www.combinatorics.ir

Transactions on Combinatorics
ISSN (print): 2251-8657, ISSN (on-line): 2251-8665
Vol. 5 No. 4 (2016), pp. 21-34.

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ON THE NEW EXTENSION OF DISTANCE-BALANCED GRAPHS

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Communicated by Ali Reza Ashrafi

Abstract. In this paper, we initially introduce the concept of n-distance-balanced property which is considered as the generalized concept of distance-balanced property. In our consideration, we also define the new concept locally regularity in order to find a connection between n-distance-balanced graphs and their lexicographic product. Furthermore, we include a characteristic method which is practicable and can be used to classify all graphs with i-distance-balanced properties for $i = 2, 3$ which is also relevant to the concept of total distance. Moreover, we conclude a connection between distance-balanced and 2-distance-balanced graphs.

1. Introduction and preliminaries

During a few decades ago, a great interest has been devoted to the study of graph theory. Recently, a new class of graphs so-called distance-balanced graphs have been introduced and then investigated in several papers (see [2]-[8], [12]-[14]).

We know that equilibrium in communication networks and its preservation in inevitable events is so important and vital. In network optimization, we sometimes need to remove some links to prevent additional costs. One of the most important flaws in the concept of distance-balanced is that a removal of an edge always destroys the property of being distance-balanced (see [11, Proposition 3.1]). Throughout this paper, we initially introduce the concept of n-distance-balanced property which reduces this problem to some extent.

In the following, we recall some basic definitions which will be needed further on. Let G be a finite, undirected and connected graph with diameter d, and let $V(G)$ and $E(G)$ denote the vertex

MSC(2010): Primary: 05C12; Secondary: 05C40.

Keywords: n-distance-balanced property, lexicographic product, total distance.

Received: 03 February 2016, Accepted: 17 March 2016.

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set and the edge set of G, respectively. For $u, v \in V(G)$, we let $d(u,v) = d_G(u,v)$ denote the minimal path-length distance between u and v. For a pair of adjacent vertices u, v of G we denote

$$W_{uv}^G = \{x \in V(G) | d(x,u) < d(x,v)\}.$$

Similarly, we can define W_{vu}^G. These quantities studied in earlier literature, especially with relation to the Wiener index (of trees) and the Szeged index [4, 5, 8]. Also, consider the notion

$${}_uW_v^G = \{x \in V(G) | d(x,u) = d(x,v)\}.$$

Note that for any edge $uv \in E(G)$ the sets W_{vu}^G, W_{uv}^G and ${}_uW_v^G$ form a partition of $V(G)$ but if G is connected and bipartite graph, then the vertex set of G is partitioned into only W_{vu}^G and W_{uv}^G. As these notions play a crucial role in metric graph theory, Jerebic and Klavžar [11] introduced the following class of graphs based on these sets.

Definition 1.1 ([11]). *We say that G is distance-balanced whenever for an arbitrary pair of adjacent vertices u and v of G there exists a positive integer γ_{uv}, such that*

$$|W_{uv}^G| = |W_{vu}^G| = \gamma_{uv}$$

Jerebic and Klavžar [11] obtained some basic properties of these graphs and studied on this concept with respect to symmetry conditions in graphs and local operations on graphs. They also proved that distance-balanced Cartesian and lexicographic products of graphs can be characterized. Motivated by the results of [6], Handa considered distance-balanced partial cubes and proved that they are 3-connected, with the exception of cycles and the complete graph of order two.

The aim of this paper is to introduce the notion of n-distance-balanced graph as an extended version of the notion of distance-balanced, to provide examples of such graphs and discuss relations between distance-balanced and n-distance-balanced properties from an special point of view, as well as to prove some other results regarding these graphs. To do this, we first discuss some basic properties of n-distance-balanced graphs in Section 2, and then we study n-distance-balanced graphs in the framework of lexicographic product in Section 3. In Section 4, some characteristic results for i-distance-balanced ($i = 2, 3$) are presented using the concept of total distance and existence of either odd or even circles. Such results let us to discuss on classifying the n-distance-balanced graphs for $n > 3$ in the later investigations which also require to have a connection between such graphs and the concept of total distance. Finally, in Section 5, we highlight the most significant parts of obtained results.

2. Basic properties of n-distance-balanced graphs and examples

Throughout of this section, we initially introduce a large class of graphs including distance-balanced graphs which are called n-distance-balanced graphs and then we present some results and examples concerning with this concept.

Definition 2.1. *A connected graph G is called n-distance-balanced if and only if for each $u, v \in V(G)$ with $d(u, v) = n$ we have $|W^G_{u\underline{n}v}| = |W^G_{v\underline{n}u}|$ where*

$$W^G_{u\underline{n}v} = \{x \in V(G) | d(x, u) < d(x, v)\}.$$

Similarly, we can define $W^G_{v\underline{n}u}$. Also, consider the notion

$${}_uW^G_{\underline{n}}{}_v = \{x \in V(G) | d(x, u) = d(x, v)\}.$$

Similar to distance-balanced property we easily see that all three sets as above form a partition for $V(G)$. Besides, 1-distance-balanced property coincides to distance-balanced property and so the collection of all n-distance-balanced graphs are larger than the set of all distance-balanced graphs. In the following we first present some results for the graphs with n-distance-balanced property inspired by the results related to distance-balanced property which are easily obtained and left without proof (see also [11]) .

Proposition 2.2. *A graph G of diameter d is n-distance-balanced if and only if*

$$|N[a]/N[b]| + \sum_{k=2}^{d-1} |N_k(a)/N_{k-1}(b)| = |N[b]/N[a]| + \sum_{k=2}^{d-1} |N_k(b)/N_{k-1}(a)|$$

holds for all $a, b \in V(G)$ with $d(a, b) = n$ where

$$N_k(x) = \{y | d(x, y) = k\}, \quad N_k[x] = \{y | d(x, y) \le k\}, \; x, y \in V(G), \; 1 \le k \le d,$$

and for $k = 1$, these notations are replaced by $N(x)$ and $N[x]$, respectively.

Corollary 2.3. *Let G be a regular graph of diameter d. Then G is n-distance-balanced if and only if*

$$\sum_{k=2}^{d-1} |N_k(a)/N_{k-1}(b)| = \sum_{k=2}^{d-1} |N_k(b)/N_{k-1}(a)|$$

holds for all $a, b \in V(G)$ with $d(a, b) = n$.

Corollary 2.4. *Let G be a graph with diameter $d = 2$. Then G is an n-distance-balanced graph if and only if all vertices of distance $n \le 2$ in G have the same degree, that is,*

$$\forall a, b \in V(G), \; d(a, b) = n \implies deg(a) = deg(b). \tag{2.1}$$

Remark 2.5. *Comparing Corollary 2.4 as above with Corollary 2.3 in* [11]*, which states all distance-balanced graphs of diameter 2 are only regular graphs, we see that unlike distance-balanced graphs, any n-distance-balanced graph with diameter 2 does not need to be regular and only some certain vertices should be with the same degree.*

In the following we present some well-known non-regular 2-distance-balanced graphs with diameter 2 which are satisfied in Corollary 2.4.

Example 2.6. *Complete bipartite graphs (or bicliques) $K_{m,n}$ ($m \neq 1$ or $n \neq 1$) are class of graphs with diameter 2 which satisfy (2.1) and so are 2-distance-balanced. Therefore, all stars S_k with $k > 1$ as complete bipartite graphs $K_{1,k}$ are 2-distance-balanced graphs. Some of stars are shown in Figure 1. Notice that the mentioned graphs are not included in the class of distance-balanced graphs.*

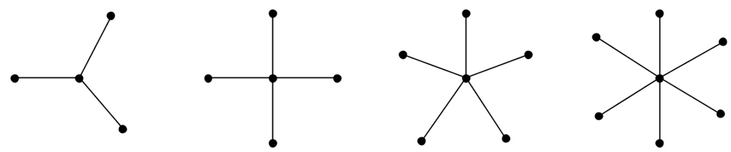

FIGURE 1. The star graphs S_3, S_4, S_5 and S_6

Example 2.7. *Friendship graph (or Dutch windmill graph or n-fan) F_n is a planar graph with $2n+1$ vertices and $3n$ edges. This graph is 2-distance-balanced but not distance-balanced. The friendship graph F_n can be constructed by joining n copies of the cycle graph C_3 with a common vertex (see also [7]). In the following some of such graphs have been shown.*

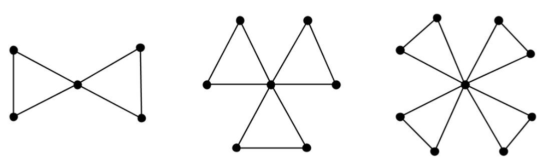

FIGURE 2. friendship graphs F_2, F_3 and F_4

Example 2.8. *Wheel graph W_n with n vertices ($n \geq 4$) is a 2-distance-blanced graph formed by connecting a single vertex to all vertices of an $(n-1)$-cycle.*

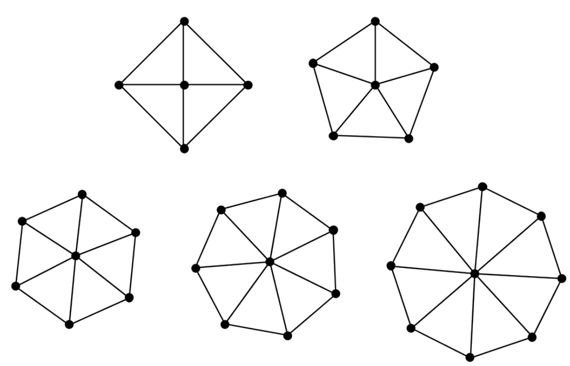

FIGURE 3. The wheel graphs W_5, W_6, W_7, W_8 and W_9

Example 2.9. *In the final example, we present two well-known 2-distance-balance graphs. The diamond graph as a complete graph K_4 minus one edge and the butterfly graph which can be constructed* *by joining two copies of the cycle graph C_3 with a common vertex (and is therefore isomorphic to the friendship graph F_2) both are 2-distance-balance graphs. Note that similar to the previous examples the mentioned graphs have diameter 2 and are not distance-balanced.*

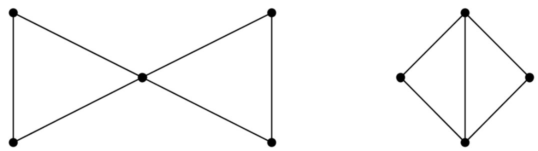

FIGURE 4. The butterfly graph and the diamond graph

We remark that there are several graphs which have the both distance-balanced and 2-distanced properties, simultaneously. Here in Figure 5 we give an example for non-regular graph with both distance-balanced and 2-distance-balanced properties.

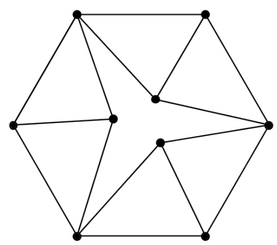

FIGURE 5. Distance-balanced and 2-distance-balanced non-regular graph.

As observed in the examples, Corollary 2.4 is so efficient and can help us to classify all 2-distance-blanced graphs with diameter 2 but the graphs with $d = n > 2$ is still discussable. We now give a sufficient condition for n-distance-balance graphs which is inspired by Proposition 2.4 in [11] (with similar proof) and concerned in connection between symmetry conditions and distance-balanced property.

Proposition 2.10. *Let G be a graph. If for any vertices a, b of $V(G)$ with $d(a, b) = n$ there exists an automorphism φ of G such that $\varphi(a) = b$ and $\varphi(b) = a$, then G is n-distance-balanced.*

Definition 2.11. *A graph G is said n-vertex-transitive if for any given vertices v_1 and v_2 of G with $d(v_1, v_2) = n$, there is an automorphism $f : V(G) \to V(G)$ such that $f(v_1) = v_2$.*

To determine n-vertex-transitive graphs, loosely speaking, all vertices with distance n look the same. Moreover, following Definition 2.11 we easily see that any vertex-transitive graph is n-vertex-transitive for all n. In the next section, we introduce a class of n-vertex-transitive graphs for $n > 1$ while they are not vertex-transitive.

Corollary 2.12. *Suppose that graph G is n-vertex-transitive. Then G is n-distance-balanced.*

2.1. **Affection of removal an edge on n-distance-balanced property.** As we know that a removal an arbitrary edge in a distance-balanced graph G implies loss of distance-balanced property, the question arises, is this proposition true for n-distance-balanced property for each positive n or not. In general, we can not find a suitable answer for this but applying Corollary 2.4 we are able to access more information about 2-distance-balanced graphs. We easily observe that removal an edge may not cause loss of 2-distance-balanced property and this is possible if we pick a certain vertex. In order to prevent ambiguity, we call such edge a *2-distance-balanced preserving removal edge.* See the following example.

Example 2.13. *Complete graphs K_3, K_4 and the diamond graph are 2-distance-balanced but there is a 2-distance-balanced preserving removal edge e (see Figure 6) which does not destroy the 2-distance-balanced property and it refers to the nature of this concept.*

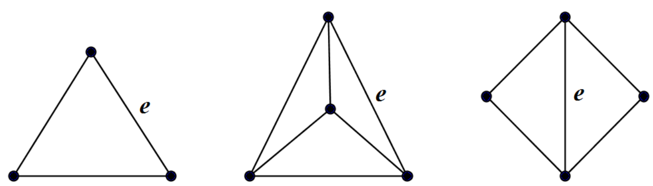

FIGURE 6. 2-distance-balanced preserving removal edge in K_3, K_4 and the diamond graph

Remark 2.14. *Concerning with discussion as above we see that removal a certain edge may be ineffective on 2-distance-balanced property. As a motivation, we notice that comparing to distance-balanced property this is practically considered an advantage. For instance, in communicational structures we may eliminate some certain links to prevent further costs so that it does not affect on the network balance in the sense of distance-balanced property. This fact about 2-distance-balanced property may help us to reach this goal.*

Problem 2.15. *Taking a look to all previous examples and using Corollary 2.4 a question is arose that " is there any 2-distance-balanced graph with $d = 2$ and without a central vertex (a vertex with degree $V(G) - 1$)? ".*

3. n-Distance-balanced lexicographic product graphs

In this section, we determine which lexicographic products are n-distance-balanced. In order to obtain a characteristic result for n-distance-balanced lexicographic product graphs we need to introduce a new class of graphs containing the regular graphs and so the strongly regular graphs. We recall that the lexicographic product of graphs G and H is the graph $G \circ H$ with the vertex set $V(G) \times V(H)$ and the edge set:

$$E(G \circ H) = \{(a,u)(b,v) \mid ab \in E(G), \text{ or } a = b \text{ and } uv \in E(H)\}.$$

Definition 3.1. *Let G be a connected graph. Then G is called locally regular graph if any non-adjacent vertices in G have the same degree.*

For example, the complete bipartite graph $K_{m,n}$ for $m \neq n$, wheel graph W_n for $n \geq 5$ and friendship graph F_n for $n \geq 2$ are locally regular graphs but not regular. Moreover, we easily can construct a partially regular graph from a given r-regular graph while has not regularity. To do this, consider an r-regular graph G with $V(G) \neq r+1$, then the graph $G+v$ obtained by adding vertex v and $E(G+v) = E(G) \cup \{uv | u \in V(G)\}$ is a locally regular graph but not regular. In Figure 7, we see the structure of a locally regular graph using Peterson graph while the obtained graph is not regular.

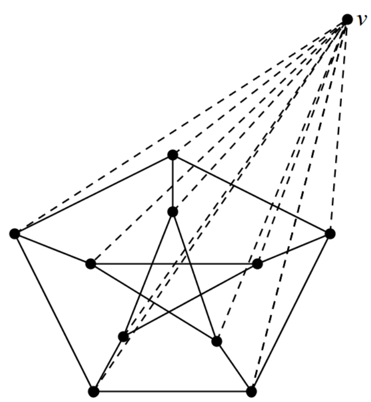

FIGURE 7. A locally regular graph deduced by Peterson graph and vertex v

Moreover, we remark that we can add more vertices such as $v_1, v_2, \ldots, v_k$ to regular graph G for arbitrary $k \in \mathbb{N}$ and obtain locally regular graphs with sufficiently large size.

Remark 3.2. *Following the construction of the class of locally regular graphs as above, we see that if G is vertex-transitive, then $G+v$ is n-vertex-transitive ($n > 1$) and is not vertex-transitive since its automorphism group does not act transitively upon its vertices including the vertex v. Using this fact together with Corollary 2.12 we obtain so many n-distance-balanced graphs. Similar to the notion of n-vertex-transitive graph, if we define n-symmetric graph based on symmetry, then $G+v$ is n-symmetric ($n > 1$) while it is not symmetric. Therefore, we see that the technique mentioned as above can be applied in the other classes of graphs with new notions.*

Remark 3.3. *Following Corollary 2.4 we observe that the only 2-distance-balanced graphs with $d = 2$ are locally regular graphs, and vice versa.*

Theorem 3.4. *Let G and H be connected graphs. Then $G \circ H$ is n-distance-balanced if and only if G is n-distance-balanced and H is regular for $n = 1$ and locally regular for $n > 1$.*

Proof. Obviously, for $n = 1$ since both concepts of 1-distance-balanced and distance-balanced graphs are the same hence the statement and Theorem 4.2 in [11] are so and there is nothing to prove for this case. For $n > 1$, let $G \circ H$ be n-distance-balanced. Bring in mind that

$$d_{G\circ H}((g,h),(\acute{g},\acute{h})) = \begin{cases} d_G(g,\acute{g}); & \text{if } g \neq \acute{g}, \\ 1; & \text{if } g = \acute{g} \text{ and } h\acute{h} \in E(H), \\ 2; & \text{if } g = \acute{g} \text{ and } h\acute{h} \notin E(H). \end{cases}$$

Suppose $(a,x),(a,y) \in V(G \circ H)$, $x \neq y$ and $d((a,x),(a,y)) = n$, then $xy \notin E(H)$. On the other hand, for $(u,v) \in V(G) \times V(H)$ we have the following implication

$$d((u,v),(a,x)) < d((u,v),(a,y)) \implies u \neq a, \ \ vx \in E(H), \ \ vy \notin E(H), \tag{3.1}$$

similarly,

$$d((u,v),(a,y)) < d((u,v),(a,x)) \implies u \neq a, \ \ vy \in E(H), \ \ vx \notin E(H). \tag{3.2}$$

Since $G \circ H$ is n-distance-balanced by applying (3.1) and (3.2) we get

$$\left|\left\{v \mid vx \in E(H)\right\}\right| = \left|\left\{v \mid vy \in E(H)\right\}\right|.$$

Since $xy \notin E(H)$, the recent equality shows that any non-adjacent vertices of H have the same degree which means H is a locally regular graph. Now consider $(a,x),(b,y) \in V(G \circ H)$ such that $a \neq b$ and $d((a,x),(b,y)) = n$. Then $d(a,b) = n$ and also we have

$$(u,v) \in W^{G\circ H}_{(a,x)\underline{n}(b,y)}, \ \ u \notin \{a,b\} \iff u \in W^G_{a\underline{n}b}, \tag{3.3}$$

similarly

$$(u,v) \in W^{G\circ H}_{(b,y)\underline{n}(a,x)}, \ \ u \notin \{a,b\} \iff u \in W^G_{b\underline{n}a}. \tag{3.4}$$

These facts together with the n-distance-balanced property of $G \circ H$ imply that G is an n-distance-balanced graph. We note that in the previous implications the case $u \in \{a,b\}$ can never happen. More precisely, by taking $u = a$ (similarly $u = b$) we get

$$vx \in E(G) \implies (a,v) \in W^{G\circ H}_{(a,x)\underline{n}(b,y)}, \ \ (a,v) \notin W^{G\circ H}_{(b,y)\underline{n}(a,x)}$$

$$vx \notin E(G) \implies \begin{cases} \left((a,v) \in W^{G\circ H}_{(a,x)\underline{n}(b,y)} \iff n > 2\right), \\ \left((a,v) \in {}_{(a,x)}\underset{\underline{n}}{W}{}^{G\circ H}_{(b,y)} \iff n = 2\right), \\ (a,v) \notin W^{G\circ H}_{(b,y)\underline{n}(a,x)}, \end{cases}$$

which is inferred from both that $G \circ H$ can not be n-distance-balanced. Now, let G be n-distance-balanced and H is locally regular. Consider $(a,x),(b,y) \in V(G \circ H)$, $a \neq b$ and $d((a,x),(b,y)) = n$, then $d(a,b) = n$ and we again obtain both relations (3.3) and (3.4) which shows that $G \circ H$ is n-distance-balanced. Indeed, for such case $G \circ H$ is n-distance-balanced if and only if G is n-distance-balanced. Suppose that $(a,x),(a,y) \in V(G \circ H)$, $x \neq y$ and $d((a,x),(a,y)) = n$, then $xy \notin E(H)$ and

$$(u,v) \in W^{G \circ H}_{(a,x)\underline{n}(a,y)} \iff u = a,\ xv \in E(H) \text{ and } yv \notin E(H),$$

also

$$(u,v) \in W^{G \circ H}_{(a,y)\underline{n}(a,x)} \iff u = a,\ yv \in E(H) \text{ and } xv \notin E(H).$$

Therefore,

$$W^{G \circ H}_{(a,x)\underline{n}(a,y)} = \Big\{(a,v) \mid xv \in E(H),\ yv \notin E(H)\Big\},$$

$$W^{G \circ H}_{(a,y)\underline{n}(a,x)} = \Big\{(a,v) \mid yv \in E(H),\ xv \notin E(H)\Big\},$$

which together with the fact that H is locally regular implies that $G \circ H$ is n-distance-balanced and the proof is completed. □

Taking $G = mK_1$ of isolated vertices with $V(G) = m$, an immediate consequence can be obtained by following an argument similar to that used in the proof of Theorem 3.4.

Corollary 3.5. *Suppose that H is a connected graph. Then $mK_1 \circ H$ is n-distance-balanced if and only if H is regular for $n = 1$ and partially regular for $n > 1$.*

Remark 3.6. *Corollary 3.5 shows that we can construct n-distance-balanced graphs with any order by locally regular graphs.*

4. Recognition of n-distance-balanced graphs

One possible way to attack the characterization problem for n-distance-balanced graphs at least for some n is to try to classify such graphs based on the existence of odd or even circles in graph and using the concept of total distance. As the most significant part of the paper this section is dedicated to such investigation where the n-distance-balanced graphs would be characterized for some n. However, for larger integers n the classification becomes very complicated so soon and currently beyond our reach and may be infeasible by the concept of total distance.

Theorem 4.1 ([1])**.** *Let graph G be a connected graph. Then G is distance-balanced if and only if $|\{td(u) \mid u \in V(G)\}| = 1$ where $td(u)$ denotes the total distance of u in G and is defined by*

$$td(u) := d(u, V(G)) = \sum_{v \in V(G)} d(u,v).$$

Similarly, one can present some results for n-distance-balanced graphs with certain n. The following result plays the role of a prototype in our discussion of this section which is inspired by Theorem 4.1.

Definition 4.2. *The connected graph G has the property $(\triangle_n)$ if for any $u, v \in V(G)$ with $d(u, v) = n$ and $x \in W^G_{u\underline{n}v}$ the shortest path P_{xv} connecting x to v does not contain P_{xu} as the shortest path connecting x to u.*

We remark that the property $(\triangle_n)$ as a sufficient condition is needed for the existence of circle in the proof of the following theorems.

Theorem 4.3. *Suppose that G is a connected graph and one of the following conditions holds:*

- **(i):** *G has no odd circle, that is, G is bipartite;*
- **(ii):** *G has the property $(\triangle_2)$ with no even circle.*

Then G is 2-distance-balanced if and only if for all $u, v \in V(G)$ with $d(u, v) = 2$ we have $d(u, V(G)) = d(v, V(G))$.

Proof. To prove it under condition (i), suppose that u, v are pair of vertices in G such that $d(u, v) = 2$, then $d(u, V(G)) = d(v, V(G))$ is equivalent to

$$\sum_{x\in W^G_{u\underline{2}v}} d(u,x) + \sum_{x\in W^G_{v\underline{2}u}} d(u,x) + \sum_{x\in {}_uW^G_{\underline{2}v}} d(u,x) = \sum_{x\in W^G_{u\underline{2}v}} d(v,x) + \sum_{x\in W^G_{v\underline{2}u}} d(v,x) + \sum_{x\in {}_uW^G_{\underline{2}v}} d(v,x)$$

which shows that

$$\sum_{x\in W^G_{u\underline{2}v}} (d(u,x) - d(v,x)) = \sum_{x\in W^G_{v\underline{2}u}} (d(v,x) - d(u,x)). \tag{4.1}$$

Now we prove that $d(u, x) - d(v, x) = -2$ for all $x \in W^G_{u\underline{2}v}$. Let $x \in W^G_{u\underline{2}v}$ and $d(x, u) = l(R)$ where R is the shortest path connecting x to u. Then

$$l(R) = d(x, u) < d(x, v) \leq l(R) + 2$$

which yields that $d(x, v) = l(R) + 1$ or $l(R) + 2$. If $d(x, v) = l(R) + 1$ then there is an odd circle with length $2l(R) + 1$ or $2l(R) + 3$ (see Figure 8). This is a contradiction since G is bipartite and thus the claim is proved. We note that if path R crosses Q or P then we obtain some new cycles which at least one of them is odd.

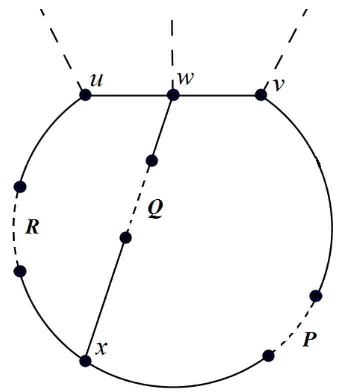

FIGURE 8. vetices u, v and possible paths connecting x to u, v in G

Similarly, we have $d(v,x) - d(u,x) = -2$ for all $x \in W^G_{v\underline{2}u}$. Now (4.1) shows that

$$\sum_{x \in W^G_{u\underline{2}v}} (-2) = \sum_{x \in W^G_{v\underline{2}u}} (-2)$$

and it holds if and only if G is 2-distance-balanced which completes the proof for the case (i). Now, let G has the property $(\triangle_2)$ with no even circle. Then following the proof in the previous case we observe that $d(x,v) = l(R)+1$ if $d(x,u) = l(R)$ for all $x \in W^G_{u\underline{2}v}$. Because if $d(x,v) = l(R)+2$ then there is a path P with length $l(R)+2$ connecting x to v (or path Q with length $l(R)+1$ connecting x to w) which shows that G has an even circle with length $2l(R)+4$ (or $2l(R)+2$) and this is a contradiction (see Figure 8). According to Figure 9, we remark that if $Q = P_3 + P_4$ (or P) crosses $R = P_1 + P_2$ then $l(P_2) \leq l(P_4)$ since R is the shortest path connecting x to u. On the other hand, considering both paths $P_2 + P_3$ and $P_3 + P_4$ we observe that $l(P_4) \leq l(P_2)$ which means that $P_2 + P_4$ is an even circle and that is a contradiction.

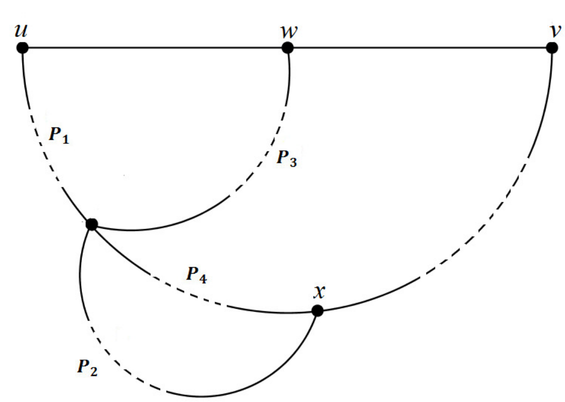

FIGURE 9. vetices u, v and the intersecting paths $R = P_1 + P_2$ and $Q = P_3 + P_4$

Therefore,

$$\sum_{x\in W^G_{u\underline{2}v}}(d(u,x)-d(v,x))=\sum_{x\in W^G_{v\underline{2}u}}(d(v,x)-d(u,x))\Longleftrightarrow\sum_{x\in W^G_{u\underline{2}v}}(-1)=\sum_{x\in W^G_{v\underline{2}u}}(-1)$$

and the consequence follows. □

Unlike simple look, Theorem 4.3 plays a crucial role to classify a group of the distanced-balanced graphs with specified properties. Using the both recent theorems we have an immediate consequence as follows.

Corollary 4.4. *Suppose that G is bipartite and 2-distance-balanced. Then G is distance-balanced.*

Remark 4.5. *Based on Corollary 4.4 we observe that for any bipartite and 2-distance-balanced graph G we have that G_e as a graph obtained from G by subdividing $e\in E(G)$ is distance-balanced if and only if G is a cycle (see* [11, Proposition 3.3]*).*

In the following we give a similar condition for characterization of 3-distance-balanced graphs based on the concept of total distance. For 3-distance-balanced graphs, the procedure is very much analogous to that for 2-distance-balanced graphs.

Theorem 4.6. *Suppose that G is a connected graph and has the property $(\triangle_3)$ with no even circle. Then G is 3-distance-balanced if and only if for all $u,v\in V(G)$ with $d(u,v)=3$ we have $d(u,V(G))=d(v,V(G))$.*

Proof. Inspired by the proof of Theorem 4.3 we claim that

$$\sum_{x\in W^G_{u\underline{3}v}}(d(u,x)-d(v,x))=\sum_{x\in W^G_{v\underline{3}u}}(d(v,x)-d(u,x))\Longleftrightarrow\sum_{x\in W^G_{u\underline{3}v}}(-2)=\sum_{x\in W^G_{v\underline{3}u}}(-2). \tag{4.2}$$

To prove this assertion we let $x\in W^G_{u\underline{3}v}$ and $d(x,u)=l(P)$ where P is the shortest path connecting x to u. Since $l(P)=d(u,x)<d(v,x)\le l(P)+3$ hence $d(v,x)\in\{l(P)+1,l(P)+2,l(P)+3\}$. If we have $d(v,x)=l(P)+1$ then following the property $(\triangle_3)$ and using the paths Q,R or S there is an even circle with length $2l(P),2l(P)+2$ or $2l(P)+4$, respectively, which is a contradiction (see Figure 10). Similarly, if $d(x,v)=l(P)+3$ then using the paths Q,R or S there is an even circle with length $2l(P)+2,2l(P)+4$ or $2l(P)+6$, respectively, which is again a contradiction and so the claim is proved. Similar to the proof of the former theorem we notice that the path P does not intersect Q,R or S and the proof is completed.

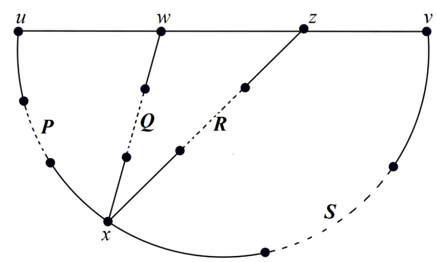

Figure 10. vetices u, v and possible distinct paths connecting x to u, v in G

$\square$

Remark 4.7. *If one may follow the process as above for n-distance-balanced graphs, then for $x \in W^G_{u\underline{n}v}$ and $d(x,u) = l(P)$ we get $d(v,x) \in \{l(P)+1, l(P)+2, \ldots, l(P)+n\}$ which all n cases should to be considered.*

5. Conclusions

We have shown that it can be introduced a new sense of distance-balanced property so-called n-distance-balanced and it seems this concept has more advantages than distance-balanced property. Because first the distance-balanced graphs are included in the class of n-distance-balanced graph. Second, unlike distance-balanced property removal a certain edge may not ruin the n-distance-balanced property. Moreover, when a graph is a model for a real-life problem (say in economy, communication networks, or some natural phenomena) we may encounter optimization problem and have to eliminate some redundant costs and the recent property of n-distance-balanced is helpful in such cases. Thus, in many situations the design of n-distance-balanced networks is highly desirable.

For further research, using the nicely distance-balanced and strongly distance-balanced properties which both recently have been introduced one can define new concepts, similar to the technique we applied and obtain results (see [1],[9]-[11]).

Acknowledgement

We are grateful to the referees for their suggestions and helpful remarks.

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