www.combinatorics.ir

Transactions on Combinatorics
ISSN (print): 2251-8657, ISSN (on-line): 2251-8665
Vol. 1 No. 3 (2012), pp. 39-46.

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THE COMMON MINIMAL DOMINATING SIGNED GRAPH

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Communicated by Alireza Abdollahi

Abstract. In this paper, we define the common minimal dominating signed graph of a given signed graph and offer a structural characterization of common minimal dominating signed graphs. In the sequel, we also obtained switching equivalence characterizations: $\overline{S} \sim CMD(S)$ and $CMD(S) \sim N(S)$, where $\overline{S}$, $CMD(S)$ and $N(S)$ are complementary signed graph, common minimal signed graph and neighborhood signed graph of S respectively.

1. Introduction

For standard terminology and notion in graph theory we refer the reader to Harary [8]; the non-standard will be given in this paper when required. We treat only finite simple graphs without self loops and isolates.

Within the rapid growth of the Internet and the Web, and in the ease with which global communication now takes place, connectedness took an important place in modern society. Global phenomena, involving social networks, incencitives and the behavior of people based on the links that connect us appear in a regular manner. Motivated by these developments, there is a growing multidisciplinary interest to understand how highly connected systems operate [7].

In social sciences we often deal with relations of opposite content, e.g., "love"- "hatred", "likes"- "dislikes", "tells truth to"-"lies to" etc. In common use opposite relations are termed positive and

MSC(2010): Primary: 05C15;

Keywords: Signed graphs, Balance, Switching, Complement, Common minimal dominating signed graph, Neighborhood signed graph, Negation.

Received: 6 September 2012, Accepted: 12 October 2012.

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negative relations. A signed graph is one in which relations between entities may be of various types in contrast to an unsigned graph where all relations are of the same type. In signed graphs edge-coloring provides an elegant and uniform representation of the various types of relations where every type of relation is represented by a distinct color.

In the case where precisely one relation and its opposite are under consideration, then instead of two colors, the signs $+$ and $-$ are assigned to the edges of the corresponding graph in order to distinguish a relation from its opposite. In the case where precisely one relation and its opposite are under consideration, then instead of two colors, the signs + and - are assigned to the edges of the corresponding graph in order to distinguish a relation from its opposite. Formally, a signed graph $S = (G, \sigma)$ is a graph $G = (V, E)$ together with a function $\sigma : E \rightarrow \{+, -\}$, which associates each edge with the sign $+$ or $-$. In such a signed graph, a subset A of $E(G)$ is said to be positive if it contains an even number of negative edges, otherwise is said to be negative. A signed graph S is balanced if each cycle of S is positive. Otherwise it is unbalanced.

The theory of balance goes back to Heider [11] who asserted that a social system is balanced if there is no tension and that unbalanced social structures exhibit a tension resulting in a tendency to change in the direction of balance. Since this first work of Heider, the notion of balance has been extensively studied by many mathematicians and psychologists.

In 1956, Cartwright and Harary [4] provided a mathematical model for balance through graphs. Their cornerstone result states that a signed graph is balanced if and only if in each cycle the number of negative edges is even. The following theorem of Harary gives an equivalent definition of a balanced signed graph.

Proposition 1.1. (**Harary** [9])
A signed graph is balanced if and only if its vertex set can be partitioned into two classes (one of the two classes may be empty) so that every edge joining vertices within a class is positive and every edge joining vertices between classes is negative.

Since then, mathematicians have written numerous papers on the topic of signed graphs. Many of these papers demonstrate the connection between signed graphs and different subjects: circuit design (Barahona [2]), coding theory (Solé and Zaslavsky [29]), physics (Toulouse [31]) and social psychology (Abelson and Rosenberg [1]). While these subjects seem unrelated, balance plays an important role in each of these fields.

Four years after Harary's paper, Abelson and Rosenberg [1], wrote a paper in which they discuss algebraic methods to detect balance in a signed graphs. It was one of the first papers to propose a measure of imbalance, the "complexity" (which Harary called the "line index of balance"). Abelson and Rosenberg introduced an operation that changes a signed graph while preserving balance and they proved that this does not change the line index of imbalance. For more new notions on signed graphs refer the papers ([15, 16, 19, 20], [22]-[28]).

A *marking* of S is a function $\mu : V(G) \to \{+, -\}$; A signed graph S together with a marking μ is denoted by S_μ. Given a signed graph S one can easily define a marking μ of S as follows: For any vertex $v \in V(S)$,

$$\mu(v) = \prod_{uv \in E(S)} \sigma(uv),$$

the marking μ of S is called *canonical marking* of S. In a signed graph $S = (G, \sigma)$, for any $A \subseteq E(G)$ the *sign* $\sigma(A)$ is the product of the signs on the edges of A.

The following characterization of balanced signed graphs is well known.

Proposition 1.2. (**E. Sampathkumar** [18]) *A signed graph $S = (G, \sigma)$ is balanced if, and only if, there exists a marking μ of its vertices such that each edge uv in S satisfies $\sigma(uv) = \mu(u)\mu(v)$.*

Let $S = (G, \sigma)$ be a signed graph. *Complement* of S is a signed graph $\overline{S} = (\overline{G}, \sigma')$, where for any edge $e = uv \in \overline{G}$, $\sigma'(uv) = \mu(u)\mu(v)$. Clearly, $\overline{S}$ as defined here is a balanced signed graph due to Proposition 1.2.

The idea of switching a signed graph was introduced in [1] in connection with structural analysis of social behavior and also its deeper mathematical aspects, significance and connections may be found in [34].

Switching S with respect to a marking μ is the operation of changing the sign of every edge of S to its opposite whenever its end vertices are of opposite signs. The signed graph obtained in this way is denoted by $\mathcal{S}_\mu(S)$ and is called *μ-switched signed graph* or just *switched signed graph*. Two signed graphs $S_1 = (G, \sigma)$ and $S_2 = (G', \sigma')$ are said to be *isomorphic*, written as $S_1 \cong S_2$ if there exists a graph isomorphism $f : G \to G'$ (that is a bijection $f : V(G) \to V(G')$ such that if uv is an edge in G then $f(u)f(v)$ is an edge in G') such that for any edge $e \in E(G)$, $\sigma(e) = \sigma'(f(e))$. Further a signed graph $S_1 = (G, \sigma)$ *switches* to a signed graph $S_2 = (G', \sigma')$ (or that S_1 and S_2 are *switching equivalent*) written $S_1 \sim S_2$, whenever there exists a marking μ of S_1 such that $\mathcal{S}_\mu(S_1) \cong S_2$. Note that $S_1 \sim S_2$ implies that $G \cong G'$, since the definition of switching does not involve change of adjacencies in the underlying graphs of the respective signed graphs.

Two signed graphs $S_1 = (G, \sigma)$ and $S_2 = (G', \sigma')$ are said to be *weakly isomorphic* (see [30]) or *cycle isomorphic* (see [33]) if there exists an isomorphism $\phi : G \to G'$ such that the sign of every cycle Z in S_1 equals to the sign of $\phi(Z)$ in S_2. The following result is well known (See [33]):

Proposition 1.3. (**T. Zaslavsky** [33]) *Two signed graphs S_1 and S_2 with the same underlying graph are switching equivalent if, and only if, they are cycle isomorphic.*

In [21], the authors also introduced the switching and cycle isomorphism for signed digraphs.

2. Common Minimal Dominating Signed Graph

Mathematical study of domination in graphs began around 1960, there are some references to domination-related problems about 100 years prior. In 1862, de Jaenisch [6] attempted to determine the minimum number of queens required to cover an $n \times n$ chess board. In 1892, W. W. Rouse Ball [17] reported three basic types of problems that chess players studied during that time.

The study of domination in graphs was further developed in the late 1950s and 1960s, beginning with Berge [3] in 1958. Berge wrote a book on graph theory, in which he introduced the "coefficient of external stability", which is now known as the domination number of a graph. Oystein Ore [14] introduced the terms "dominating set" and "domination number" in his book on graph theory which was published in 1962. The problems described above were studied in more detail around 1964 by brothers Yaglom and Yaglom [32]. Their studies resulted in solutions to some of these problems for rooks, knights, kings, and bishops. A decade later, Cockayne and Hedetniemi [5] published a survey paper, in which the notation $\gamma(G)$ was first used for the domination number of a graph G. Since this paper was published, domination in graphs has been studied extensively and several additional research papers have been published on this topic.

Let $G = (V, E)$ be a graph. A set $D \subseteq V$ is a dominating set of G, if every vertex in $V - D$ is adjacent to some vertex in D. A dominating set D of G is minimal, if for any vertex $v \in D$, $D - \{v\}$ is not a dominating set of G (See, Ore [14]).

Kulli and Janakiram [12] introduced a new class of intersection graphs in the field of domination theory. The common minimal dominating graph $CMD(G)$ of a graph G is the graph having same vertex set as G with two vertices adjacent in $CMD(G)$ if, and only if, there exists a minimal dominating set in G containing them.

In this paper, we introduce a natural extension of the notion of common minimal dominating graph to the realm of signed graphs since this appears to have interesting connections with complementary signed graph and neighborhood signed graph.

The *common minimal dominating signed graph* $CMD(S)$ of a signed graph $S = (G, \sigma)$ is a signed graph whose underlying graph is $CMD(G)$ and sign of any edge uv in $CMD(S)$ is $\mu(u)\mu(v)$, where μ is the canonical marking of S. Further, a signed graph $S = (G, \sigma)$ is called common minimal dominating signed graph, if $S \cong CMD(S')$ for some signed graph S'. In this paper we will give a structural characterization of which signed graphs are common minimal dominating signed graph.

The following result indicates the limitations of the notion $CMD(S)$ introduced above, since the entire class of unbalanced signed graphs is forbidden to be common minimal dominating signed graphs.

Proposition 2.1. *For any signed graph $S = (G, \sigma)$, its common minimal dominating signed graph $CMD(S)$ is balanced.*

Proof. Since sign of any edge uv in $CMD(S)$ is $\mu(u)\mu(v)$, where μ is the canonical marking of S, by Proposition 1.2, $CMD(S)$ is balanced. □

For any positive integer k, the k^{th} iterated common minimal dominating signed graph $CMD(S)$ of S is defined as follows:

$$CMD^0(S) = S, \, CMD^k(S) = CMD(CMD^{k-1}(S))$$

Corollary 2.2. *For any signed graph $S = (G, \sigma)$ and any positive integer k, $CMD^k(S)$ is balanced.*

In [12], the authors characterized graphs for which $CMD(G) \cong \overline{G}$.

Proposition 2.3. (**Kulli and Janakiram** [12])
For any graph $G = (V, E)$, $CMD(G) \cong \overline{G}$ if, and only if, every minimal dominating set of G is independent.

We now characterize signed graphs whose common minimal dominating signed graphs and complementary signed graphs are switching equivalent.

Proposition 2.4. *For any signed graph $S = (G, \sigma)$, $\overline{S} \sim CMD(S)$ if, and only if, every minimal dominating set of G is independent.*

Proof. Suppose $\overline{S} \sim CMD(S)$. This implies, $\overline{S} \cong CMD(S)$ and hence by Proposition 2.3, every minimal dominating set of G is independent.

Conversely, suppose that every minimal dominating set of G is independent. Then $\overline{S} \cong CMD(S)$ by Proposition 2.3. Now, if S is a signed graph with every minimal dominating set of underlying graph G is independent, by the definition of complementary signed graph and Proposition 2.1, $\overline{S}$ and $CMD(S)$ are balanced and hence, the result follows from Proposition 1.3. □

In [16], Rangarajan et al. introduced neighborhood signed graph of a signed graph as follows:

The *neighborhood signed graph* $N(S)$ of a signed graph $S = (G, \sigma)$ is a signed graph whose underlying graph is $N(G)$ and sign of any edge uv is $N(S)$ is $\mu(u)\mu(v)$, where μ is the canonical marking of S. Further, a signed graph $S = (G, \sigma)$ is called neighborhood signed graph, if $S \cong N(S')$ for some signed graph S'. The following result restricts the class of neighborhood graphs.

Proposition 2.5. (**Rangarajan et al.** [16])
For any signed graph $S = (G, \sigma)$, its neighborhood signed graph $N(S)$ is balanced.

We now characterize signed graphs whose common minimal dominating signed graphs and neighborhood signed graphs are switching equivalent. In case of graphs the following result is due to Kulli and Janakiram [13].

Proposition 2.6. (**Kulli and Janakiram** [13])
If G is a $(p-2)$-regular graph with $p \geq 6$, then $CMD(G) \cong N(G)$.

Proposition 2.7. *For any signed graph $S = (G, \sigma)$, $CMD(S) \sim N(S)$ if, and only if, G is a $(p-2)$-regular graph with $p \geq 6$.*

Proof. Suppose $CMD(S) \sim N(S)$. This implies, $CMD(G) \cong N(G)$ and hence by Proposition 2.6, we see that the graph G must be $(p-2)$-regular graph with $p \geq 6$.

Conversely, suppose that G is $(p-2)$-regular graph with $p \geq 6$. Then $CMD(G) \cong N(G)$ by Proposition 2.6. Now, if S is a signed graph with underlying graph as $(p-2)$-regular graph with $p \geq 6$, by Propositions 2.1 and 2.5, $CMD(S)$ and $N(S)$ are balanced and hence, the result follows from Proposition 1.3. □

The notion of *negation* $\eta(S)$ of a given signed graph S defined in [10] as follows: $\eta(S)$ has the same underlying graph as that of S with the sign of each edge opposite to that given to it in S. However, this definition does not say anything about what to do with nonadjacent pairs of vertices in S while applying the unary operator $\eta(.)$ of taking the negation of S.

Proposition 2.4 & 2.7 provides easy solutions to other signed graph switching equivalence relations, which are given in the following results.

Corollary 2.8. *For any signed graph $S = (G, \sigma)$, $\overline{\eta(S)} \sim CMD(S)$
(or $\overline{S} \sim CMD(\eta(S))$) if, and only if, every minimal dominating set of G is independent.*

Corollary 2.9. *For any signed graph $S = (G, \sigma)$, $\overline{\eta(S)} \sim CMD(\eta(S))$ if, and only if, every minimal dominating set of G is independent.*

Corollary 2.10. *For any signed graph $S = (G, \sigma)$, $CMD(S) \sim N(\eta(S))$
(or $CMD(\eta(S)) \sim N(S)$) if, and only if, G is a $(p-2)$-regular graph with $p \geq 6$.*

Corollary 2.11. *For any signed graph $S = (G, \sigma)$, $CMD(\eta(S)) \sim N(\eta(S))$ if, and only if, G is a $(p-2)$-regular graph with $p \geq 6$.*

For a signed graph $S = (G, \sigma)$, the $CMD(S)$ is balanced (Proposition 2.1). We now examine, the conditions under which negation of $CMD(S)$ is balanced.

Proposition 2.12. *Let $S = (G, \sigma)$ be a signed graph. If $CMD(G)$ is bipartite then $\eta(CMD(S))$ is balanced.*

Proof. Since, by Proposition 2.1, $CMD(S)$ is balanced, each cycle C in $CMD(S)$ contains even number of negative edges. Also, since $CMD(G)$ is bipartite, all cycles have even length; thus, the number of positive edges on any cycle C in $CMD(S)$ is also even. Hence $\eta(CMD(S))$ is balanced. □

3. Characterization of Common Minimal Dominating Signed Graphs $CMD(S)$

The following result characterize signed graphs which are common minimal dominating signed graphs.

Proposition 3.1. *A signed graph $S = (G, \sigma)$ is a common minimal dominating signed graph if, and only if, S is balanced signed graph and its underlying graph G is a $CMD(G)$.*

Proof. Suppose that S is balanced and its underlying graph G is a common minimal dominating graph. Then there exists a graph H such that $CMD(H) \cong G$. Since S is balanced, by Proposition 1.2, there exists a marking μ of G such that each edge uv in S satisfies $\sigma(uv) = \mu(u)\mu(v)$. Now consider the signed graph $S' = (H, \sigma')$, where for any edge e in H, $\sigma'(e)$ is the marking of the corresponding vertex in G. Then clearly, $CMD(S') \cong S$. Hence S is a common minimal dominating signed graph.

Conversely, suppose that $S = (G, \sigma)$ is a common minimal dominating signed graph. Then there exists a signed graph $S' = (H, \sigma')$ such that $CMD(S') \cong S$. Hence by Proposition 2.1, S is balanced. □

Problem 3.2. Characterize signed graphs for which
i). $\overline{S} \cong CMD(S)$
ii). $CMD(S) \cong N(S)$.

Acknowledgments

The authors would like to thank referee for his valuable comments. Also, the authors are grateful to Sri. B. Premnath Reddy, Chairman, Acharya Institutes, for his constant support and encouragement for research and development.

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