www.combinatorics.ir

Transactions on Combinatorics
ISSN (print): 2251-8657, ISSN (on-line): 2251-8665
Vol. 1 No. 3 (2012), pp. 47-56.

www.ui.ac.ir

THE EIGENVALUES AND ENERGY OF INTEGRAL CIRCULANT GRAPHS

MOHSEN MOLLAHAJIAGHAEI

Communicated by Dariush Kiani

Abstract. A graph is called *circulant* if it is a Cayley graph on a cyclic group, i.e. its adjacency matrix is circulant. Let D be a set of positive, proper divisors of the integer $n > 1$. The integral circulant graph $ICG_n(D)$ has the vertex set $\mathbb{Z}_n$ and the edge set $\mathrm{E}(ICG_n(D)) = \{\{a, b\}; gcd(a - b, n) \in D\}$. Let $n = p_1 p_2 \cdots p_k m$, where $p_1, p_2, \cdots, p_k$ are distinct prime numbers and $gcd(p_1 p_2 \cdots p_k, m) = 1$. The open problem posed in paper [A. Ilić, The energy of unitary Cayley graphs, Linear Algebra Appl., 431 (2009) 1881–1889] about calculating the energy of an arbitrary integral circulant $ICG_n(D)$ is completely solved in this paper, where $D = \{p_1, p_2, \ldots, p_k\}$.

1. Introduction

The interest of circulant graphs in graph theory and applications has grown during the last two decades, they appeared in coding theory, VLSI design, Ramsey theory and other areas. Recently there is vast research on the interconnection schemes based on circulant topology graphs represent an important class of interconnection networks in parallel and distributed computing [10].

Let D be a set of positive, proper divisors of the integer $n > 1$. The integral circulant graph $ICG_n(D)$ has vertex set $\mathbb{Z}_n$ and edge set $\mathrm{E}(ICG_n(D)) = \{\{a, b\}; gcd(a - b, n) \in D\}$. Let A_G be the adjacency matrix of a simple graph G, and $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of the matrix A_G. The *energy* of G is defined as the sum of absolute values of its eigenvalues, $E(G) = \sum_{i=1}^{n} |\lambda_i|$. This concept was introduced first by Gutman in [7] and afterwards it has been studied intensively in the literature [8], [9] and [15]. The graph G is said to be *hyperenergetic* if $E(G) > 2n - 2$. Hyperenergetic graphs are important because molecular graphs with maximum energy pertain to maximality stable π-electron

MSC(2010): Primary: 05C50.

Keywords: Graph; Integral circulant graph; Eigenvalue; Energy.

Received: 10 October 2012, Accepted: 16 October 2012.

systems. It has been proven that for every $n \geq 8$, there exists a hyperenergetic graph of order n [3]. If the distinct eigenvalues of A_G are $\lambda_1 < \lambda_2 < \cdots < \lambda_r$, and their multiplicities are $m_1, m_2, \ldots, m_r$, respectively, then we shall write

$$Spec(G) = \begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_r \\ m_1 & m_2 & \cdots & m_r \end{pmatrix}.$$

A graph is called *circulant* if it is a Cayley graph on a cyclic group, i.e. its adjacency matrix is circulant. A graph is *integral* if all its eigenvalues are integers.

Let R be a finite commutative ring with unity $1 \neq 0$. Its unit group of all invertible elements is denoted by $R^{\times}$. The *unitary Cayley graph of* R, $G_R = cay(R, R^{\times})$, is the Cayley graph whose vertex set is R and edge set is $\{\{a, b\} : a, b \in R \text{ and } a - b \in R^{\times}\}$. For some other recent papers on unitary Cayley graphs, we refer the reader to [1], [16], [17], [18] and [19]. By definition $G_{\mathbb{Z}_n} \simeq ICG_n(\{1\})$.

Let $n = p_1 p_2 \cdots p_k m$, where $p_1, p_2, \ldots, p_k$ are distinct prime numbers and $gcd(p_1 p_2 \cdots p_k, m) = 1$. The open problem posed in paper [11] about calculating the energy of an arbitrary integral circulant graph $ICG_n(D)$ is completely solved in this paper, where $D = \{p_1, p_2, \ldots, p_k\}$. Moreover, we obtain the spectrum of $ICG_n(D)$, where $D = \{p_1, p_2, \ldots, p_k\}$.

2. **Basic Notations and Properties**

We use $V(G)$ to denote the vertex set of a graph G. The *category product* $G_1 \otimes G_2$ of two graphs G_1 and G_2 is the graph with vertex set $V(G_1 \otimes G_2) := V(G_1) \times V(G_2)$, specified by putting (u, v) adjacent to (u', v') if and only if u is adjacent to u' in G_1 and v is adjacent to v' in G_2. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of G_1, and $\mu_1, \mu_2, \ldots, \mu_m$ be the eigenvalues of G_2. Then the eigenvalues of $G_1 \otimes G_2$ are $\lambda_i \mu_j$, for $1 \leq i \leq n$ and $1 \leq j \leq m$ by Theorem 2.3.4 of [5].

If G_1 and G_2 are graphs, their *Cartesian product* $G_1 \square G_2$ has vertex set $V(G_1) \times V(G_2)$, where (u, v) adjacent to (u', v') if and only if $u = u'$ and v is adjacent to v', or $v = v'$ and u is adjacent to u' [6].

The p-sum G of graphs $G_1, G_2, \ldots, G_n$ is a graph whose set of vertices is the cartesian product of the sets of vertices of the graphs $G_1, G_2, \ldots, G_n$, and If x_i and y_i are vertices of the graphs G_i $(i = 1, \ldots, n)$, the vertices $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ of the p-sum G are adjacent if and only if exactly p of the n pairs $(x_i, y_i)(i = 1, \ldots, n)$ are pairs of adjacent vertices in the corresponding graphs G_i, and $x_i = y_i$ for the remaining $n - p$ pairs [4].

Let $\mathscr{B}$ be a set of n-tuple $(\beta_1, \ldots, \beta_n)$ of symbol 0 and 1, which does not contain the n-tuple $(0, \ldots, 0)$. The *NEPS* (Non-complete Extended p-Sum) with basis $\mathscr{B}$ of the graphs $G_1, G_2, \ldots, G_n$ is the graph whose set of vertices is the cartesian product of the sets of vertices of the graphs $G_1, G_2, \ldots, G_n$, where two vertices $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ are adjacent if and only if there is an n-tuple $(\beta_1, \ldots, \beta_n)$ in $\mathscr{B}$ such that $x_i = y_i$ holds exactly when $\beta_i = 0$, and x_i is adjacent to y_i in G_i exactly when $\beta_i = 1$.
The p-sum, is obtained if $\mathscr{B}$ consists of all possible n-tuples with exactly p 1's. The NEPS with basis $\mathscr{B} = \{(1, 1)\}$ and $\mathscr{B} = \{(1, 0), (0, 1)\}$ of graphs G_1 and G_2 is category product and cartesian product, respectively [4].

The following is about the energy of category product of graphs.

Lemma 2.1. *If $G \simeq H \otimes K$, then $E(G) = E(H)E(K)$.*

3. Spectrum and Energy of $ICG_n(D)$

Various properties of integral circulant graphs were investigated in [2], [12], [13], [14], [20] and [21]. In this section, we compute spectrum and energy of $ICG_n(\{1,p\})$. Also we find some new classes of hyperenergetic graphs.

Theorem 3.1. *Let m, n be natural numbers such that $m|n$ and $gcd(m, n/m) = 1$ and let D be a set of proper divisors of $n > 1$. If $d|m$ for all $d \in D$, then $ICG_n(D) \simeq ICG_m(D) \otimes ICG_{n/m}(\{1\})$.*

Proof. Let $G = ICG_n(D)$ and $H = ICG_m(D) \otimes ICG_{n/m}(\{1\})$. Let $\psi : V(G) \longrightarrow V(H)$ be defined by $\psi(a) = (a, a)$. It is obvious that ψ is one-to-one and onto. Let $\{a, b\} \in \mathrm{E}(G)$. Thus by our assumption $gcd(a-b, n) \in D$ and so $gcd(a-b, m) \in D$, thus $gcd(a-b, n/m) = 1$. Therefore, $\{\psi(a), \psi(b)\} \in \mathrm{E}(H)$. Conversely, if $\{\psi(a), \psi(b)\} \in \mathrm{E}(H)$, then $gcd(a-b, n/m) = 1$ and $gcd(a-b, m) \in D$. Therefore, $gcd(a-b, n) \in D$. Then, $\{a, b\} \in \mathrm{E}(G)$. Thus, $ICG_n(D) \simeq ICG_m(D) \otimes ICG_{n/m}(\{1\})$. □

Lemma 3.2. [16, Lemma 2.3] *Let R be a finite commutative ring, where $R = R_1 \times R_2 \times \cdots \times R_s$ and R_i is a local ring with the maximal ideal M_i of size m_i for all $i \in \{1, 2, \ldots, s\}$. Then the eigenvalues of G_R are*

(i) $(-1)^{|C|}\dfrac{|R^\times|}{\prod_{j\in C}|R_j^\times|/m_j}$ with multiplicity $\prod_{j\in C}|R_j^\times|/m_j$ for all subsets C of $\{1, 2, \ldots, s\}$, and

(ii) 0 with multiplicity $|R| - \prod_{i=1}^{s}\left(1 + \dfrac{|R_i^\times|}{m_i}\right)$.

Corollary 3.3. *Let $n = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$, where p_i are distinct prime numbers. Then the eigenvalues of $ICG_n(\{1\})$ are*

(i) $(-1)^{|C|}\dfrac{\varphi(n)}{\prod_{j\in C}(p_j - 1)}$ with multiplicity $\prod_{j\in C}(p_j - 1)$ for all subsets C of $\{1, 2, \ldots, r\}$, and

(ii) 0 with multiplicity $n - p_1p_2 \ldots p_r$.

Proof. It is clear $\mathbb{Z}_n \simeq \mathbb{Z}_{p_1^{\alpha_1}} \times \mathbb{Z}_{p_2^{\alpha_2}} \times \cdots \times \mathbb{Z}_{p_r^{\alpha_r}}$, and $\mathbb{Z}_{p_i^{\alpha_i}}$ are local rings with the maximal ideal $p_i\mathbb{Z}_{p_i^{\alpha_i}}$, for all $i = 1, 2, \ldots, r$. The result follows directly from Lemma 3.2. □

Let $n = p^\alpha p_1^{\alpha_1} \cdots p_r^{\alpha_r}$, where p and p_i are distinct prime numbers, and $m = n/p^\alpha$. Then by Theorem 3.1 for finding spectrum and energy of $ICG_n(\{1,p\})$ it is enough to check the $ICG_{p^\alpha}(\{1,p\})$.

Theorem 3.4. *Let $n = pm$ where, p is a prime number and $gcd(m, p) = 1$. If λ_i are the distinct eigenvalues of $ICG_m(\{1\})$ with multiplicity t_i, then the eigenvalues of $ICG_n(\{1,p\})$ are $p\lambda_i$ and 0 with multiplicity t_i and $(p-1)m$ respectively.*

Proof. By Theorem 3.1 we have $ICG_n(\{1,p\}) \simeq ICG_p(\{1,p\}) \otimes ICG_m(\{1\})$. By definition, $ICG_p(\{1,p\})$ is a complete graph with a loop on every vertex. Therefore $A_{ICG_p(\{1,p\})} = J_p$, where J_p is the $p \times p$

all 1-matrix. So the eigenvalues of $ICG_n(\{1,p\})$ are $p\lambda_i$ and 0 with multiplicity t_i and $(p-1)m$ respectively. □

Lemma 3.5. [11, Theorem 2.3] *The energy of unitary Cayley graph $ICG_n(\{1\})$ equals $2^k\varphi(n)$, where k is the number of distinct prime factors dividing n.*

Lemma 3.6. [11, Theorem 2.4] *Let $n = p_1^{\alpha_1}p_2^{\alpha_2}\dots p_k^{\alpha_k}$. The unitary Cayley graph $ICG_n(\{1\})$ is hyperenergetic if and only if $k > 2$ or $k = 2$ and $p_1 > 2$.*

Theorem 3.7. *Let $n = pm$ where, p is prime number and $gcd(m,p) = 1$. Then $E(ICG_n(\{1,p\})) = pE(ICG_m(\{1\}))$, and $ICG_n(\{1,p\})$ is hyperenergetic if and only if $ICG_m(\{1\})$ is hyperenergetic.*

Proof. By Theorems 3.1, 3.4 and Lemma 2.1, it follows that $E(ICG_n(\{1,p\})) = pE(ICG_m(\{1\}))$. By the definition, $ICG_n(\{1,p\})$ is hyperenergetic if and only if $pE(ICG_m(\{1\})) > 2mp-2$, if and only if $\frac{p}{2}E(ICG_m(\{1\})) \geq mp$ and so $E(ICG_m(\{1\})) > 2m-2$. □

The following corollary follows directly from Theorem 3.7.

Corollary 3.8. [11, Theorem 4.1] *Let $n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$. For a two-element set of divisors $D = \{1, p_i\}$ where $\alpha_i = 1$, it holds*

$$E(ICG_n(\{1,p_i\})) = p_iE(G_{\mathbb{Z}_{\frac{n}{p_i}}}) = 2^{k-1}p_i\varphi(\frac{n}{p_i}).$$

Theorem 3.9. *Let $n = p^km$, where p is a prime number, $k > 1$ and $gcd(m,p) = 1$. If λ_i are eigenvalues of $ICG_m(\{1\})$ with multiplicity t_i, then the eigenvalues of $ICG_n(\{1,p\})$ are $-p^{k-2}\lambda_i$, $(p^k - p^{k-2})\lambda_i$ and 0 with multiplicity $(p^2-1)t_i$, t_i and $(p^k-p^2)m$, respectively.*

Proof. By Theorem 3.1, $ICG_n(\{1,p\}) \simeq ICG_{p^k}(\{1,p\}) \otimes ICG_m(\{1\})$. Let $V_i = \{(i-1)p^2+1, (i-1)p^2+2, \dots, ip^2\}$, for $i = 1, \dots, p^{k-2}$. Thus $V(G)$ is partitioned into V_i. Then the adjacency matrix of $ICG_{p^k}(\{1,p\})$ is

$$\begin{array}{c} \\ V_1 \\ \vdots \\ V_{p^{k-2}} \end{array} \begin{array}{c} \begin{array}{ccc} V_1 & \cdots & V_{p^{k-2}} \end{array} \\ \left(\begin{array}{c|c|c} J_{p^2} - I_{p^2} & \cdots & J_{p^2} - I_{p^2} \\ \hline \vdots & \ddots & \vdots \\ \hline J_{p^2} - I_{p^2} & \cdots & J_{p^2} - I_{p^2} \end{array}\right) \end{array}.$$

Thus $A_{ICG_{p^k}(\{1,p\})} = J_{p^{k-2}} \otimes J_{p^2} - I_{p^2}$. Therefore $A_{ICG_n(\{1,p\})} = J_{p^{k-2}} \otimes J_{p^2} - I_{p^2} \otimes A_{ICG_m(\{1\})}$. Thus the eigenvalues of $ICG_n(\{1,p\})$ are $-p^{k-2}\lambda_i$, $(p^k - p^{k-2})\lambda_i$ and 0 with multiplicity $(p^2-1)t_i$, t_i and $(p^k-p^2)m$, respectively. □

Theorem 3.10. *Let $n = p^km$, where p is a prime number, $k > 1$ and $gcd(m,p) = 1$. Thus $E(ICG_n(\{1,p\}) = 2(p^k - p^{k-2})E(ICG_m(\{1\}))$.*

Proof. By Theorems 3.1, 3.9 and Lemma 2.1, it follows that

$E(ICG_n(\{1,p\}) = (p^{k-2}(p^2-1) + (p^k - p^{k-2}))E(ICG_m(\{1\}))$.

So, $E(ICG_n(\{1,p\}) = 2(p^k - p^{k-2})E(ICG_m(\{1\}))$. □

Theorem 3.11. *Let $n = p^k m$, where p is a prime number, $k > 1$ and $gcd(m, p) = 1$. Then $ICG_n(\{1, p\})$ is hyperenergetic if and only if m has at least two distinct prime factors or m is an odd number.*

Proof. We know that $ICG_n(\{1, p\})$ is hyperenergetic if and only if $2(p^k - p^{k-2})E(ICG_m(\{1\})) > 2n-2$ equivalently, $(p^k - p^{k-2})E(ICG_m(\{1\})) \geq n$, it holds if and only if, $\frac{(p^2-1)}{p^2}2^r\varphi(m) \geq m$, where r is the number of prime factors of m, hence it is equivalent to $r \geq 2$ or m is an odd number. □

In what follows, we state some facts about the *NEPS* graphs.

Theorem 3.12. [4, Theorem 2.21] *The NEPS, G with basis $\mathscr{B}$ of graphs $G_1, G_2, \ldots, G_k$ whose adjacency matrices are $A_{G_1}, \ldots, A_{G_k}$ has the following adjacency matrix:*

$$A = \sum_{\beta \in \mathscr{B}} A_{G_1}^{\beta_1} \otimes \ldots \otimes A_{G_k}^{\beta_k}.$$

Theorem 3.13. [4, Theorem 2.23] *For $i = 1, 2, \ldots, k$, let G_i be a graph with n_i vertices, and let $\lambda_{i_1}, \ldots, \lambda_{i_{n_i}}$ be the spectrum of G_i. Then the spectrum of the NEPS with basis $\mathscr{B}$ of the graphs $G_1, G_2, \ldots, G_k$ consists of all possible values of $\Lambda_{i_1,\ldots,i_k}$, where*

$$\Lambda_{i_1,\ldots,i_n} = \sum_{\beta \in \mathscr{B}} \lambda_{1_{i_1}}^{\beta_1} \ldots \lambda_{k_{i_k}}^{\beta_k}.$$

$(i_j = 1, 2, \ldots, k_j; j = 1, 2, \ldots, k)$

Corollary 3.14. *Let G be the NEPS with basis $\mathscr{B} = \{(0, 1, \ldots, 1), (1, 0, 1, \ldots, 1), \ldots,$ $(1, \ldots, 1, 0)\}$ of graphs $G_1, G_2, \ldots, G_k$.*

(i) Adjacency matrix of G is $I \otimes A_{G_2} \otimes A_{G_3} \otimes \cdots \otimes A_{G_k} + A_{G_1} \otimes I \otimes A_{G_3} \otimes \cdots \otimes A_{G_k} + \cdots + A_{G_1} \otimes A_{G_2} \otimes \cdots \otimes A_{G_{k-1}} \otimes I$.

(ii) The eigenvalues of G are of the form $\lambda_2\lambda_3 \cdots \lambda_k + \lambda_1\lambda_3 \cdots \lambda_k + \cdots + \lambda_1\lambda_2 \cdots \lambda_{k-1}$, where $\lambda_1, \ldots, \lambda_k$ are eigenvalues of $G_1, \ldots, G_k$, respectively.

Remark 3.15. *It is clear that, there is a one-to-one correspondence relation between NEPS of complete graphs of prime orders and integral circulant graph of square free order.*

Theorem 3.16. *Let $n = p_1p_2 \cdots p_k m$, where $p_1, p_2, \ldots, p_k$ are distinct prime numbers such that $gcd(p_1p_2 \ldots p_k, m) = 1$. Thus adjacency matrix of $ICG_n(\{p_1, p_2, \ldots, p_k\})$ is $(I_{p_1} \otimes (J-I)_{p_2} \otimes (J-I)_{p_3} \otimes \cdots \otimes (J-I)_{p_k} + (J-I)_{p_1} \otimes I_{p_2} \otimes (J-I)_{p_3} \otimes \cdots \otimes (J-I)_{p_k} + \cdots + (J-I)_{p_1} \otimes (J-I)_{p_2} \otimes \cdots \otimes (J-I)_{p_{k-1}} \otimes I_{p_k}) \otimes A_{ICG_m(\{1\})}$.*

Proof. By Theorem 3.1, $ICG_n(\{p_1, p_2, \ldots, p_k\}) \simeq ICG_{p_1p_2\ldots p_k}(\{p_1, p_2, \ldots, p_k\}) \otimes ICG_m(\{1\})$. It is clear that $\mathbb{Z}_{p_1p_2\ldots p_k} \simeq \mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \ldots \times \mathbb{Z}_{p_k}$. So, $a = (a_1, \ldots, a_k)$ is adjacent to $b = (b_1, \ldots, b_k)$ if and only if a and b are the same in exactly one component. Therefore $ICG_{p_1p_2\ldots p_k}(\{p_1, p_2, \ldots, p_k\})$ is a *NEPS* graph with basis $\mathscr{B} = \{(0, 1, \ldots, 1), (1, 0, 1, \ldots, 1), \ldots, (1, \ldots, 1, 0)\}$ of graphs $ICG_{p_1}(\{1\}), \ldots, ICG_{p_k}(\{1\})$. It is clear that $ICG_{p_i}(\{1\})$ is the complete graph with p_i vertices. So by

Corollary 3.14, the adjacency matrix of $ICG_n(\{p_1,p_2,\ldots,p_k\})$ is $(I_{p_1}\otimes(J-I)_{p_2}\otimes(J-I)_{p_3}\otimes\cdots\otimes(J-I)_{p_k}+(J-I)_{p_1}\otimes I_{p_2}\otimes(J-I)_{p_3}\otimes\cdots\otimes(J-I)_{p_k}+\cdots+(J-I)_{p_1}\otimes(J-I)_{p_2}\otimes\cdots\otimes(J-I)_{p_{k-1}}\otimes I_{p_k})\otimes A_{ICG_m(\{1\})}$. □

In the next theorem we compute $Spec(ICG_n(\{p,q\}))$.

Theorem 3.17. *Let $n=pqm$ where p,q are distinct prime numbers, and $gcd(m,pq)=1$. If λ_i are eigenvalues of $ICG_m(\{1\})$ with multiplicity t_i, then the eigenvalues of $ICG_n(\{p,q\})$ are $-2\lambda_i$, $(p-2)\lambda_i$, $(q-2)\lambda_i$ and $(p+q-2)\lambda_i$ with multiplicity $(p-1)(q-1)t_i$, $(q-1)t_i$, $(p-1)t_i$ and t_i, respectively.*

Proof. By Theorem 3.1, $ICG_n(\{p,q\})\simeq ICG_{pq}(\{p,q\})\otimes ICG_m(\{1\})$. By Lemma 3.14 and Theorem 3.16, we see that eigenvalues of $ICG_{pq}(\{p,q\})$ are -2, $p-2$, $q-2$ and $p+q-2$ with multiplicity $(p-1)(q-1)$, $(q-1)$, $(p-1)$ and 1, respectively. So, the eigenvalues of $ICG_n(\{p,q\})$ are $-2\lambda_i$, $(p-2)\lambda_i$, $(q-2)\lambda_i$ and $(p+q-2)\lambda_i$ with multiplicity $(p-1)(q-1)t_i$, $(q-1)t_i$, $(p-1)t_i$ and t_i, respectively. □

In the following theorem, we calculate $E(ICG_n(\{p,q\}))$.

Theorem 3.18. *Let $n=pqm$, where p,q are distinct prime numbers, and $gcd(m,pq)=1$. Then $E(ICG_n(\{p,q\}))=4(p-1)(q-1)E(ICG_m(\{1\}))$.*

Proof. By Theorem 3.1, $ICG_n(\{p,q\})\simeq ICG_{pq}(\{p,q\})\otimes ICG_m(\{1\})$.
By Theorem 3.17, $E(ICG_{pq}(\{p,q\}))=4(p-1)(q-1)$. Therefore by Lemma 2.1, $E(ICG_n(\{p,q\}))=4(p-1)(q-1)E(ICG_m(\{1\}))$. □

Theorem 3.18 yields the following corollary.

Corollary 3.19. [11, Theorem 4.2] *Let n be a square-free number, $n=p_1p_2\cdots p_k$. Then the energy of integral circulant graph $ICG_n(\{p_i,p_j\})$ does not depend on the choice of p_i and p_j, $E(ICG_n(\{p_i,p_j\}))=2^k\varphi(n)=2^k\prod_{i=1}^{k}(p_i-1)$.*

Accordingly we may summarize our result as follows.

Theorem 3.20. *Let $n=pqm$, where p,q are distinct prime numbers, and $gcd(m,pq)=1$. Then $ICG_n(\{p,q\})$ is hyperenergetic if and only if $ICG_n(\{1\})$ is hyperenergetic.*

Proof. By Lemma 3.6, $E(ICG_{pq}(\{1\}))=2^2\varphi(pq)$. By Theorem 3.1 and Lemma 2.1, $E(ICG_n(\{1\}))=E(ICG_{pq}(\{1\}))E(ICG_m(\{1\}))$, so $E(ICG_n(\{1\}))=4(p-1)(q-1)E(ICG_m(\{1\}))$. Therefore By Theorem 3.18, $E(ICG_n(\{p,q\}))=E(ICG_n(\{1\}))$. □

4. Computing Spectrum and Energy of $ICG_n(p_1,\ldots,p_k)$

In this section, we compute spectrum and energy of $ICG_n(p_1,\ldots,p_k)$. Also we find some new classes of hyperenergetic graphs.

Theorem 4.1. *Let $n = p_1 p_2 \cdots p_k$, where p_i are distinct prime numbers. If $S_i = \{p_i - 1, -1\}$ and $G = ICG_n(\{p_1, p_2, \ldots, p_k\})$, then*

$$E(G) = \sum_{(a_1,\ldots,a_k)\in S_1\times\ldots\times S_k} |\sum_{i=1}^{k} \frac{1}{a_i}|(\varphi(n)).$$

Proof. By Theorem 3.16 and Lemma 3.14, the eigenvalues of G are $\sum_{i=1}^{k}(\frac{\prod_{j=1}^{k} a_j}{a_i})$, where $a_t \in S_t$ with multiplicity $(\frac{\prod_{j=1}^{k}(p_j-1)}{|\prod_{j=1}^{k} a_j|})$. So,

$$E(G) = \sum_{(a_1,\ldots,a_k)\in S_1\times\ldots\times S_k} |\sum_{i=1}^{k}\Big(\frac{\prod_{j=1}^{k} a_j}{a_i}\Big)|\Big(\frac{\prod_{j=1}^{k}(p_j-1)}{|\prod_{j=1}^{k} a_j|}\Big). \tag{4.1}$$

Thus,

$$E(G) = \sum_{(a_1,\ldots,a_k)\in S_1\times\ldots\times S_k} |\sum_{i=1}^{k} \frac{1}{a_i}|(\varphi(n)). \tag{4.2}$$

□

Theorem 4.2. *Let $n = p_1 p_2 \ldots p_k m$, where p_i are distinct prime numbers such that $gcd(m, p_1 p_2 \ldots p_k) = 1$ and $p_i > k$ for all $i = 1, \ldots, k$. Then*

$$E(ICG_n(\{p_1, \ldots, p_k\})) = 2^t\Big(k2^{k-1} - (2^{k-1} - 2)\sum_{i=1}^{k}(\frac{1}{p_i - 1})\Big)\varphi(n),$$

where t is the number of prime factors of m.

Proof. We know that $p_i > k$, so

$$\sum_{i=1}^{k}(\frac{1}{p_i - 1}) \leq 1. \tag{4.3}$$

If

$$\alpha = \sum_{(a_1,\ldots,a_k)\in S_1\times\ldots\times S_k} |\sum_{i=1}^{k}(\frac{1}{a_i})|,$$

where $S_i = \{-1, p_i - 1\}$, then by (4.3) we have,

$$\alpha = \sum_{i=1}^{k}(\frac{1}{p_i - 1}) - \sum_{\substack{(a_1,\ldots,a_k)\in S_1\times\ldots\times S_k \\ (a_1,\ldots,a_k)\neq(p_1-1,\ldots,p_k-1)}} \sum_{i=1}^{k}(\frac{1}{a_i}).$$

Let

$$S = \sum_{\substack{(a_1,\ldots,a_k)\in S_1\times\ldots\times S_k \\ (a_1,\ldots,a_k)\neq(p_1-1,\ldots,p_k-1)}} \sum_{i=1}^{k}(\frac{1}{a_i}).$$

The number of occurrences of -1 in S is $k2^{k-1}$, and coefficients of $\frac{1}{p_r - 1}$ in S is $2^{k-1} - 1$, where $1 \leq r \leq k$. So,

$$\alpha = \Big(k2^{k-1} - (2^{k-1} - 2)\sum_{i=1}^{k}(\frac{1}{p_i - 1})\Big).$$

Thus by (4.2), we have

$$E(ICG_{p_1\dots p_k}(\{p_1,\dots,p_k\})) = \Big(k2^{k-1} - (2^{k-1}-2)\sum_{i=1}^{k}(\frac{1}{p_i-1})\Big)\varphi(p_1\dots p_k). \tag{4.4}$$

By Theorem 3.1, $ICG_n(\{p_1,\dots,p_k\}) \simeq ICG_{p_1\dots p_k}(\{p_1,\dots,p_k\}) \otimes ICG_m(\{1\})$, so by Lemmas 2.1, 3.6 and formula (4.4),

$$E(ICG_n(\{p_1,\dots,p_k\})) = 2^t\Big(k2^{k-1} - (2^{k-1}-2)\sum_{i=1}^{k}(\frac{1}{p_i-1})\Big)\varphi(n).$$

□

Remark 4.3. *Let $n = pqrm$, where p, q, r are distinct prime numbers such that $gcd(m, pqr) = 1$. If λ_i are eigenvalues of $ICG_m(\{1\})$ with multiplicities t_i, then the eigenvalues of $ICG_n(\{p, q, r\})$ are $3\lambda_i$, $(-2r+3)\lambda_i$, $(-2p+3)\lambda_i$, $(-2q+3)\lambda_i$, $(pq-2p-2q+3)\lambda_i$, $(pr-2p-2r+3)\lambda_i$, $(qr-2q-2r+3)\lambda_i$, and $(pq+pr+qr-2p-2q-2r+3)\lambda_i$ with multiplicities $(p-1)(r-1)(q-1)t_i$, $(p-1)(q-1)t_i$, $(r-1)(q-1)t_i$, $(p-1)(r-1)t_i$, $(r-1)t_i$, $(q-1)t_i$, $(p-1)t_i$ and t_i, respectively.*

Remark 4.4. *Let $n = pqrm$, where p, q, r are distinct prime numbers such that $gcd(m, pqr) = 1$ and $r > q > p > 2$. Then*

$$E(ICG_n(\{p,q,r\})) = (12pqr - 14pr - 14pq - 14qr + 16p + 16q + 16r - 18)E(ICG_m(\{1\})).$$

Lemma 4.5. *Let $p_1, \dots, p_k$ are distinct prime numbers where $k > 2$. If, $p_i > k$ for all $i = 1, \dots, k$, then $2^{k-1}\prod_{i=1}^{k}(p_i-1) > 2(\prod_{i=1}^{k} p_i) - 2$.*

Proof. It is enough to prove that

$$2^{k-2}\prod_{i=1}^{k}(p_i-1) \geq \prod_{i=1}^{k} p_i. \tag{4.5}$$

By induction on k. If $k = 3$, then $p_i \geq 5$, so $\sqrt[3]{2} > \frac{5}{4} > \frac{p_i}{p_i-1} = 1 + \frac{1}{p_i-1}$, for $i = 1, 2, 3$. So $2(p_1-1)(p_2-1)(p_3-1) > p_1p_2p_3$. It is clear that $2(p_i-1) > p_i$, thus $2^{k-2}\prod_{i=1}^{k}(p_i-1) \geq \prod_{i=1}^{k} p_i$. □

Lemma 4.6. *Let $n = q_1^{r_1}q_2^{r_2}\dots q_k^{r_k}$, where q_i are distinct prime numbers for all $i = 1, \dots, k$. Then $2^k\varphi(n) \geq n$.*

Proof. We know that, $2\varphi(q_i^{r_i}) = 2(q_i-1)q_i^{r_i-1} \geq q_iq_i^{r_i-1} = q_i^{r_i}$. So, $2^k\varphi(n) = \prod_{i=1}^{k} 2\varphi(q_i^{r_i}) \geq \prod_{i=1}^{k} q_i^{r_i} = n$. □

Theorem 4.7. *Let $n = p_1p_2\dots p_km$, where p_i are distinct prime numbers such that $gcd(m, p_1p_2\dots p_k) = 1$ and $p_i > k$ for all $i = 1, \dots, k$. Then $ICG_n(\{p_1,\dots,p_k\})$ is hyperenergetic.*

Proof. By Theorem 4.2, $ICG_n(\{p_1,\dots,p_k\})$ is hyperenergetic if and only if $2^t\Big(k2^{k-1} - (2^{k-1}-2)\sum_{i=1}^{k}(\frac{1}{p_i-1})\Big)\varphi(n) > 2n-2$. It holds if and only if $2^t\Big(k2^{k-2} - (2^{k-2}-1)\sum_{i=1}^{k}(\frac{1}{p_i-1})\Big)\varphi(n) \geq n = p_1p_2\dots p_km$.

By Lemma 4.6, $2^t\varphi(m) \geq m$. So, it is enough to prove that, $(k2^{k-2} - (2^{k-2}-1)\sum_{i=1}^{k}(\frac{1}{p_i-1}))\varphi(p_1p_2\dots p_k) \geq p_1p_2\dots p_k$. By Lemma 4.5, it is enough to prove the following,

$$\Big((k-1)2^{k-2} - (2^{k-2}-1)\sum_{i=1}^{k}(\frac{1}{p_i-1})\Big)\varphi(p_1p_2\dots p_k) \geq 0. \tag{4.6}$$

It is clear that,

$$(k-1)2^{k-1} - (2^{k-1}-2) \geq 0. \tag{4.7}$$

Since $p_i > k$ for each $1 \leq i \leq k$, it follows that

$$\sum_{i=1}^{k}(\frac{1}{p_i-1}) \leq 1. \tag{4.8}$$

So, we have that, $\Big((k-1)2^{k-2} - (2^{k-2}-1)\sum_{i=1}^{k}(\frac{1}{p_i-1})\Big)\varphi(p_1p_2\dots p_k) \geq 0$. Then, $ICG_n(\{p_1,\dots,p_k\})$ is hyperenergetic. □

Acknowledgments

The author is grateful to the referees whose many comments have significantly helped to improve the presentation of this article.

References

[1] R. Akhtar, M. Boggess, T. Jackson-Henderson, I. Jiménez, R. Karpman, A. Kinzel and D. Pritikin, On the unitary Cayley graph of a finite ring, *Electron. J. Combin.*, **16** (2009) Research Paper 117.

[2] S. Blackburn and I. Shparlinski, On the average energy of circulant graphs, *Linear Algebra Appl.* **428** (2008) 1956-1963.

[3] R. A. Brualdi, Energy of a Graph, http://www.public.iastate.edu/lhogben/energyB.pdf

[4] D. Cvetković, M. Doob and H. Sachs, *Spectra of Graphs*, Academic Press, New York, 1980.

[5] D. M. Cvetković, P. Rowlinson and S. Simić, *Eigenspaces of Graphs*, Cambridge University Press, Cambridge, 1997.

[6] C. Godsil and G. Royle, *Algebraic Graph Theory*, Springer-Verlag, New York, 2001.

[7] I. Gutman, The energy of a graph, *Ber. Math. Stat. Sekt. Forschungszent. Graz.*, **103** (1978) 1-22.

[8] I. Gutman, *The energy of a graph: old and new results*, In: A. Betten, A. Kohnert, R. Laue and A. Wassermann, Editors, Algebraic Combinatorics and Applications, Springer-Verlag, Berlin, (2001) 196-211.

[9] I. Gutman, D. Kiani, M. Mirzakhah and B. Zhou, On incidence energy of a graph, *Linear Algebra Appl.*, **431** (2009) 1223-1233.

[10] F. K. Hwang, A survey on multi-loop networks, *Theor. Comput. Sci.*, **299** (2003) 107-121.

[11] A. Ilić, The energy of unitary Cayley graphs, *Linear Algebra Appl.*, **431** (2009) 1881-1889.

[12] A. Ilić, Distance spectra and distance energy of integral circulant graphs, *Linear Algebra Appl.*, **433** (2010) 1005-1014.

[13] A. Ilić and M. Bašić, On the chromatic number of integral circulant graphs, *Comp. Math. Appl.*, **60** (2010) 144-150.

[14] A. Ilić and M. Bašić, New results on the energy of integral circulant graphs, *Appl. Math. Comput.*, **218** (2011) 3470–3482.

[15] M. R. Jooyandeh, D. Kiani and M. Mirzakhah, Incidence energy of a graph, *MATCH Commun. Math. Comput. Chem.*, **62** (2009) 561-572.

[16] D. Kiani, M. Molla Haji Aghaei, Y. Meemark and B. Suntornpoch, Energy of unitary Cayley graphs and gcd-graphs, *Linear Algebra Appl.*, **435** (2011) 1336-1343.

[17] D. Kiani and M. Molla Haji Aghaei, On the unitary Cayley graphs of a ring, *Electron. J. Combin.*, **19** no. 2 (2012) P10.

[18] W. Klotz and T. Sander, Some properties of unitary Cayley graphs, *Electron. J. Combin.*, **14** (2007), Reasearch paper 45.

[19] H. N. Ramaswamy and C. R. Veena, On the Energy of Unitary Cayley Graphs, *Electron. J. Combin.*, **16** (2009) Note 24.

[20] J. W. Sander and T. Sander, The energy of integral circulant graphs with prime power order, *Appl. Anal. Discrete Math.*, **5** (2011) 22-36.

[21] W. So, Integral circulant graphs, *Discrete Math.*, **306** (2006) 153-158.

Mohsen Mollahajiaghaei

Department of Mathematics, Amirkabir University of Technology, P.O.Box 15914, Tehran, Iran

`Email: mhmaghaei@yahoo.com`