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SUBGROUP INTERSECTION GRAPH OF FINITE ABELIAN GROUPS

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Abstract. Let G be a finite group with the identity e. The subgroup intersection graph $\Gamma_{SI}(G)$ of G is the graph with vertex set $V(\Gamma_{SI}(G)) = G - e$ and two distinct vertices x and y are adjacent in $\Gamma_{SI}(G)$ if and only if $|\langle x\rangle \cap \langle y\rangle| > 1$, where $\langle x\rangle$ is the cyclic subgroup of G generated by $x \in G$. In this paper, we obtain a lower bound for the independence number of subgroup intersection graph. We characterize certain classes of subgroup intersection graphs corresponding to finite abelian groups. Finally, we characterize groups whose automorphism group is the same as that of its subgroup intersection graph.

1. Introduction

The role of algebra and graph theory in the field of Discrete Mathematics has been rapidly increasing over several decades. The tools of each have been used in the other to explore and investigate problems in deep. Especially the Cayley graph constructed out of a finite group has been greatly and extensively used in parallel computers to provide networks to routing problems. There are many papers on assigning a graph to a group. For example, Cayley graph, non-commuting graph, power graph, etc.([7, 1, 4, 2]) are some of them to mention in this regard. Let G be a finite group. The power graph [4, 2] of G is the graph with vertex set G and two vertices x and y are adjacent if either $x = y^i$ or $y = x^j$ for some positive integers i and j. In [8], T. Tamizh Chelvam and M. Sattanathan introduced the subgroup intersection graph $\Gamma_{SI}(G)$ of a group G as follows: Given a finite group G, we associate the simple graph $\Gamma_{SI}(G)$ whose vertex set is $G - e$ and two distinct vertices x and y are adjacent in $\Gamma_{SI}(G)$ if and only if $|\langle x\rangle \cap \langle y\rangle| > 1$, where $\langle x\rangle$ is the cyclic subgroup of G generated by $x \in G$.

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Note that the power graph of G is a subgraph of the subgroup intersection graph $\Gamma_{SI}(G)$. Having introduced $\Gamma_{SI}(G)$, certain fundamental properties of this new graph are studied [8].

By a graph Γ, we mean an undirected graph with no loops and multiple edges. For any graph $\Gamma = (V, E)$, we denote the sets of the vertices by V and edges by E. A subset X of the vertices of Γ is called an *independent set* of Γ if the induced subgraph on X has no edges. The maximum size of an independent set in a graph Γ is called the *independence number* of Γ and denoted by $\beta_0(\Gamma)$. A *planar* graph is a graph that can be embedded in the plane so that no two edges intersect geometrically except at vertices. *Unicyclic* graphs are graphs which are connected and have just one cycle. Two graphs Γ_1 and Γ_2 are *isomorphic* (written $\Gamma_1 \cong \Gamma_2$) if there exists a one-to-one correspondence between their vertex sets which preserves adjacency. An isomorphism from a graph Γ to itself is called an *automorphism* of Γ.

Let G be a group with the identity element e. The number of elements of a group is called its *order* and it is denoted by $O(G)$. The order $O(g)$ of an element $g \in G$ is the smallest positive integer n such that $g^n = e$. If no such integer exists, we say g has an infinite order. We state the following theorems for the sake of use in the subsequent discussions.

Theorem 1.1. [3] *K_5 and $K_{3,3}$ are non-planar.*

Theorem 1.2. [8] *Let G be a finite group. Then $\Gamma_{SI}(G)$ is a complete graph if and only if G has a unique subgroup of order p and $O(G) = p^m$ for some prime number p and positive integer m.*

Theorem 1.3. [8] *Let G be a finite abelian group of order p^n for some prime number p. Then $\Gamma_{SI}(G)$ is a finite union of complete graphs.*

Now we have the following corollary.

Corollary 1.4. *Let G be an elementary abelian group of order p^n. Then $\Gamma_{SI}(G) \cong \bigcup_m K_{p-1}$ where $m = \frac{p^n-1}{p-1}$.*

In this paper, we obtain a lower bound for the independence number of the subgroup intersection graph. We characterize certain subgroup intersection graphs corresponding to finite abelian groups. Finally, we characterize groups whose automorphism group is the same as that of its subgroup intersection graph.

2. Independence number of Γ_{SI}

In this section, we obtain the independence number of $\Gamma_{SI}(G)$ corresponding to a finite group G.

Theorem 2.1. *Let G be a finite group of order $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$, where $p_1, p_2, \dots p_k$ are distinct primes and $\alpha_1, \alpha_2, \dots, \alpha_k$ are positive integers. Then the independence number $\beta_0(\Gamma_{SI}(G)) \geq k$.*

Proof. Since each p_i divides $O(G)$, there exist $a_i \in G$ such that $O(a_i) = p_i$, for $1 \leq i \leq k$. Note that $\langle a_i \rangle \cap \langle a_j \rangle = \{e\}$ for all $i \neq j$. From this $\{a_1, a_2, \dots, a_n\}$ is an independent set of $\Gamma_{SI}(G)$ and hence the result follows. $\square$

The following lemma is trivial.

Lemma 2.2. *Let G be a finite group of order $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$, where $p_1, p_2, \dots, p_k$ are distinct primes, $\alpha_1, \alpha_2, \dots, \alpha_k$ are positive integers. Assume that G has unique subgroups of orders $p_1, p_2, \dots, p_k$. Two non identity elements $a, b \in G$ are non-adjacent in $\Gamma_{SI}(G)$ if and only if $gcd(O(a), O(b)) = 1$.*

Theorem 2.3. *Let G be a finite group of order $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$, where $p_1, p_2, \dots p_k$ are distinct primes and $\alpha_1, \alpha_2, \dots, \alpha_k$ are positive integers. The independence number $\beta_0(\Gamma_{SI}(G)) = k$ if and only if G has the unique subgroup of order p_i for every $i = 1, 2, \dots, k$.*

Proof. Assume that G has the unique subgroup of order p_i for $i = 1, 2, \dots, k$. If $\beta_0(\Gamma_{SI}(G)) > k$, then G has an independent set A with at least $k+1$ elements. By Lemma 2.2, the orders of elements in A are pairwise relatively prime. Since $O(G)$ has exactly k distinct prime divisors, we cannot find $k+1$ elements in G whose orders are pairwise relatively prime. Therefore $\beta_0(\Gamma_{SI}(G)) \leq k$, whereas by Theorem 2.1, $\beta_0(\Gamma_{SI}(G)) \geq k$. Hence $\beta_0(\Gamma_{SI}(G)) = k$.

Conversely, assume that $\beta_0(\Gamma_{SI}(G)) = k$. Let a_i be an element in G such that $O(a_i) = p_i$ for $i = 1, 2, \dots, k$. Suppose G has two distinct subgroups of order p_i for some i say H_i and K_i. Then $H_i = \langle a_i \rangle$ and $K_i = \langle b_i \rangle$, for some $a_i, b_i \in G$. Since $H_i \cap K_i = \{e\}$, the set $\{a_1, a_2, \dots, a_k, b_i\}$ is an independent set in $\Gamma_{SI}(G)$ with $k+1$ elements, which is a contradiction. Hence G has the unique subgroup of order p_i for $i = 1, 2, \dots, k$. $\square$

3. Subgroup intersection graph of finite abelian groups

In this section, we study about the subgroup intersection graph of finite abelian groups. We characterize certain subgroup intersection graphs corresponding to finite abelian groups. Finally, we characterize groups whose automorphism group is the same as that of its subgroup intersection graph.

Theorem 3.1. *Let G be a finite cyclic group. Then $\Gamma_{SI}(G)$ is planar if and only if $O(G) \leq 6$.*

Proof. Assume that $\Gamma_{SI}(G)$ is planar. Suppose $p|O(G)$ for some prime $p \geq 7$. Clearly G has an element a of order p. By Theorem 1.2, the subgraph induced by $\langle a \rangle$ is K_{p-1}. Since $p \geq 7$, $\Gamma_{SI}(G)$ is non-planar, a contradiction. Therefore $O(G) = 2^{n_1} 3^{n_2} 5^{n_3}$, where $n_1 \geq 0$, $n_2 \geq 0$, $n_3 \geq 0$ are integers. Suppose there exists $r \in \{10, 12, 15, 18\}$ such that $r|O(G)$. Since G is cyclic, G has an element x of order r and the subgraph induced by $\langle x \rangle$ must contain K_5 and so $\Gamma_{SI}(G)$ is non-planar, a contradiction. This gives that $O(G)$ must be either 6 or 2^n or 3^n or 5^n for some $n \geq 1$. Suppose $O(G)$ is either 3^n or 5^n for some $n \geq 2$. Then either 9 or 25 must divide $O(G)$ and G has an element x of order either 9 or 25 respectively. By Theorem 1.2, the subgraph induced by $\langle x \rangle$ is either K_8 or K_{24}, which are non-planar, a contradiction. Suppose $O(G) = 2^n$ for some $n \geq 3$. In this case G contains an element x of order 8 and the subgraph induced by $\langle x \rangle$ is K_7, which is non-planar, a contradiction. Hence $O(G) \leq 6$.

Conversely, assume that $O(G) \leq 6$. Note that, for $O(G) \leq 5$, $\Gamma_{SI}(G) \cong K_{O(G)-1}$ is planar. One can see from the Figure 2.1 that $\Gamma_{SI}(\mathbb{Z}_6)$ is also planar.

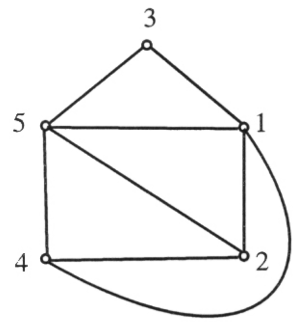

Figure 2.1: $\Gamma_{SI}(\mathbb{Z}_6)$

□

Lemma 3.2. *Let G be an abelian group of order either 12 or 18. Then $\Gamma_{SI}(G)$ is non-planar.*

Proof. By Theorem 3.1, it is enough to consider the case that G is non-cyclic. Let G be a non-cyclic abelian group of order 12. Then $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$. Note that G has the unique subgroup of order 3 and G contains 6 elements of order 6. From this, K_5 is a subgraph of $\Gamma_{SI}(G)$ and so $\Gamma_{SI}(G)$ is non-planar.

Let G be a non-cyclic abelian group of order 18 and so $G \cong \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3$. Note that G has the unique subgroup of order 2 and G contains 8 elements of order 6. From this, K_5 is a subgraph of $\Gamma_{SI}(G)$ and hence $\Gamma_{SI}(G)$ is non-planar. □

Lemma 3.3. *Let G be an abelian group of order 8. Then $\Gamma_{SI}(G)$ is planar if and only if G is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.*

Proof. Assume that $\Gamma_{SI}(G)$ is planar. By Theorem 3.1, G is non-cyclic and so $G \cong \mathbb{Z}_2 \times \mathbb{Z}_4$ or $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Note that $\Gamma_{SI}(\mathbb{Z}_2 \times \mathbb{Z}_4)$ is $K_5 \cup 2K_2$ and so $\Gamma_{SI}(\mathbb{Z}_2 \times \mathbb{Z}_4)$ is non-planar. Hence G is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Converse is trivially true. □

Theorem 3.4. *Let G be a finite abelian group. Then $\Gamma_{SI}(G)$ is planar if and only if G is isomorphic to $\mathbb{Z}_4$ or $\mathbb{Z}_6$ or $\mathbb{Z}_5 \times \mathbb{Z}_5 \times \cdots \times \mathbb{Z}_5$ or $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \cdots \times \mathbb{Z}_3$ or $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$.*

Proof. Assume that $\Gamma_{SI}(G)$ is planar. Suppose $p|O(G)$ for some prime $p \geq 7$. Then $K_m (m \geq 6)$ is a subgraph of $\Gamma_{SI}(G)$ and hence $\Gamma_{SI}(G)$ is non-planar, which is a contradiction. Therefore $O(G) = 2^{n_1}3^{n_2}5^{n_3}$, where $n_1 \geq 0$, $n_2 \geq 0$, $n_3 \geq 0$ are integers. Suppose $O(G) = 2^{n_1}3^{n_2}$ for two integers $n_1 \geq 2$ and $n_2 \geq 1$. Since G is abelian and $12|O(G)$, G must have a subgroup of order 12. By Lemma 3.2, $\Gamma_{SI}(G)$ is non-planar, a contradiction. Similarly we get contradiction in the following cases.

(i) $n_1 \geq 1$ and $n_2 \geq 2$

(ii) $n_1 \geq 1$ and $n_3 \geq 1$

(iii) $n_3 \geq 1$ and $n_2 \geq 1$

In view of these observations, $O(G)$ is either 6 or 2^n or 3^n or 5^n for some integer $n \geq 1$.

Suppose G is not an elementary abelian group of order p^n, where $p = 3, 5$. Since G is abelian, $\mathbb{Z}_{p^2}$ must be a subgroup of G and hence K_{p^2-1} is a subgraph of $\Gamma_{SI}(G)$, a contradiction. Suppose $O(G) = 2^n$, for some $n \in \mathbb{Z}^+$ and G is not isomorphic to either $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$ or $\mathbb{Z}_4$. Then $\mathbb{Z}_2 \times \mathbb{Z}_4$ must be a subgroup of G and by Lemma 3.3, $\Gamma_{SI}(G)$ is non-planar, a contradiction. Hence G is isomorphic to $\mathbb{Z}_4$ or $\mathbb{Z}_6$ or $\mathbb{Z}_5 \times \mathbb{Z}_5 \times \cdots \times \mathbb{Z}_5$ or $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \cdots \times \mathbb{Z}_3$ or $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$.

Converse part is true from Corollary 1.4 and Theorem 3.1. □

Theorem 3.5. *Let G be a finite abelian group. $\Gamma_{SI}(G)$ is a unicyclic graph if and only if $G \cong \mathbb{Z}_4$.*

Proof. Let $G \cong \mathbb{Z}_4$. By Theorem 1.2, $\Gamma_{SI}(G) \cong K_3$ and hence $\Gamma_{SI}(G)$ is a unicyclic graph.

Conversely, assume that $\Gamma_{SI}(G)$ is a unicyclic graph. Suppose there exists a prime $p > 3$ such that $p|O(G)$. Then G has an element of order p and so K_{p-1} is a subgraph of $\Gamma_{SI}(G)$, which is a contradiction. This implies that $O(G) = 2^n 3^m$ for some non negative integers n and m. Assume that $n \geq 1$ and $m \geq 1$. Since G is abelian, G has an element of order 6 and so K_4 is a subgraph of $\Gamma_{SI}(G)$, which is a contradiction. Therefore $O(G)$ is either 2^n or 3^n for some positive integer n.

Suppose G is an elementary abelian group of order 3^n for some $n > 0$. By Corollary 1.4, $\Gamma_{SI}(G) \cong \underset{\ell}{\cup} K_2$ where $\ell = \frac{3^n-1}{2}$ and so $\Gamma_{SI}(G)$ is not unicyclic. On the other hand for some $n > 1$, $\mathbb{Z}_{3^n}$ is a subgroup of G. This implies that K_{3^n-1} is a subgraph of $\Gamma_{SI}(G)$, which is a contradiction. Therefore $O(G) = 2^n$.

Suppose G is an elementary abelian group of order 2^n. Then $\Gamma_{SI}(G)$ is totally disconnected, which is a contradiction.

Suppose for some $n > 2$, $\mathbb{Z}_{2^n}$ is a subgroup of G. Then K_{2^n-1} is a subgraph of $\Gamma_{SI}(G)$, which is a contradiction.

Suppose for $n > 2$, $\mathbb{Z}_4 \times \mathbb{Z}_2$ is a subgroup of G. Now the subgraph induced by $\mathbb{Z}_4 \times \mathbb{Z}_2$ contains more than two cycles, which is a contradiction. Hence $O(G) = 4$ and $G \cong \mathbb{Z}_4$. □

Since any group isomorphism between two groups G_1 and G_2 is a graph isomorphism between the corresponding $\Gamma_{SI}(G_1)$ and $\Gamma_{SI}(G_2)$, we have the following.

Lemma 3.6. *Let G_1 and G_2 be two groups. If $G_1 \cong G_2$, then $\Gamma_{SI}(G_1) \cong \Gamma_{SI}(G_2)$.*

Corollary 3.7. *Let G be a finite group. Then $Aut(G) \subseteq Aut(\Gamma_{SI}(G))$.*

Remark 3.8. The converse of Lemma 3.6 is not true. Consider the group $(\mathbb{Z}_8, +_8)$ and quaternion group Q_8 of order 8. Note that $\mathbb{Z}_8 \ncong Q_8$, whereas $\Gamma_{SI}(\mathbb{Z}_8) \cong K_7 \cong \Gamma_{SI}(Q_8)$.

In view of the above observation, we now characterize groups G for which $Aut(G) = Aut(\Gamma_{SI}(G))$. For an integer $n \geq 1$, S_n is the symmetric group of degree n and U_n is the multiplicative group of units in the ring $\mathbb{Z}_n$.

Theorem 3.9. *For a finite group G, $Aut(G) = Aut(\Gamma_{SI}(G))$ if and only if G is either $\mathbb{Z}_2$ or $\mathbb{Z}_3$ or Klein's 4-group.*

Proof. Observe that $\Gamma_{SI}(\mathbb{Z}_2) = K_1$, $\Gamma_{SI}(\mathbb{Z}_3) = K_2$ and $\Gamma_{SI}(\mathbb{Z}_2 \times \mathbb{Z}_2) = \overline{K_3}$. From these observations, we have $Aut(\Gamma_{SI}(\mathbb{Z}_2)) = S_1$, $Aut(\Gamma_{SI}(\mathbb{Z}_3)) = S_2$ and $Aut(\Gamma_{SI}(\mathbb{Z}_2 \times \mathbb{Z}_2)) = S_3$. On the other hand $Aut(\mathbb{Z}_2) = S_1$, $Aut(\mathbb{Z}_3) = S_2$ and $Aut(\mathbb{Z}_2 \times \mathbb{Z}_2) = S_3$. Hence $Aut(G) = Aut(\Gamma_{SI}(G))$ where G is either $\mathbb{Z}_2$ or $\mathbb{Z}_3$ or Klein's 4-group.

Conversely assume that $Aut(G) = Aut(\Gamma_{SI}(G))$. Since the map $x \longmapsto x^{-1}$ is a graph automorphism, it is also a group automorphism and so G is abelian. Suppose G has two elements x and y such that $o(x) = o(y) \geq 3$ and $\langle x \rangle \cap \langle y \rangle = \{e\}$. Since the element xy is a non self inverse element, there is a graph automorphism fixing x and y and mapping xy to its inverse but no group automorphism can do this. This proves that G is either cyclic or an elementary abelian group of order 2^n.

Case(i). G is cyclic.

Suppose G has two elements x and y such that $o(x) = p_1$ and $o(y) = p_2$ for two distinct primes p_1 and p_2. Since $o(xy) = p_1p_2$, the element xy is a non self inverse element of G. As observed above, there is a graph automorphism fixing x and y and mapping xy to its inverse but no group automorphism can do this. Hence G must be a cyclic group of order p^n, for some prime p and a positive integer $n \geq 1$. From this we have a graph automorphism mapping an element of order p^n to an element of order p, which is impossible for a group automorphism. Therefore G is a cyclic group of order p for some prime p and in such a case $\Gamma_{SI}(G) = K_{p-1}$ and $Aut(\Gamma_{SI}(G)) = S_{p-1}$. But $Aut(G) = U_p$ and $|U_p| = p - 1$. By the assumption that $Aut(G) = Aut(\Gamma_{SI}(G))$, we get that $(p - 1)! = p - 1$ which is possible only when $p = 2$ or $p = 3$. Hence G is either $\mathbb{Z}_2$ or $\mathbb{Z}_3$.

Case(ii). G is an elementary abelian group of order 2^n.

In this case $\Gamma_{SI}(G) = \overline{K_{2^n-1}}$ and hence $Aut(\Gamma_{SI}(G)) = S_{2^n-1}$, whereas $Aut(G)$ is a general linear group $GL(n, 2)$ and these two groups are isomorphic if and only if $n = 2$. Hence G is the Klein's 4-group.

$\square$

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