Transactions on Combinatorics
ISSN (print): 2251-8657, ISSN (on-line): 2251-8665
Vol. 01 No. 2 (2012), pp. 59-65.

www.combinatorics.ir
www.ui.ac.ir

THE ORDER DIFFERENCE INTERVAL GRAPH OF A GROUP

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Communicated by Bijan Taeri

ABSTRACT. In this paper we introduce the concept of order difference interval graph $\Gamma_{ODI}(G)$ of a group G. It is a graph $\Gamma_{ODI}(G)$ with $V(\Gamma_{ODI}(G)) = G$ and two vertices a and b are adjacent in $\Gamma_{ODI}(G)$ if and only if $o(b) - o(a) \in [o(a), o(b)]$. Without loss of generality, we assume that $o(a) \leq o(b)$. In this paper we obtain several properties of $\Gamma_{ODI}(G)$, upper bounds on the number of edges of $\Gamma_{ODI}(G)$ and determine those groups whose order difference interval graph is isomorphic to a complete multipartite graph.

1. Introduction

The study of algebraic structures, using the properties of graphs, has become an exciting research topic in the past three decades, leading to many fascinating results and questions. There are many papers on assigning a graph to a ring or group and investigating algebraic properties of ring or group using the associated graph, for instance, see [4]. In this paper, to any group G, we assign a graph and investigate algebraic properties of the group using the graph theoretical concepts. We need the following definitions and notations. Terms not defined here are used in the sense of [1, 2] and [3].

We consider simple graphs which are undirected, with no loops or multiple edges. For any graph Γ, we denote the sets of the vertices and edges of Γ by $V(\Gamma)$ and $E(\Gamma)$, respectively. The degree $deg_\Gamma(v)$ of a vertex v in Γ is the number of edges incident to v and if the graph is understood, then we denote $deg_\Gamma(v)$ simply by $deg(v)$. The order of Γ is defined by $|V(\Gamma)|$ and the maximum and minimum degrees will be denoted, respectively, by $\Delta(\Gamma)$ and $\delta(\Gamma)$. A graph Γ is regular if the degrees of all vertices of Γ are the same. Unicyclic graphs are graphs which are connected and have just one cycle.

MSC(2010): Primary: 05C25; Secondary: 05C15.
Keywords: Order difference interval graph, Unicyclic graph, Eulerian, generating set.
Received: 25 May 2012, Accepted: 5 August 2012.
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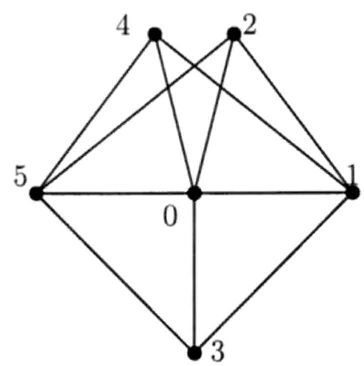
Let G be a group with identity e. The number of elements of a group is called its *order* and it is denoted by $o(G)$. The order of an element g in a group is the smallest positive integer n such that $g^n = e$ and is denoted by $o(g)$. If no such integer exists, we say g has infinite order. If A is any subset of a group G, the subgroup generated by A, $[A]$ is defined as the intersection of all subgroups of G containing A. If $[A]= G$, then A is said to be a generating set of G.

2. Main results

Definition 2.1. *Let G be a finite group. Then order difference interval graph $\Gamma_{ODI}(G)$ of a group G is a graph with $V(\Gamma_{ODI}(G)) = G$ and two vertices a and b are adjacent in $\Gamma_{ODI}(G)$ if and only if $o(b) - o(a) \in [o(a), o(b)]$, where $[x, y]$ denotes the closed interval. Without loss of generality, we assume that $o(a) \leq o(b)$. Here $o(a)$ and $o(b)$ denote the orders of a and b, respectively.*

Example 2.2. *Let $G = (\mathbb{Z}_6, +_6)$. Then $\Gamma_{ODI}(G)$ is given in Figure 1.*

Figure 1



Proposition 2.3. *Let G be a group with $o(G) = n$. Then $\Delta(\Gamma_{ODI}(G)) = n - 1$.*

Proof. Identity e is an element of order 1 in G and hence $deg(e) = n - 1$ in $\Gamma_{ODI}(G)$. □

Proposition 2.4. *Let G be any group and $e \neq a \in G$. If $deg(a) = o(G) - 1$, then a is the unique element of order 2 in G. In general, the converse is not true. Moreover, the converse is true if G has no element of order 3.*

Proof. First, we show that a is a self-inverse element. If not, then since $o(a) = o(a^{-1})$, so a and a^{-1} are non-adjacent in $\Gamma_{ODI}(G)$ so that $deg(a) \leq o(G) - 2$, which is a contradiction. Hence a is a self-inverse element.

Now, we show that a is unique. Suppose that G has more than one self-inverse element, say a, b. Clearly, a and b are non-adjacent in $\Gamma_{ODI}(G)$ and so $deg(a) < o(G) - 1$, which is a contradiction. So a is the only self-inverse element in G.

Now, we show that the converse is not true. Let $G \cong \mathbb{Z}_6$. Then $\Gamma_{ODI}(G)$ is given in Example 2.2. Clearly, 3 is the only self-inverse element in G but $deg(3) = 3 < o(G) - 1$. So the converse of the proposition is not true.

Finally, we show that the converse is true whenever G has no element of order 3. Let a be the unique element of order 2 in G. Let $b \in G$ be such that $b \neq a, e$. Then by hypothesis, $o(b) \neq 3$. Therefore $o(b) - o(a) \geq 2$ so that a is adjacent to b. Since b is arbitrary, $deg(a) = o(G) - 1$. □

Corollary 2.5. *Let G be any group. Then the identity element in G is the only element of maximum degree in $\Gamma_{ODI}(G)$ if and only if either $o(G)$ is odd or the number of self-inverse elements in G is at least 2.*

Corollary 2.6. *Let G be any group and $g \neq e$ be any element in G. Then $deg(g) = deg(e)$ in $\Gamma_{ODI}(G)$ if and only if g is a unique self inverse element in G and no element in G is of order 3.*

Two integers n and m are called *relatively prime* if the greatest common divisor (g. c. d) of m and n is 1.

Lemma 2.7. *Let a be a generator element in group G of order n. Then a is adjacent to all the non-generator elements of G in the graph $\Gamma_{ODI}(G)$.*

Proof. Let a be a generator element and b be a non-generator element in G. Since a is generator of G, it follows that $o(a) = n$. Now, $o(b) = m \leq \frac{n}{2}$. It is clear that $o(a) - o(b) \geq \frac{n}{2}$ and so a is adjacent to b. Since a and b are arbitrary, a is adjacent to all the non-generator elements of G in $\Gamma_{ODI}(G)$. □

Proposition 2.8. *Let G be a cyclic group with $o(G) = n$. Then at least $\phi(n)$ vertices have same degree in $\Gamma_{ODI}(G)$. Moreover, $deg(g) = n - \phi(n)$ if g is a generator of G, where $\phi(n)$ is the number of positive integers less than n and relatively prime to it.*

Proof. First, we show that if a and b are elements with same order in G, then $deg(a) = deg(b)$ in $\Gamma_{ODI}(G)$. Let $deg(a) = m$. Then a is adjacent to $a_1, a_2, \ldots, a_m$ and so $o(a_i) - o(a) \geq o(a)$, $1 \leq i \leq m$. By hypothesis, $o(a_i) - o(b) \geq o(b)$, $1 \leq i \leq m$ and so $deg(b) \geq m$. Suppose that $deg(b) > m$. Then by a similar argument, we get $deg(a) > m$, which is a contradiction. Therefore, $deg(a) = deg(b)$.

Since G is cyclic, number of generators in G is $\phi(n)$ and so $\phi(n)$ elements in G have same order. So at least $\phi(n)$ vertices have same degree in $\Gamma_{ODI}(G)$. Let g be a generator element of G. Then it follows from Lemma 2.7 that $deg(g) = n - \phi(n)$. □

Proposition 2.9. *For any group G, $\Gamma_{ODI}(G)$ is complete if and only if $o(G) = 2$.*

Proof. If $o(G) = 2$, then $\Gamma_{ODI}(G) \cong K_2$. Suppose $\Gamma_{ODI}(G)$ is complete. If $o(G) \geq 3$ and every element in G is a self-inverse element, then every element of G is of order 2 and so $\Gamma_{ODI}(G)$ cannot be complete. If there exists at least one element which is not a self-inverse element, the element and its inverse have the same order in G and hence non-adjacent in $\Gamma_{ODI}(G)$. These contradictions show that $o(G) = 2$. □

Corollary 2.10. *Let G be any group with $n \geq 3$ elements. Then $\Gamma_{ODI}(G)$ cannot be regular.*

Proof. This follows from Propositions 2.3 and 2.9. □

Theorem 2.11. *Let $o(G) = p^\alpha$, p a prime. Then G has $(k+1)$-elements with different orders if and only if $\Gamma = \Gamma_{ODI}(G)$ is a $(k+1)$-complete partite graph.*

Proof. First note that $p^j - p^i \in [p^i, p^j], i < j, i, j \in \mathbb{N}$. Let $a_1, a_2, \ldots, a_{k+1} \in G$ be the elements of different orders and let $M_i = \{b \in G : o(a_i) = o(b)\}, 1 \leq i \leq k+1$. Clearly, each $M_i \neq \emptyset$ since $a \in M_i$. Also, $M_i \cap M_j = \emptyset$ for all $i \neq j$ and $M_1 \cup M_2 \cup \ldots \cup M_{k+1} = V(\Gamma_{ODI}(G))$. No two elements in M_i are adjacent and $p^j - p^i \in [p^i, p^j]$ for all $i \neq j$. Hence G is a complete $(k+1)$-partite graph. Conversely, suppose G is a $(k+1)$-complete partite graph. Then clearly, the $(k+1)$- partitions have different orders since otherwise we can find $a_i \in M_i$ and $a_j \in M_j$ such that a_i and a_j are non-adjacent. Therefore the group G has $(k+1)$-elements with different orders. □

Proposition 2.12. *For any group G, $\Gamma_{ODI}(G)$ can never be a unicyclic graph.*

Proof. Suppose that $\Gamma_{ODI}(G)$ is unicyclic. Since $\Delta(\Gamma_{ODI}(G)) = n - 1$ and e is the vertex with degree Δ, $\Gamma_{ODI}(G) - e$ has exactly one edge e'. Let $e' = ab$. Then $o(a)$ and $o(b)$ are different with $o(b) - o(a) \in [o(a), o(b)]$. Now, at least one of a and b should not be a self-inverse element since otherwise $0 = o(b) - o(a) \notin [o(a), o(b)]$. Let $a \neq a^{-1}$. But $o(a) = o(a^{-1})$ and so a^{-1}, b are also adjacent in $\Gamma_{ODI}(G) - e$, which is a contradiction. □

Proposition 2.13. *Let G be any group. There exists $a, b \in G - e$ such that a is adjacent to b in $\Gamma_{ODI}(G)$ if and only if it contains a cycle. Moreover, G has a smallest cycle of length 3 and also the girth of $\Gamma_{ODI}(G)$ is 3.*

Proof. Suppose that a and b are adjacent. Since e is adjacent to a and b, $\Gamma_{ODI}(G)$ contains a cycle of length 3. Conversely, consider the cycle C_3, say $exye$. Clearly x and y are adjacent in C_3. So x and y are adjacent in $\Gamma_{ODI}(G)$. □

Proposition 2.14. *Let G be any group. Then $\Gamma_{ODI}(G)$ can never be Eulerian.*

Proof. If $o(G)$ is even, then by Proposition 2.3, $deg(e)$ is odd and so the result is obvious.

Next, assume that $o(G)$ is odd. Let $a_1, a_2, \ldots, a_{k+1} \in G$ be the elements of different orders and let $M_i = \{b \in G : o(a_i) = o(b)\}$, $1 \leq i \leq k+1$. Also each M_i is non-empty since $a \in M_i$. Now, let $M_1 = \{e\}$. The cardinality of each M_i, $i \neq 1$, is even since every element in $G - e$ is non-self inverse. Moreover $M_i \cap M_j = \emptyset$ for all $i \neq j$ and $M_1 \cup M_2 \cup \ldots \cup M_{k+1} = V(\Gamma_{ODI}(G))$. Clearly, no two elements in M_i are adjacent and every element of M_i is adjacent to each element of some M_j. In particular, $deg(a)$ in $\Gamma_{ODI}(G) - \{e\}$ is even for $a \in M_i$. Since a is also adjacent to e in $\Gamma_{ODI}(G)$, It follows that $deg(a)$ is odd. Therefore $\Gamma_{ODI}(G)$ is non-Eulerian. □

Theorem 2.15. *Let G be any group. If either* (i) *$o(G) = p, p \geq 2$, p is prime or* (ii) *$o(a) = 2$ for all $a \in G - e$, then $\Gamma_{ODI}(G) \cong K_{1,o(G)-1}$. But converse is not true.*

Proof. (i) Suppose that $o(G) = p$. Let $a, b \in G - e$. Therefore $o(a) = o(b) = o(G)$ and so every element in G except identity is a generator. Therefore $0 = o(b) - o(a) \notin [o(a), o(b)]$. But identity is adjacent to every vertex in $\Gamma_{ODI}(G)$ which shows that $\Gamma_{ODI}(G)$ is $K_{1,o(G)-1}$.

(ii) Suppose $o(a) = 2$ for all $a \in G - e$. Then order difference of any two elements in G except identity is zero. So any two vertices in $\Gamma_{ODI}(G)$ except identity is non-adjacent and hence by Proposition 2.3, $\Gamma_{ODI}(G)$ is $K_{1,o(G)-1}$.

Consider the group $G = D_6$. Then $\Gamma_{ODI}(G)$ is $K_{1,5}$. But neither $o(G)$ is a prime nor order of every element in $G - e$ is two. □

The following two corollaries immediately follows from Theorem 2.15.

Corollary 2.16. *Let G be any group. If either* (i) $o(G) = p, p \geq 2$, *p is prime or* (ii) $o(a) = 2$ *for all $a \in G - e$, then $\Gamma_{ODI}(G)$ is a tree. The converse is not true.*

Corollary 2.17. *Let G be any group. If either* (i) $o(G) = p, p \geq 2$, *p is prime or* (ii) $o(a) = 2$ *for all $a \in G - e$, then $\Gamma_{ODI}(G)$ has exactly $o(G) - 1$ pendent vertices. The converse is not true.*

Proposition 2.18. *Let G be any group with $o(G) = p^\alpha$, where p is a prime number and $\alpha \in \mathbb{N}$. Then every element in $G - e$ has the same order if and only if $\Gamma_{ODI}(G) \cong K_{1,o(G)-1}$. Moreover, that order is p.*

Proof. By hypothesis, all elements in $G - e$ have the same order and so difference between order of any two elements in $G - e$ is zero. So every two elements in G except identity are non-adjacent. Hence the result. Conversely, suppose that $\Gamma_{ODI}(G) \cong K_{1,o(G)-1}$. Then by Theorem 2.11, number of elements of different orders in G except identity is 1 and so the result follows. Moreover, by Cauchy Theorem the order is p. □

Observation 2.19. *In general, converse of Proposition* 2.18 *is not true. Consider the group $G = D_6$. Then $\Gamma_{ODI}(G)$ is $K_{1,5}$. But every element in $G - e$ is not of same order.*

Proposition 2.20. *For any group G, $\Gamma_{ODI}(G)$ is a path if and only if $o(G) = 2$ or* 3.

Proof. Suppose that $o(G) = 2$ *or* 3. Then $\Gamma_{ODI}(G)$ is isomorphic to a path, namely P_2 or P_3. Suppose that $\Gamma_{ODI}(G)$ is path. Then $1 \leq deg(a) \leq 2$ for all $a \in V(\Gamma_{ODI}(G))$. If $o(G) \geq 4$, then by Proposition 2.3, $deg(e) = o(G) - 1 \geq 3$ and so $o(G) = 2$ or 3. □

Theorem 2.21. *Let G be any group. Then $\Gamma_{ODI}(G)$ is a bipartite graph if and only if it is a star.*

Proof. If $\Gamma_{ODI}(G)$ is a star, then it is bipartite. Conversely, assume that $\Gamma_{ODI}(G)$ is a bipartite graph. Suppose that $\Gamma_{ODI}(G)$ is not a star. Then there exists at least two vertices u, v in $\Gamma_{ODI}(G)$ that have degree at least 2. So we get a cycle $e - u - v - e$. This a cycle of length 3, which is a contradiction as G is bipartite. □

Corollary 2.22. *Let G be any group. Then $\Gamma_{ODI}(G)$ is a tree if and only if it is a star.*

A *coloring* of a graph Γ is an assignment of colors to its vertices so that no two adjacent vertices have the same color. The *chromatic number* $\chi(G)$ is defined as the minimum n for which Γ has a $n-$coloring.

Proposition 2.23. *Let G be any group. Then $2 \leq \chi(\Gamma_{ODI}(G)) \leq k+1$ where $k+1$ = number of M_i and M_i is defined as Theorem 2.14. Moreover, these bounds are sharp.*

Proof. If $i = 1$, then $\Gamma_{ODI}(G)$ is a star. So it is a bipartite graph and hence $\chi(\Gamma_{ODI}(G)) = 2$.

If $2 \leq i \leq k$, then G has at least $k+1$ elements with different orders. Since every vertex of M_i is adjacent to some vertex of M_j, $\chi(\Gamma_{ODI}(G) - e)$ is at most k. Also, e is adjacent to all the vertices in $V(\Gamma_{ODI}(G))$ so that $\chi(\Gamma_{ODI}(G))$ is at most $k+1$. Therefore we get $2 \leq \chi(\Gamma_{ODI}(G)) \leq k+1$.

To observe the sharpness of the bounds, consider $G_1 = \{\mathbb{Z}_3, +_3\}$. Then $\Gamma_{ODI}(G_1) \cong P_3$ and hence $\chi(\Gamma_{ODI}(G_1)) = \chi(P_3) = 2$. Now, consider $G_2 = \{\pm 1, \pm i\}$. Then $\Gamma_{ODI}(G_2) \cong K_4 - e$. Therefore $\chi(\Gamma_{ODI}(G_1)) = \chi(K_4 - e) = 3 = k+1$, where $k = 2$. □

Theorem 2.24. *Let G be any group. Then,*
(1) $\chi(\Gamma_{ODI}(G)) = 2$ if and only if it is a star.
(2) $\Gamma_{ODI}(G) \cong K_{|M_1|,\ldots,|M_{k+1}|}$ if and only if $\chi(\Gamma_{ODI}(G)) = k+1$, where $k+1$ = number of elements of different order in G.

Proof. (1) If $\chi(\Gamma_{ODI}(G)) = 2$, then $\Gamma_{ODI}(G)$ is a bipartite graph and so by Theorem 2.21, $\Gamma_{ODI}(G)$ is a star. Conversely, assume that $\Gamma_{ODI}(G)$ is a star. Then by Theorem 2.21, $\Gamma_{ODI}(G)$ is a bipartite graph and hence $\chi(\Gamma_{ODI}(G)) = 2$.
(2) Here M_i is defined as Theorem 2.14. Suppose that $\Gamma_{ODI}(G) \cong K_{|M_1|,|M_2|,\ldots,|M_{k+1}|}$. Then $\chi(\Gamma_{ODI}(G)) = k+1$. Conversely, assume that $\chi(\Gamma_{ODI}(G)) = k+1$, where $k+1$ = number of elements of different order in G. Now, Let $M_1 = \{e\}$ and $M_i = \{a \in G - e : o(a_i) = o(b) \ for \ all \ b \in G\}$ $i = 2, 3, \ldots, k+1$. Then any two elements of different order in G are adjacent since otherwise $\chi(\Gamma_{ODI}(G)) \leq k$. Thus $\Gamma_{ODI}(G) \cong K_{|M_1|,|M_2|,\ldots,|M_{k+1}|}$. □

Theorem 2.25. *Let G be any group with $o(G) = n = p_1^{n_1} p_2^{n_2} \ldots p_k^{n_k}$. Let q be the number of edges of the graph $\Gamma_{ODI}(G)$. Then $q \leq \frac{1}{2}[n^2 - n + M - (|M_1|^2 + |M_2|^2 + \ldots + |M_k|^2)]$ where $M = |M_1| + |M_2| + \ldots + |M_k|$. Moreover, equality holds if and only if $\Gamma_{ODI}(G)$ is a complete $(k+1)$-partite graph.*

Proof. Clearly, $M_i's$ are mutually non-adjacent. Hence the maximum number of edges of G equals $\binom{n}{2} - \{\binom{|M_1|}{2} + \ldots + \binom{|M_k|}{2}\} = [\frac{n(n-1)}{2} - \{\frac{|M_1|(|M_1|-1)}{2} + \ldots + \frac{|M_k|(|M_k|-1)}{2}\}] = \frac{1}{2}[n^2 - n - \{|M_1|^2 + |M_2|^2 + \ldots + |M_k|^2\} + (|M_1| + \ldots + |M_k|)] = \frac{1}{2}[n^2 - n + M - (|M_1|^2 + \ldots + |M_k|^2)]$. Hence $q \leq \frac{1}{2}[n^2 - n + M - (|M_1|^2 + \ldots + |M_k|^2)]$. Moreover, $q = \binom{n}{2} - \{\binom{|M_1|}{2} + \ldots + \binom{|M_k|}{2}\}$ if and only if each M_i is adjacent to each M_j, $i \neq j$ if and only if $\Gamma_{ODI}(G)$ is a complete $(k+1)$-partite graph since otherwise some M_i is non-adjacent to some M_j, $i \neq j$. □

The following two theorems are direct consequences of the fact that every group isomorphism preserves the order of each element.

Theorem 2.26. *If G_1, G_2 are two groups such that $G_1 \cong G_2$, then $\Gamma_{ODI}(G_1) \cong \Gamma_{ODI}(G_2)$.*

Theorem 2.27. *Let G be any group. Then $Aut(G) \subseteq Aut(\Gamma_{ODI}(G))$.*

Remark 2.28. *The converse of Theorem* 2.27 *is not true. Consider the group* $G = (\mathbb{Z}_5, +_5)$. *Here* $\Gamma_{ODI}(G) \cong K_{1,4}$. *Define* $g : \Gamma_{ODI}(G) \to \Gamma_{ODI}(G)$ *such that* $g(0) = 0$, $g(1) = 2$, $g(2) = 3$, $g(3) = 4$, $g(4) = 1$. *Clearly* g *is an automorphism of* $\Gamma_{ODI}(G)$. *But* $g(1 +_5 2) = g(3) = 4$ *and* $g(1) +_5 g(2) = 2 +_5 3 = 0$. *Hence* $g(1 +_5 2) \neq g(1) +_5 g(2)$ *so that* g *is not an automorphism of* G.

Acknowledgments

The authors are much indebted to the referees for some valuable comments and constructive suggestions.

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