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**Transactions on Combinatorics**

ISSN (print): 2251-8657, ISSN (on-line): 2251-8665

Vol. 1 No. 3 (2012), pp. 1-4.

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## A NOTE ON THE TOTAL DOMINATION SUPERCRITICAL GRAPHS

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Communicated by Alireza Abdollahi

**ABSTRACT.** Let  $G$  be a connected spanning subgraph of  $K_{s,s}$  and let  $H$  be the complement of  $G$  relative to  $K_{s,s}$ . The graph  $G$  is  $k$ -supercritical relative to  $K_{s,s}$  if  $\gamma_t(G) = k$  and  $\gamma_t(G + e) = k - 2$  for all  $e \in E(H)$ . The 2002 paper by T.W. Haynes, M.A. Henning and L.C. van der Merwe, “Total domination supercritical graphs with respect to relative complements” that appeared in *Discrete Mathematics*, 258 (2002), 361-371, presents a theorem (Theorem 11) to produce  $(2k + 2)$ -supercritical graphs relative to  $K_{2k+1, 2k+1}$  of diameter 5, for each  $k \geq 2$ . However, the families of graphs in their proof are not the case. We present a correction of this theorem.

### 1. Introduction

Let  $G = (V(G), E(G))$  be a simple graph of order  $n$ . We denote the *open neighborhood* of a vertex  $v$  of  $G$  by  $N_G(v)$ , or just  $N(v)$ , and its *closed neighborhood* by  $N_G[v] = N[v]$ . For a vertex set  $S \subseteq V(G)$ ,  $N(S) = \cup_{v \in S} N(v)$  and  $N[S] = \cup_{v \in S} N[v]$ . A set of vertices  $S$  in  $G$  is a *total dominating set*, (or just TDS), if  $N(S) = V(G)$ . The *total domination number*,  $\gamma_t(G)$  of  $G$ , is the minimum cardinality of a total dominating set of  $G$ . For Graph Theory notation and terminology in general we follow [2].

Haynes, Henning and Van der Merwe in [3] studied *total domination supercritical graphs with respect to relative complements*. Let  $G$  be a connected spanning subgraph of  $K_{s,s}$  and let  $H$  be the complement of  $G$  relative to  $K_{s,s}$ . The graph  $G$  is  $k$ -supercritical relative to  $K_{s,s}$  if  $\gamma_t(G) = k$  and  $\gamma_t(G + e) = k - 2$  for all  $e \in E(H)$ . They presented a construction to produce 6-supercritical graphs of diameter 5.

For  $k \geq 2$ , let  $\mathcal{G}_k$  be the class of all graphs  $G$  such that  $G \in \mathcal{G}_k$  if and only if  $G$  is formed as follows. Form  $G$  from  $k$  copies of the cycle  $C_6$  by identifying an edge, say  $ab$ , common to every cycle. Let

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MSC(2010): Primary: 05C69.

Keywords: Total domination, Supercritical, Diameter.

Received: 13 July 2012, Accepted: 23 September 2012.

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$A = N(a) - \{b\}$  and  $B = N(b) - \{a\}$ , and label the vertices of  $A$  and  $B$  as  $A = \{a_1, a_2, \dots, a_k\}$  and  $B = \{b_1, b_2, \dots, b_k\}$  such that  $a_i$  and  $b_i$  are in the  $i$ th copy of  $C_6$ . Finally, for each  $i \neq j$ , add exactly one of the edges  $a_i b_j$  and  $a_j b_i$ . Clearly,  $G$  is a bipartite spanning subgraph of  $K_{2k+1, 2k+1}$  and  $\text{diam}(G) = 5$ .

**Theorem 1.1.** (Theorem 11 of [3]) For each  $k \geq 2$ ,  $G \in \mathcal{G}_k$  is a  $(2k + 2)$ -supercritical graph relative to  $K_{2k+1, 2k+1}$  of diameter 5.

We show that Theorem 1.1 is not correct, and the construction presented for  $2k + 2$ -supercritical graphs of diameter 5 is incorrect, and then we give a corrected construction which produces  $2k + 2$ -supercritical graphs of diameter 5. We remark that in [1] the authors corrected the construction presented in the first paragraph of page 370 of [3].

## 2. Main results

Consider the construction presented before Theorem 1.1. For  $i \in \{1, \dots, k - 1\}$ , let  $b_i \in N(a_{i+1})$ , and  $b_k \in N(a_1)$ . Then clearly  $S = \{a_i, b_i : i = 1, 2, \dots, k\}$  is a TDS for  $G$ , implying that  $\gamma_t(G) = 2k$ . Thus the above construction does not produce  $(2k + 2)$ -supercritical graphs.

We will now give a corrected construction.

- For  $k \geq 2$ , let  $\mathcal{H}_k$  be the class of all graphs  $G$  such that  $G \in \mathcal{H}_k$  if and only if  $G$  is formed as follows. Form  $G$  from  $k$  copies of the cycle  $C_6$  by identifying an edge, say  $ab$ , common to every cycle. Let  $A = N(a) - \{b\}$  and  $B = N(b) - \{a\}$ , and label the vertices of  $A$  and  $B$  as  $A = \{a_1, a_2, \dots, a_k\}$  and  $B = \{b_1, b_2, \dots, b_k\}$  such that  $a_i$  and  $b_i$  are in the  $i$ th copy of  $C_6$ . Finally, for  $i = 1, 2, \dots, k - 1$ , join  $a_i$  to every vertex in  $\{b_{i+1}, b_{i+2}, \dots, b_k\}$ . Clearly,  $G$  is a bipartite spanning subgraph of  $K_{2k+1, 2k+1}$  and  $\text{diam}(G) = 5$ .

**Theorem 2.1.** For each  $k \geq 2$ ,  $G \in \mathcal{H}_k$  is a  $(2k + 2)$ -supercritical graph relative to  $K_{2k+1, 2k+1}$  of diameter 5.

*Proof.* Let  $k \geq 2$  and  $G \in \mathcal{H}_k$ . Clearly  $\text{diam}(G) = 5$ . Let  $C_6^1, C_6^2, \dots, C_6^k$  be the  $k$  copies of  $C_6$ , and  $V(C_6^i) = \{a, b, a_i, b_i, c_i, d_i\}$ , and  $E(C_6^i) = \{ab, aa_i, a_i d_i, d_i c_i, c_i b_i, b_i b\}$  for  $i = 1, 2, \dots, k$ .

We first show that  $\gamma_t(G) = 2k + 2$ . Let  $S$  be a  $\gamma_t(G)$ -set. For each  $i$ , since  $d_i$  is totally dominated by  $S$ , we find that  $S \cap \{a_i, c_i\} \neq \emptyset$ , and since  $c_i$  is totally dominated by  $S$ , we find that  $S \cap \{d_i, b_i\} \neq \emptyset$ , and so  $|S \cap (V(C_6^i) - \{a, b\})| \geq 2$ . Thus  $|S| \geq 2k$ . We show that  $|S| = 2k + 2$ .

Suppose that  $|S| = 2k + 1$ . Then  $|S \cap \{a, b\}| \leq 1$ . We consider the following cases.

Case 1.  $|S \cap \{a, b\}| = 1$ . Then  $|S \cap (V(C_6^i) - \{a, b\})| = 2$  for any  $i \in \{1, 2, \dots, k\}$ . Without loss of generality assume that  $a \in S$  and  $b \notin S$ . Since  $a$  is totally dominated by  $S$ , we find that there is an integer  $j \in \{1, 2, \dots, k\}$  such that  $a_j \in S$ . If  $c_j \in S$ , then  $S \cap \{b_j, d_j\} \neq \emptyset$ , since  $c_j$  is totally dominated by  $S$ . Then  $|S \cap (V(C_6^j) - \{a, b\})| = 3 > 2$ , a contradiction. Thus  $c_j \notin S$ . Since  $b_j$  is totally dominated by  $S$ , there is an integer  $t < j$  such that  $a_t \in S$ . As before we find that  $c_t \notin S$ , and there is an integer  $l < t$  such that  $a_l \in S$ . By continuing this process, we obtain that  $a_1 \in S$ . Since  $b_1$  is totally

dominated by  $S$ , we find that  $c_1 \in S$ , and so  $S \cap \{b_1, d_1\} \neq \emptyset$ . Thus,  $|S \cap (V(C_6^1) - \{a, b\})| = 3 > 2$ , a contradiction.

Case 2.  $|S \cap \{a, b\}| = 0$ . Then there is an integer  $m \in \{1, 2, \dots, k\}$  such that  $|S \cap (V(C_6^m) - \{a, b\})| = 3$ , and

$$(2.1) \quad \text{for any } i \neq m, \quad |S \cap (V(C_6^i) - \{a, b\})| = 2.$$

Claim 1.  $m \notin \{1, k\}$ .

Proof of Claim 1. Assume that  $m = 1$ . Then  $|S \cap (V(C_6^1) - \{a, b\})| = 3$  and

$$(2.2) \quad \text{for any } i \in \{2, 3, \dots, k\}, \quad |S \cap (V(C_6^i) - \{a, b\})| = 2.$$

Since  $b_1$  is totally dominated by  $S$ ,  $c_1 \in S$ , and since  $c_1$  is totally dominated by  $S$ ,  $S \cap \{b_1, d_1\} \neq \emptyset$ . Since  $a$  is totally dominated by  $S$ , there is an integer  $i$  such that  $a_i \in S$ . If  $i \neq 1$ , then  $a_1 \notin S$ . Since  $c_i$  is totally dominated by  $S$ ,  $S \cap \{b_i, d_i\} \neq \emptyset$ , and since  $b_i$  is totally dominated by  $S$ ,  $a_t \in S$  for some integer  $t < i$ . By continuing this process as seen in Case 1, we obtain that  $a_1 \in S$ , a contradiction. Thus  $i = 1$ , and  $a_1 \in S$ .

Since  $b$  is dominated by  $S$ , there is an integer  $j$  such that  $b_j \in S$ . Assume that  $j \neq 1$ . Then  $b_1 \notin S$ . Since  $d_j$  is totally dominated by  $S$ ,  $S \cap \{a_j, c_j\} \neq \emptyset$ , and since  $a_j$  is totally dominated by  $S$ ,  $b_n \in S$ , for some  $n > j$ . By continuing this process we obtain that  $b_k \in S$ .

Since  $a_k$  is totally dominated by  $S$ ,  $d_k \in S$ , and since  $d_k$  is totally dominated by  $S$ ,  $S \cap \{a_k, c_k\} \neq \emptyset$ . Then  $|S \cap (V(C_6^k) - \{a, b\})| = 3 > 2$  contradicting (2.2). Thus  $j = 1$  and  $b_1 \in S$ .

Now  $a_1, b_1, c_1 \in S$ , and  $d_1 \notin S$ . Since  $a_1$  is totally dominated by  $S$ , there is an integer  $l$  such that  $b_l \in S$ , and since  $d_l$  is totally dominated by  $S$ ,  $S \cap \{a_l, c_l\} \neq \emptyset$ . Since  $a_l$  is totally dominated by  $S$ ,  $b_p \in S$  for some integer  $p$ , and by continuing this process we obtain that  $b_k \in S$ . Since  $a_k$  is totally dominated by  $S$ ,  $d_k \in S$ , and since  $d_k$  is totally dominated by  $S$ ,  $S \cap \{a_k, c_k\} \neq \emptyset$ . Thus  $|S \cap (V(C_6^k) - \{a, b\})| = 3 > 2$ , contradicting (2.2). Thus  $m \neq 1$ . The proof for  $m \neq k$  is similar.  $\square$

Since  $a$  is totally dominated by  $S$ , there is an integer  $j \in \{1, 2, \dots, k\}$  such that  $a_j \in S$ , and since  $b$  is totally dominated by  $S$ , there is an integer  $l \in \{1, 2, \dots, k\}$  such that  $b_l \in S$ . Since  $c_j$  is totally dominated by  $S$ ,  $S \cap \{d_j, b_j\} \neq \emptyset$ .

We show that  $j = m$ . If  $j \neq m$ , then by (2.1), for  $b_j$  to be totally dominated by  $S$ , there is an integer  $t$  such that  $t < j$  and  $a_t \in S$ . If  $t > 1$ , we do with  $b_t$  similarly to  $b_j$ , and thus we may assume that  $t = 1$ , and so  $a_1 \in S$ . Since  $b_1$  is totally dominated by  $S$ , we find that  $c_1 \in S$ , and since  $c_1$  is dominated by  $S$ ,  $S \cap \{b_1, d_1\} \neq \emptyset$ . Then  $|S \cap (V(C_6^1) - \{a, b\})| = 3 > 2$ , and so  $m = 1$ . But by Claim 1,  $m \notin \{1, m\}$ , a contradiction. Thus  $j = m$ .

Similarly,  $l = m$ . Thus  $j = l = m$ . Since  $b_m$  is totally dominated by  $S$ ,  $S \cap \{c_m, a_t\} \neq \emptyset$  for some  $t < m$ , and since  $a_m$  is totally dominated by  $S$ ,  $S \cap \{d_m, b_n\} \neq \emptyset$  for some  $n > m$ . Without loss of generality assume that  $c_m \in S$ . As before, we can see that  $a_1 \in S$ . Since  $b_1$  is totally dominated by

$S$  we find that  $c_1 \in S$ , and since  $c_1$  is totally dominated by  $S$  we find that  $S \cap \{b_1, d_1\} \neq \emptyset$ . Thus  $|S \cap (V(C_6^1) - \{a, b\})| = 3 > 2$ , a contradiction.

We conclude that  $|S| \geq 2k + 2$ . On the other hand  $\{a, b, a_i, b_i : i = 1, 2, \dots, k\}$  is a TDS for  $G$ , implying that  $\gamma_t(G) = 2k + 2$ .

Now we show that  $\gamma_t(G + e) = 2k$  for all  $e \in E(H)$ , where  $H$  is the complement of  $G$  relative to  $K_{2k+1, 2k+1}$ . Since  $\gamma_t(G) = 2k + 2$ , for any  $e \in E(H)$ , it is obvious that  $\gamma_t(G + e) \geq 2k$ . Thus for any  $e \in E(H)$ , it is sufficient to present a TDS for  $G + e$  of cardinality  $2k$ .

If  $e = ac_i$  for some  $i$ , then  $\{a, c_i, d_j, c_j : j = 1, 2, \dots, k, j \neq i\}$  is a TDS for  $G + e$ . If  $e = a_i b_i$  for some  $i$ , then  $\{a_i, b_i, d_j, c_j : j = 1, 2, \dots, k, j \neq i\}$  is a TDS for  $G + e$ . If  $e = b_i a_j$  for some  $i, j$  with  $i < j$ , then  $\{a_i, b_i, a_j, b_j, c_l, d_l : l = 1, 2, \dots, k, l \neq i, j\}$  is a TDS for  $G + e$ . If  $e = b_i c_l$  for some  $i, l$  with  $i \neq l$ , then  $\{b_i, c_l, a, a_i, c_j, d_j : j = 1, 2, \dots, k, j \neq i, l\}$  is a TDS for  $G + e$ . If  $e = b d_i$  for some  $i$ , then  $\{b, d_i, c_j, d_j : j = 1, 2, \dots, k, j \neq i\}$  is a TDS for  $G + e$ . If  $e = a_j d_i$  for some  $i, j$  with  $i \neq j$ , then  $\{a_j, d_i, b, b_j, c_l, d_l : l = 1, 2, \dots, k, l \neq i, j\}$  is a TDS for  $G + e$ . Finally if  $e = d_i c_l$  for some  $i, l$  with  $i \neq l$ , then  $\{a, b, d_i, c_l, c_j, d_j : j = 1, 2, \dots, k, j \neq i, l\}$  is a TDS for  $G + e$ . Note that the other possibilities for  $e$  are similarly verified.

□

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