R_3$, where R_1, R_2 and R_3 are local rings with at most two proper ideals.*

Proof. By Theorem 2.8, $R = R_1 \times \cdots \times R_k$, where each R_i is a local ring. If $k \geq 4$, then $\mathcal{C}(R)$ contain $K_{3,3}$ as a subgraph which is a contradiction and so

(1) If $k = 2$, then $\mathcal{C}(R)$ is a complete bipartite graph. So one of its factor has at most three ideals.
(2) If $k = 3$, then each factor has at most three ideals.

□

Remark 3.4. *By the above theorem if $\mathcal{C}(R)$ is a planar graph then it is a star graph or $K_{2,n}$ or it is isomorphic to $G_1 = \mathcal{C}(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$, $G_2 = \mathcal{C}(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4)$ or $G_3 = \mathcal{C}(\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_4)$, by Theorem 2.16. See the following figure:*

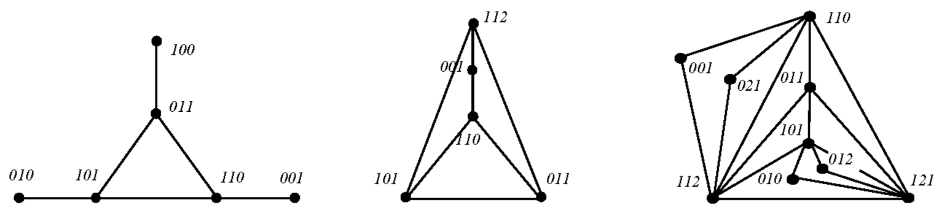

FIGURE 1. the graphs G_1, G_2 and G_3

Remark 3.5. *Let G be a graph. Clearly, if we remove all vertices of degree one, then the genus of graph dose not change. Now, suppose that uvw is a path of G and $d_G(v) = 2$. Let L be a graph such that $V(L) = V(G) \setminus \{v\}$ and $E(L) = E(G) \cup \{uw\}$. Then it is easy to see that $\gamma(L) = \gamma(G)$.*

The girth of a graph G is the minimum of the lengths of all cycles in G, and is denoted by $gr(G)$. If G is acyclic, that is, if G has no cycles, then we write $gr(G) = \infty$. It has been proved in [5] that if G is a connected graph (but not acyclic) having n vertices and m edges, then

$$\gamma(G) \geq \frac{m(k-2)}{2k} - \frac{n}{2} + 1 \tag{3.1}$$

, where $k = gr(G)$.

Theorem 3.6. *Let R be a non-local ring. Then $\gamma(\mathcal{C}(R)) = 1$ if and only if $\mathcal{C}(R)$ is isomorphic to $\mathcal{C}(S)$ where S is one of the followings rings:*

(1) $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_8$.
(2) $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{16}$.
(3) $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{32}$.
(4) $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{64}$.
(5) $\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_8$.
(6) $\mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_4$.
(7) $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

Proof. By Theorem 3.2, $R \cong \Pi_{i=1}^{k} R_i$ where R_i's are local rings with finitely many ideals. With no loss of generality, we can assume that $n_1 \leq \cdots \leq n_k$ are the number of ideals of R_i. If $k \geq 5$, then $\mathcal{C}(R)$ contains $K_{3,7}$, a contradiction and so $k \leq 4$. Suppose that $k = 4$. If $n_3 \geq 3$, then $\mathcal{C}(R)$ contains $K_{5,5}$, which is a contradiction and so $n_1 = n_2 = n_3 = 2$. We claim that $n_4 < 4$. Because otherwise, $\mathcal{C}(R)$ contains $K_{3,7}$, a contradiction and we conclude that $n_4 = 2, 3$. Hence $\mathcal{C}(R) \cong \mathcal{C}(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$ or $\mathcal{C}(R) \cong \mathcal{C}(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4)$, by Theorem 2.16. We claim that $\gamma(\mathcal{C}(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)) = 1$ and $\gamma(\mathcal{C}(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4)) \geq 2$. To prove the claim, first consider $G = \mathcal{C}(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$. Clearly, G has 4 vertices of degree one. Remove these vertices from G and rename it by L. The genus of L is equal to that of G, by Remark 3.5. The following figure shows that the graph L is a toroidal graph and so $\gamma(\mathcal{C}(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)) = 1$.

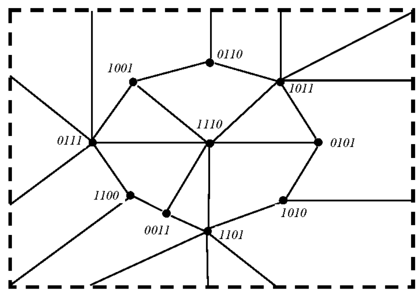

FIGURE 2. the graph L

Now, Let $H = \mathcal{C}(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4)$. Consider the graphs H' in the following figure. One can see that H' is a subgraph of H. For all the path $x_1x_2x_3$, where $d_{H'}(x_2) = 2$, we remove the vertex x_2 and add the edge x_1x_3. Then rename the obtained graph by H''. By Remark 3.5, $\gamma(H') = \gamma(H'')$. Note that $|V(H'')| = 8$, $|E(H'')| = 25$ and $g(H'') = 3$. By (3.1), we have $\gamma(H) \geq \gamma(H'') \geq 2$ and so the claim is proved.

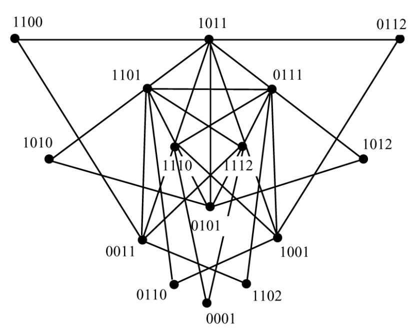

FIGURE 3. the graph H'

Now, suppose that $k = 3$. If $n_2 \geq 4$, then $\mathcal{C}(R)$ contains $K_{3,7}$, which is a contradiction. Thus we have $n_2 = 2, 3$. If $n_1 = n_2 = 2$, then $n_3 \leq 7$. Since $\mathcal{C}(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$ and $\mathcal{C}(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4)$ are planar, we will show that $\mathcal{C}(R) \cong \mathcal{C}(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_8)$, $\mathcal{C}(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{16})$, $\mathcal{C}(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{32})$ or $\mathcal{C}(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{64})$. By Remark 3.5, it is enough to prove that $\gamma(\mathcal{C}(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{64})) = 1$. Let L be a graph obtained from $\mathcal{C}(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{64})$ by removing all the vertices of degree one. The following figure shows that $\gamma(L) = 1$. So $\gamma(\mathcal{C}(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{64})) = 1$.

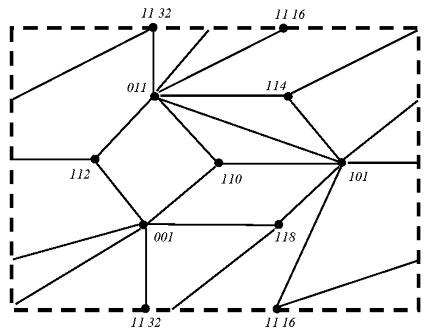

FIGURE 4. the graph L

Now, we assume that $n_1 = 2, n_2 = 3$. Hence $n_3 = 3$ or 4. So $\mathcal{C}(R) \cong \mathcal{C}(\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_4)$ or $\mathcal{C}(\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_8)$. By Remark 3.5, it suffices to show that $G = \mathcal{C}(\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_8)$ is toroidal. Remove all vertices of degree one and rename the obtained graph by G'. The figure 5 proves that $\gamma(\mathcal{C}(\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_8)) = 1$.

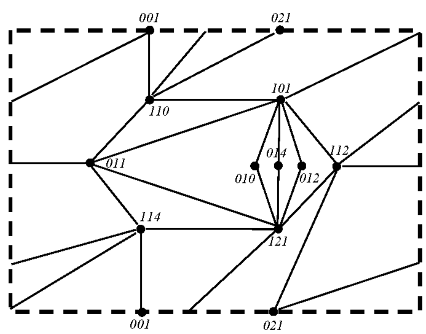

FIGURE 5. the graph G'

The last case is $n_1 = n_2 = n_3 = 3$ which implies $\mathcal{C}(R) \cong \mathcal{C}(\mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_4)$. Let $G = \mathcal{C}(\mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_4)$. Let $L = (G - \{v : d_G(v) = 2\}) \cup \{(011, 211), (101, 121), (110, 112)\}$. One can see that $\gamma(G) = \gamma(L)$. Since L is a subgraph of K_6, we have $\gamma(L) = 1$ and we are done. □

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