A NOTE ON THE TOTAL DOMINATION SUPERCRITICAL GRAPHS

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Abstract. Let \( G \) be a connected spanning subgraph of \( K_{s,s} \) and let \( H \) be the complement of \( G \) relative to \( K_{s,s} \). The graph \( G \) is \( k \)-supercritical relative to \( K_{s,s} \) if \( \gamma_t(G) = k \) and \( \gamma_t(G + e) = k - 2 \) for all \( e \in E(H) \). The 2002 paper by T.W. Haynes, M.A. Henning and L.C. van der Merwe, “Total domination supercritical graphs with respect to relative complements” that appeared in Discrete Mathematics, 258 (2002), 361-371, presents a theorem (Theorem 11) to produce \( (2k + 2) \)-supercritical graphs relative to \( K_{2k+1,2k+1} \), for each \( k \geq 2 \). However, the families of graphs in their proof are not the case. We present a correction of this theorem.

1. Introduction

Let \( G = (V(G), E(G)) \) be a simple graph of order \( n \). We denote the open neighborhood of a vertex \( v \) of \( G \) by \( N_G(v) \), or just \( N(v) \), and its closed neighborhood by \( N_G[v] = N[v] \). For a vertex set \( S \subseteq V(G) \), \( N(S) = \bigcup_{v \in S} N(v) \) and \( N[S] = \bigcup_{v \in S} N[v] \). A set of vertices \( S \) in \( G \) is a total dominating set, (or just TDS), if \( N(S) = V(G) \). The total domination number, \( \gamma_t(G) \) of \( G \), is the minimum cardinality of a total dominating set of \( G \). For Graph Theory notation and terminology in general we follow [2].

Haynes, Henning and Van der Merwe in [3] studied total domination supercritical graphs with respect to relative complements. Let \( G \) be a connected spanning subgraph of \( K_{s,s} \) and let \( H \) be the complement of \( G \) relative to \( K_{s,s} \). The graph \( G \) is \( k \)-supercritical relative to \( K_{s,s} \) if \( \gamma_t(G) = k \) and \( \gamma_t(G + e) = k - 2 \) for all \( e \in E(H) \). They presented a construction to produce \( 6 \)-supercritical graphs of diameter 5.

For \( k \geq 2 \), let \( \mathcal{G}_k \) be the class of all graphs \( G \) such that \( G \in \mathcal{G}_k \) if and only if \( G \) is formed as follows. Form \( G \) from \( k \) copies of the cycle \( C_6 \) by identifying an edge, say \( ab \), common to every cycle. Let

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Consider the construction presented before Theorem 2.1. For $i \in \{1, \ldots, k-1\}$, let $b_i \in N(a_{i+1})$, and $b_k \in N(a_1)$. Then clearly $S = \{a_i, b_i : i = 1, 2, \ldots, k\}$ is a TDS for $G$, implying that $\gamma_t(G) = 2k$. Thus the above construction does not produce $(2k+2)$-supercritical graphs.

We will now give a corrected construction.

- For $k \geq 2$, let $\mathcal{H}_k$ be the class of all graphs $G$ such that $G \in \mathcal{H}_k$ if and only if $G$ is formed as follows. Form $G$ from $k$ copies of the cycle $C_6$ by identifying an edge, say $ab$, common to every cycle. Let $A = N(a) - \{b\}$ and $B = N(b) - \{a\}$, and label the vertices of $A$ and $B$ as $A = \{a_1, a_2, \ldots, a_k\}$ and $B = \{b_1, b_2, \ldots, b_k\}$ such that $a_i$ and $b_i$ are in the $i$th copy of $C_6$. Finally, for $i = 1, 2, \ldots, k-1$, join $a_i$ to every vertex in $\{b_{i+1}, b_{i+2}, \ldots, b_k\}$. Clearly, $G$ is a bipartite spanning subgraph of $K_{2k+1,2k+1}$ and $diam(G) = 5$.

**Theorem 2.1.** For each $k \geq 2$, $G \in \mathcal{H}_k$ is a $(2k+2)$-supercritical graph relative to $K_{2k+1,2k+1}$ of diameter 5.

**Proof.** Let $k \geq 2$ and $G \in \mathcal{H}_k$. Clearly $diam(G) = 5$. Let $C^1_6, C^2_6, \ldots, C^k_6$ be the $k$ copies of $C_6$, and $V(C^i_6) = \{a,b,a_i,b_i,c_i,d_i\}$, and $E(C^i_6) = \{ab,aa_i,a_id_i,d_ic_i,c_ib_i,bib\}$ for $i = 1, 2, \ldots, k$.

We first show that $\gamma_t(G) = 2k + 2$. Let $S$ be a $\gamma_t(G)$-set. For each $i$, since $d_i$ is totally dominated by $S$, we find that $S \cap \{a_i, c_i\} \neq \emptyset$, and since $c_i$ is totally dominated by $S$, we find that $S \cap \{d_i, b_i\} \neq \emptyset$, and so $|S \cap (V(C^i_6) - \{a, b\})| \geq 2$. Thus $|S| \geq 2k$. We show that $|S| = 2k + 2$.

Suppose that $|S| = 2k + 1$. Then $|S \cap \{a, b\}| \leq 1$. We consider the following cases.

Case 1. $|S \cap (a, b)| = 1$. Then $|S \cap (V(C^i_6) - \{a, b\})| = 2$ for any $i \in \{1, 2, \ldots, k\}$. Without loss of generality assume that $a \in S$ and $b \not\in S$. Since $a$ is totally dominated by $S$, we find that there is an integer $j \in \{1, 2, \ldots, k\}$ such that $a_j \in S$. If $c_j \in S$, then $S \cap \{b_j, d_j\} \neq \emptyset$, since $c_j$ is totally dominated by $S$. Then $|S \cap (V(C^j_6) - \{a, b\})| = 3 > 2$, a contradiction. Thus $c_j \not\in S$. Since $b_j$ is totally dominated by $S$, there is an integer $t < j$ such that $a_t \in S$. As before we find that $c_t \not\in S$, and there is an integer $l < t$ such that $a_l \in S$. By continuing this process, we obtain that $a_1 \in S$. Since $b_1$ is totally
dominated by $S$, we find that $c_1 \in S$, and so $S \cap \{b_1, d_1\} \neq \emptyset$. Thus, $|S \cap (V(C^i_6) - \{a,b\})| = 3 > 2$, a contradiction.

Case 2. $|S \cap \{a,b\}| = 0$. Then there is an integer $m \in \{1, 2, \ldots, k\}$ such that $|S \cap (V(C^m_6) - \{a,b\})| = 3$, and

$$\text{(2.1)}$$

for any $i \neq m$, $|S \cap (V(C^i_6) - \{a,b\})| = 2$.

Claim 1. $m \notin \{1,k\}$.

Proof of Claim 1. Assume that $m = 1$. Then $|S \cap (V(C^1_6) - \{a,b\})| = 3$ and

$$\text{(2.2)}$$

for any $i \in \{2,3,\ldots,k\}$, $|S \cap (V(C^i_6) - \{a,b\})| = 2$.

Since $b_1$ is totally dominated by $S$, $c_1 \in S$, and since $c_1$ is totally dominated by $S$, $S \cap \{b_1,d_1\} \neq \emptyset$. Since $a$ is totally dominated by $S$, there is an integer $i$ such that $a_i \in S$. If $i \neq 1$, then $a_1 \notin S$. Since $c_i$ is totally dominated by $S$, $S \cap \{b_i,d_i\} \neq \emptyset$, and since $b_i$ is totally dominated by $S$, $a_t \in S$ for some integer $t < i$. By continuing this process as seen in Case 1, we obtain that $a_1 \in S$, a contradiction. Thus $i = 1$, and $a_1 \in S$.

Since $b$ is dominated by $S$, there is an integer $j$ such that $b_j \in S$. Assume that $j \neq 1$. Then $b_1 \notin S$. Since $d_j$ is totally dominated by $S$, $S \cap \{a_j,c_j\} \neq \emptyset$, and since $a_j$ is totally dominated by $S$, $b_n \in S$, for some $n > j$. By continuing this process we obtain that $b_k \in S$.

Since $a_k$ is totally dominated by $S$, $d_k \in S$, and since $d_k$ is totally dominated by $S$, $S \cap \{a_k,c_k\} \neq \emptyset$. Then $|S \cap (V(C^k_6) - \{a,b\})| = 3 > 2$ contradicting (2.2). Thus $j = 1$ and $b_1 \in S$.

Now $a_1,b_1,c_1 \in S$, and $d_1 \notin S$. Since $a_1$ is totally dominated by $S$, there is an integer $l$ such that $b_l \in S$, and since $d_l$ is totally dominated by $S$, $S \cap \{a_l,c_l\} \neq \emptyset$. Since $a_l$ is totally dominated by $S$, $b_p \in S$ for some integer $p$, and by continuing this process we obtain that $b_k \in S$. Since $a_k$ is totally dominated by $S$, $d_k \in S$, and since $d_k$ is totally dominated by $S$, $S \cap \{a_k,c_k\} \neq \emptyset$. Thus $|S \cap (V(C^k_6) - \{a,b\})| = 3 > 2$, contradicting (2.2). Thus $m \neq 1$. The proof for $m \neq k$ is similar. \[ \square \]

Since $a$ is totally dominated by $S$, there is an integer $j \in \{1,2,\ldots,k\}$ such that $a_j \in S$, and since $b$ is totally dominated by $S$, there is an integer $l \in \{1,2,\ldots,k\}$ such that $b_l \in S$. Since $c_j$ is totally dominated by $S$, $S \cap \{d_j,b_j\} \neq \emptyset$.

We show that $j = m$. If $j \neq m$, then by (2.1), for $b_j$ to be totally dominated by $S$, there is an integer $t$ such that $t < j$ and $a_t \in S$. If $t > 1$, we do with $a_t$ similarly to $b_j$, and thus we may assume that $t = 1$, and so $a_1 \in S$. Since $b_1$ is totally dominated by $S$, we find that $c_1 \in S$, and since $c_1$ is dominated by $S$, $S \cap \{b_1,d_1\} \neq \emptyset$. Then $|S \cap (V(C^1_6) - \{a,b\})| = 3 > 2$, and so $m = 1$. But by Claim 1, $m \notin \{1,m\}$, a contradiction. Thus $j = m$.

Similarly, $l = m$. Thus $j = l = m$. Since $b_m$ is totally dominated by $S$, $S \cap \{c_m,a_t\} \neq \emptyset$ for some $t < m$, and since $a_m$ is totally dominated by $S$, $S \cap \{d_m,b_n\} \neq \emptyset$ for some $n > m$. Without loss of generality assume that $c_m \in S$. As before, we can see that $a_1 \in S$. Since $b_1$ is totally dominated by
$S$ we find that $c_1 \in S$, and since $c_1$ is totally dominated by $S$ we find that $S \cap \{b_1, d_1\} \neq \emptyset$. Thus $|S \cap (V(C^1_k) - \{a, b\})| = 3 > 2$, a contradiction.

We conclude that $|S| \geq 2k + 2$. On the other hand $\{a, b, a_i, b_i : i = 1, 2, \ldots, k\}$ is a TDS for $G$, implying that $\gamma_t(G) = 2k + 2$.

Now we show that $\gamma_t(G + e) = 2k$ for all $e \in E(H)$, where $H$ is the complement of $G$ relative to $K_{2k+1, 2k+1}$. Since $\gamma_t(G) = 2k + 2$, for any $e \in E(H)$, it is obvious that $\gamma_t(G + e) \geq 2k$. Thus for any $e \in E(H)$, it is sufficient to present a TDS for $G + e$ of cardinality $2k$.

If $e = ac$ for some $i$, then $\{a, c_i, d_j, c_j : j = 1, 2, \ldots, k, j \neq i\}$ is a TDS for $G + e$. If $e = ab_i$ for some $i$, then $\{a_i, b_i, d_j, c_j : j = 1, 2, \ldots, k, j \neq i\}$ is a TDS for $G + e$. If $e = b_i a_j$ for some $i, j$ with $i < j$, then $\{a_i, b_i, a_j, b_j, c_l, d_l : l = 1, 2, \ldots, k, l \neq i, j\}$ is a TDS for $G + e$. If $e = b_i c_j$ for some $i, l$ with $i \neq l$, then $\{b_i, c_l, a_i, c_j, d_j : j = 1, 2, \ldots, k, j \neq i, l\}$ is a TDS for $G + e$. If $e = bd_i$ for some $i$, then $\{b, d_i, c_j, d_j : j = 1, 2, \ldots, k, j \neq i\}$ is a TDS for $G + e$. If $e = ad_i$ for some $i, j$ with $i \neq j$, then $\{a_j, d_i, b_i, a_j, c_l, d_l : l = 1, 2, \ldots, k, l \neq i, j\}$ is a TDS for $G + e$. Finally if $e = d_i c_j$ for some $i, l$ with $i \neq l$, then $\{a, b, d_i, c_l, c_j, d_j : j = 1, 2, \ldots, k, j \neq i, l\}$ is a TDS for $G + e$. Note that the other possibilities for $e$ are similarly verified.

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References


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