ON ANNIHILATOR GRAPHS OF A FINITE COMMUTATIVE RING

SANGHITA DUTTA* AND CHANLEMKI LANONG

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Abstract. The annihilator graph $AG(R)$ of a commutative ring $R$ is a simple undirected graph with the vertex set $Z(R)^*$ and two distinct vertices are adjacent if and only if $\text{ann}(x) \cup \text{ann}(y) \neq \text{ann}(xy)$. In this paper we give the sufficient condition for a graph $AG(R)$ to be complete. We characterize rings for which $AG(R)$ is a regular graph, we show that $\gamma(AG(R)) \in \{1, 2\}$ and we also characterize the rings for which $AG(R)$ has a cut vertex. Finally we find the clique number of a finite reduced ring and characterize the rings for which $AG(R)$ is a planar graph.

1. Introduction

The study of rings using the properties of graphs lead to many interesting results. The zero-divisor graph of $R$, denoted by $\Gamma(R)$, is an undirected graph with the vertex set $Z(R)^*$ and two distinct vertices $x, y$ are adjacent if and only if $xy = 0$. The concept of a zero divisor graph goes back to I. Beck [8], who considered all elements of $R$ as the set of vertices and was mainly interested in coloring of a graph. The zero-divisor graph $\Gamma(R)$ was introduced by David F. Anderson and Philip S. Livingston [2], where it was shown among other results that $\Gamma(R)$ is connected with $\text{diam}(\Gamma(R)) \in \{0, 1, 2, 3\}$ and $\text{girth}(\Gamma(R)) \in \{3, 4\}$. Many mathematicians have studied the zero divisor graph of a ring and obtained many interesting results regarding ring theoretic properties as well as graph theoretic properties of this graph. Badawi [7] defined a graph associated with a commutative ring called the annihilator graph of a ring $R$, denoted by $AG(R)$. The vertex set of this graph is $Z(R)^*$ and two distinct vertices $x$ and $y$ are adjacent if and only if $\text{ann}(x) \cup \text{ann}(y) \neq \text{ann}(xy)$. Badawi [7] proved that $AG(R)$ is a connected graph, diameter of $AG(R)$ is atmost two, girth of $AG(R)$ is atmost four if it has a cycle and if $R$ is a

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*Corresponding author.
reduced ring then $\text{AG}(R)$ is identical to $\Gamma(R)$ if and only if the ring $R$ has exactly two minimal prime ideals. D.A Mojdeh et al. [10] found the domination number of a zero divisor graph, zero divisor graph with respect to an ideal of a ring $R$ and T. Tamish Chelvam et al. [9] found the domination number of total graph of a ring. M. Axtell et al. [6] have found the condition for a vertex $x$ to be a cut vertex of $\Gamma(R)$.

In section 2, we discuss about the existence of a vertex which is adjacent to all vertices of $\text{AG}(R)$, sufficient condition for $\text{AG}(R)$ to be a complete graph and a regular graph and we show that the domination number of $\text{AG}(R)$ is less than or equal to 2 for any finite ring. We find that if $R$ is a finite ring and $\text{AG}(R)$ has a cut vertex then $R \cong \mathbb{Z}_2 \times F$, where $F$ is a finite field with $F \not\cong \mathbb{Z}_2$. We also compute $\alpha(\text{AG}(R))$ and $\omega(\text{AG}(R))$ for some classes of rings. We show that $\text{AG}(R)$ is Hamiltonian if $R \cong A \times A$ where $A$ is a finite local ring with identity. In section 3, we characterize rings for which $\text{AG}(R)$ is planar.

Throughout the paper, all rings are assumed to be commutative ring with unity $1 \neq 0$. A ring $R$ is said to be reduced if $R$ has no non-zero nilpotent element. Let $Z(R)$ denote the set of zero-divisors of a ring $R$. If $X$ is either an element or a subset of $R$, then $\text{ann}(X)$ denotes the annihilator of $X$ in $R$, i.e., $\text{ann}(X) = \{r \in R \mid rX = 0\}$. For any subset $X$ of $R$ let $X^* = X \setminus \{0\}$. A ring $R$ is said to be decomposable if $R$ can be written as $R_1 \times R_2$, where $R_1$ and $R_2$ are rings; otherwise $R$ is said to be indecomposable.

All graphs considered in this paper are simple graphs. For a graph $G$, the degree of a vertex $v$ in $G$, denoted by $\text{deg}(v)$ is the number of edges incident to $v$. A graph $G$ is said to be regular if the degrees of all vertices of $G$ are same. A graph $G$ is said to be complete if every pair of distinct vertices are connected by an edge. A bipartite graph is a graph whose set of vertices can be partitioned into two sets $U$ and $V$ such that every edge is between a vertex of $U$ and a vertex of $V$. We denote the complete graph with $n$ vertices and complete bipartite graph with two sets of sizes $m$ and $n$ by $K_n$ and $K_{m,n}$ respectively. The complete bipartite graph $K_{1,n}$ is called a star graph. The diameter of a graph $G$ is $\text{diam}(G) = \sup\{d(x,y) : x$ and $y$ are distinct vertices of $G\}$. A vertex $a$ in a connected graph $G$ is a cut-vertex if $G$ can be expressed as a union of two subgraphs $X$ and $Y$ such that $E(X) \neq \emptyset$, $E(Y) \neq \emptyset$, $E(X) \cup E(Y) = E(G)$, $V(X) \cup V(Y) = V(G)$, $V(X) \cap V(Y) = \{a\}$, $X \setminus \{a\} \neq \emptyset$, and $Y \setminus \{a\} \neq \emptyset$. A subset $D$ of the set of vertices $V(G)$ of a graph $G$ is called a dominating set, if every vertex of $V(G) \setminus D$ is adjacent to some vertex of $D$. The minimum size of such a subset is called the domination number of $G$ and is denoted by $\gamma(G)$. A set $S \subseteq V(G)$ is an independent set of $G$, if no two vertices of $S$ are adjacent. The independence number of a graph $G$ denoted by $\alpha(G)$ is the size of the maximum independent set in $G$. A clique of a graph is a maximal complete subgraph and the number of vertices in the largest clique of a graph $G$, denoted by $\omega(G)$, is called the clique number of $G$.

A Hamiltonian cycle (resp. path) in a graph is a cycle (resp. path) including all the vertices of the graph. Similarly, an Eulerian tour or circuit (resp. trail) in a graph is a closed walk (resp. walk) including all the edges of the graph. A graph is Hamiltonian if it has a Hamiltonian cycle and it is Eulerian if it has an Eulerian tour or circuit. A graph $G$ is said to be planar if it can be drawn in the
plane so that its edges intersect only at their ends. A subdivision of a graph is a graph obtained from it by replacing edges with pairwise internally-disjoint paths.

2. Properties of $AG(R)$

In this section, we find for which ring $R$ there exist a vertex which is adjacent to all vertices of $AG(R)$ and then find some more properties of $AG(R)$. We note here the following proposition from Axtell et.al [6] which will be used frequently in this paper.

**Proposition 2.1.** [6] Let $R$ be a finite commutative ring with identity. Then the following are equivalent:
1. $Z(R)$ is an ideal;
2. $Z(R)$ is a maximal ideal;
3. $R$ is local;
4. Every $x \in Z(R)$ is nilpotent.

The following two propositions give criterion for existence of a vertex which is adjacent to all vertices of $AG(R)$ for finite rings. These propositions will be used to derive the other properties of $AG(R)$ graph.

**Proposition 2.2.** Let $R$ be a finite reduced ring. Then there exists a vertex $x \in Z(R)^*$ such that $x$ is adjacent to all vertices of $AG(R)$ if and only if $R \cong \mathbb{Z}_2 \times \mathbb{F}$ where $\mathbb{F}$ is a finite field.

**Proof.** Suppose $R$ is a finite reduced ring then we have $R \cong \mathbb{F}_1 \times \mathbb{F}_2 \times \cdots \times \mathbb{F}_n$, where each $\mathbb{F}_i$ is a finite field for $1 \leq i \leq n$.

Suppose $x = (x_1, x_2, \ldots, x_n) \in Z(R)^*$ is a vertex which is adjacent to all the vertices of $R$. First we consider $n \geq 3$ and let $e_1 = (1, 0, 0, \ldots, 0) \in Z(R)^*$. Then $xe_1 = (x_1, 0, 0, \ldots, 0)$ and so $ann(xe_1) = ann(e_1)$. Thus for $x$ and $e_1$ to be adjacent we must have $x_1 = 0$. Similarly taking $e_i = (0, 0, \ldots, 0, 1, 0, \ldots, 0)$, where 1 is in the $i^{th}$ entry, for $1 \leq i \leq n$ and continuing the same way we have $x = (0, 0, \ldots, 0)$, which is a contradiction. Hence if $n \geq 3$, there does not exist $x \in Z(R)^*$ such that $x$ is adjacent to all vertices of $AG(R)$. So we consider $n \leq 2$. If $n = 1$ then $AG(R)$ is an empty graph. Now for $n = 2$, $R \cong \mathbb{F}_1 \times \mathbb{F}_2$ and so by [3, Thereom 3.6] $AG(R) = \Gamma(R)$. But for $\Gamma(R)$, there exists $x \in Z(R)^*$ which is adjacent to all vertices of $AG(R)$ if only if $R \cong \mathbb{Z}_2 \times \mathbb{F}$ where $\mathbb{F}$ is a field or $R$ is a local ring by [2, Corrolary 2.7]. But since $R$ is a reduced ring, we must have $R \cong \mathbb{Z}_2 \times \mathbb{F}$.

If $R \cong \mathbb{Z}_2 \times \mathbb{F}$, where $\mathbb{F}$ is a field, then clearly there is a vertex adjacent to all vertices of $AG(R)$.

**Proposition 2.3.** Let $R$ be a finite non-reduced ring with identity. If $R \cong R_1 \times R_2 \times \cdots \times R_n$, where each $R_i$ are finite local rings but not field, then there exists a vertex $x \in Z(R)^*$ such that $x$ is adjacent to all vertices of $AG(R)$.

**Proof.** Assume that $R$ is a finite non-reduced ring. Then $R \cong R_1 \times R_2 \times \cdots \times R_n$, where each $R_i$ are finite local ring. Let $x = (x_1, x_2, \ldots, x_n) \in Z(R)^*$ be a vertex which is adjacent to all vertices of $AG(R)$. If atleast one of $x_i$ is zero then for $z = (1, 1, \ldots, 1, 0, 1, \ldots, 1) \in Z(R)^*$, where zero is in the $i^{th}$
position, we have \(\text{ann}(xz) = \text{ann}(x)\). So by [7, lemma 2.1(1)] \(x\) is not adjacent to \(z\). Hence, if atleast one entry in \(x\) is zero then \(x\) cannot be adjacent to every vertex of \(Z(R)^*\). Thus all entries of \(x\) must be non-zero. Suppose now, the \(k^{th}\) entry of \(x\) say \(x_k\) is invertible, i. e., there exists \(y \in R_k\) such that \(x_k y = 1\). Then for \(v = (0, 0, \ldots, 0, y, 0, \ldots, 0) \in Z(R)^*\), \(\text{ann}(xv) = \text{ann}(v)\). So \(x\) is not adjacent to some vertex of \(Z(R)^*\), which is a contradiction. So we consider that each \(R_i\) is not a field. Now assume that all entries of \(x\) are non-zero and non-unit. Let \(z = (z_1, z_2, \ldots, z_n) \in Z(R)^*\). Then \(\text{ann}(x) = \text{ann}(x_1) \times \text{ann}(x_2) \times \cdots \times \text{ann}(x_n)\) and \(\text{ann}(z) = \text{ann}(z_1) \times \text{ann}(z_2) \times \cdots \times \text{ann}(z_n)\). But as \(z \in Z(R)^*\), so there exists \(z_i's\), say \(z_k\), where \(z_k \in Z(R_k)^*\). As \(R_k\) is a local ring we have \(AG(R_k)\) is complete and therefore \(\text{ann}(x_k z_k) \neq \text{ann}(x_k) \cup \text{ann}(z_k)\). So there exists \(t_j \in \text{ann}(x_k z_k)\setminus \text{ann}(x_k) \cup \text{ann}(z_k)\). Now \(t = (0, 0, \ldots, 0, t_j, 0, \ldots, 0) \in \text{ann}(xz)\) but \(t = (0, 0, \ldots, 0, t_j, 0, \ldots, 0) \notin \text{ann}(x) \cup \text{ann}(z)\), for if \(t \in \text{ann}(x) \cup \text{ann}(z)\) then we have either \(x_j t_j = 0\) or \(z_j t_j = 0\) which is a contradiction. Hence there exists a vertex \(x \in Z(R)^*\) such that \(x\) is adjacent to all vertices of \(AG(R)\) if \(R \cong R_1 \times R_2 \times \cdots \times R_n\), where each \(R_i\) are finite local rings but not field.  

In the next proposition we characterize a finite complete \(AG(R)\) graph.

Proposition 2.4. If \(AG(R)\) is a finite complete graph then either \(R\) is a finite local ring or \(R \cong \mathbb{Z}_2 \times \mathbb{Z}_2\).

Proof. If \(AG(R)\) is finite complete graph, the set of vertices of \(AG(R)\) is same as \(\Gamma(R)\), by [1, theorem 2.2] \(R\) must be a finite ring. So let \(R \cong R_1 \times R_2 \times \cdots \times R_n\), where each \(R_i\) are finite local ring. Let \(x = (1, 0, 0, \ldots, 0) \in Z(R)^*\) and \(y = (1, 1, 0, \ldots, 0) \in Z(R)^*\). We assume that \(n \geq 3\). Then \(\text{ann}(x) = \text{ann}(xy)\) shows that \(x\) is not adjacent to \(y\), which is a contradiction. So we must have \(n \leq 2\). If \(n = 2\) then \(R \cong R_1 \times R_2\). By proposition 2.3, we have either \(R \cong \mathbb{Z}_2 \times \mathbb{F}\), where \(\mathbb{F}\) is a field, or each \(R_i\) a local ring but not a field. First we consider that atleast one of \(R_i\), say \(R_2\) is not a field. Then for \(t = (1, 0)\) and \(w = (1, x)\), where \(x \in Z(R_2)^*\), we get \(\text{ann}(t) = \text{ann}(tw)\). This shows that \((1, 0)\) is not adjacent to \((1, x)\), which is a contradiction as \(AG(R)\) is a complete graph. So we consider that both \(R_i\) are fields. But if both \(R_i\) are fields, there exists a vertex which is adjacent to all vertices of \(AG(R)\) since \(AG(R)\) is a complete graph. Hence, \(R \cong \mathbb{Z}_2 \times \mathbb{F}\). As \(AG(R)\) is a complete graph, we must have \(\mathbb{F} \cong \mathbb{Z}_2\). Now for \(n = 1\) we have \(R\) is a finite local ring. Thus, \(AG(R)\) is a finite complete graph if \(R\) is a finite local ring or \(R \cong \mathbb{Z}_2 \times \mathbb{Z}_2\).  

In the following proposition we characterize the finite rings for which \(AG(R)\) is a regular graph.

Proposition 2.5. If \(R\) is a finite ring with identity and \(AG(R)\) is a regular graph then \(R \cong \mathbb{F} \times \mathbb{F}\), i.e., \(AG(R) \cong K_{t-1,t-1}\) with \(|F| = t\) or \(R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2\) or \(R\) is a local ring or a field.

Proof. Let \(R\) be a finite commutative ring with identity and \(AG(R)\) be a regular graph. Since \(R\) is a finite ring, \(R \cong R_1 \times R_2 \times \cdots \times R_n\), where each \(R_i\) are finite local ring and \(n \geq 1\). Now if atleast one of \(R_i's\) is not a field, say \(R_1\), then consider \(e_1 = (1, 0, \ldots, 0) \in Z(R)^*\) and \(y = (y_1, 0, \ldots, 0) \in Z(R)^*\) with \(y_1 \in Z(R_1)^*\). Then clearly \(\text{deg}(y) > \text{deg}(x)\), which is a contradiction. Hence if \(n \geq 2\), each \(R_i\) must be field. So \(R \cong \mathbb{F}_1 \times \mathbb{F}_2 \times \cdots \times \mathbb{F}_n\), where \(n \geq 2\) and \(\mathbb{F}_i's\) are finite fields. If we consider \(e_1\) as above then the vertices that are adjacent to \(e_1\) in \(AG(R)\) are those vertices \(y\) such that \(e_1 y = 0\). So \(\text{deg}(e_1) = |\mathbb{F}_2||\mathbb{F}_3|\cdots|\mathbb{F}_n| - 1\) and similarly if we take \(e_2 = (0, 1, 0, \ldots, 0) \in Z(R)^*\) then \(\text{deg}(e_2) = \)
If \( |F_1|, |F_2|, \ldots, |F_n| = 1 \). As \( AG(R) \) is regular, we have \( \text{deg}(e_1) = \text{deg}(e_2) \) and so \( |F_1| = |F_2| \). Thus taking each \( e_i \) for \( 1 \leq i \leq n \), we see that all \( F_i \) have the same cardinality and hence \( R \cong F \times F \times \cdots \times F \). Let \( |F| = t \). We consider \( n \geq 3 \) and let \( z = (1, 1, 0, \ldots, 0) \). Then we have \( \text{deg}(e_1) = |F|(n-1) - 1 \) and \( \text{deg}(z) = (|F|^{(n-2)} - 1) + 2(|F| - 1)(|F|^{(n-2)} - 1) \). Now if \( n \geq 4 \) then \( \text{deg}(z) > \text{deg}(e_1) \), which is a contradiction. If \( n = 3 \) and \( |F| \geq 3 \) then also \( \text{deg}(z) > \text{deg}(e_1) \), which is a contradiction. If \( n = 3 \) and \( |F| = 2 \) then clearly \( AG(R) \) is regular with \( R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \). Now if \( n = 1 \), then \( R \) is a finite local ring or a field and clearly \( AG(R) \) is regular. For \( n = 2 \), we have \( AG(R) = \Gamma(R) \) by [7, Theorem 3.6] and for \( \Gamma(R) \) to be regular we must have \( R = F \times F \) by [5, Theorem 8] and so \( \Gamma(R) = K_{1-t-1} \).

In the following proposition we find the domination number of \( AG(R) \) graph.

**Proposition 2.6.** If \( R \) is a finite ring then \( \gamma(AG(R)) \leq 2 \).

**Proof.** Let us consider first that \( R \) is a decomposable ring with \( R \cong R_1 \times R_2 \). Now let us consider the sets \( A = \{ (x_1, 0) \mid x_1 \in R_1 \} \), \( B = \{ (0, x_2) \mid x_2 \in R_2 \} \), \( C = \{ (x_1, x_2) \mid x_1 \in Z(R_1)^*, x_2 \in R_2 \} \) and \( D = \{ (x_1, x_2) \mid x_1 \in R_1^*, x_2 \in Z(R_2)^* \} \). Then \( Z(R)^* = A \cup B \cup C \cup D \). Now we consider two vertices \( x = (1, 0) \in Z(R)^* \) and \( y = (0, 1) \in Z(R)^* \) of \( AG(R) \). Let \( z = (z_1, z_2) \in Z(R) \). If \( z_1 \in U(R_1) \) then clearly \( z \) cannot be adjacent to \( x \). Hence \( z \) is adjacent to \( x \) if \( z_1 \in Z(R_1) \) and similarly \( z \) is adjacent to \( y \) if \( z_2 \in Z(R_2) \). Now \( xz = (z_1, 0) \), \( \text{ann}(x) = B \cup \{ (0, 0) \} \), \( \text{ann}(xz) = \text{ann}(z_1, 0) = B \cup \{ (q, t) \mid q \in \text{ann}(z_1), t \in R_2 \} \). If \( z_2 \in U(R_2) \) then \( \text{ann}(z) = \{ (q, 0) \mid q \in \text{ann}(z_1) \} \) and if \( z_2 \in Z(R_2)^* \) then \( \text{ann}(z) = \{ (q_1, q_2) \mid q_1 \in \text{ann}(z_1), q_2 \in \text{ann}(z_2) \} \). Thus in all the cases we get \( \text{ann}(xz) \neq \text{ann}(x) \cup \text{ann}(z) \) and so \( x \) is adjacent to \( z \). Hence we get \( \text{Nbd}(x) = B \cup C \) and similarly we get \( \text{Nbd}(y) = A \cup D \). Therefore we have \( \text{Nbd}(x) \cup \text{Nbd}(y) = Z(R)^* \). Now for \( 1 \neq y_k \in U(R_2) \), we have \( (0, y_k) \in \text{Nbd}(x) \) but \( (0, y_k) \notin \text{Nbd}(y) \). Similarly if \( x_k \in U(R_1) \) then \( (x_k, 0) \in \text{Nbd}(y) \) but \( (x_k, 0) \notin \text{Nbd}(x) \). Thus if we take \( S = \{ x, y \} \), then \( S \) is a dominating set of \( AG(R) \). Hence for any finite commutative ring we have \( \gamma(AG(R)) \leq 2 \).

From propositions 2.2, 2.3 and 2.6, we have the following corollary.

**Corollary 2.7.** If \( R \cong R_1 \times R_2 \times \cdots \times R_n \), where each \( R_i \) are finite local ring but not fields or \( R \cong \mathbb{Z}_2 \times F \), then \( \gamma(AG(R)) = 1 \).

Next we find the criterion for the existence of a cut vertex in \( AG(R) \) graph.

**Proposition 2.8.** Let \( R \) be a finite ring such that \( AG(R) \) has a cut vertex. Then \( R \cong \mathbb{Z}_2 \times F \), where \( F \) is a finite field and \( F \notin \mathbb{Z}_2 \).

**Proof.** Let \( x \in Z(R)^* \) be a cut vertex of \( AG(R) \). Clearly \( AG(R) \) cannot be a complete graph and so \( \text{diam}(AG(R)) = 2 \). Now we have, \( AG(R) = X \cup Y \), where \( X \cap Y = \{ x \} \). As \( x \) is a cut vertex and \( \text{diam}(AG(R)) = 2 \), there exist \( a \in X \) and \( b \in Y \) which are adjacent to \( x \). So \( a - x - b \) is a path from \( a \) to \( b \) in \( AG(R) \). Now let \( c \in X \), such that \( c \) is not adjacent to \( x \) in \( AG(R) \) and as \( \text{diam}(AG(R)) = 2 \), so we have either \( c \) is adjacent to \( b \) or there exists a path \( c - d - b \) in \( AG(R) \) where \( d \neq x \). In either case we get that \( x \) is not a cut vertex of \( AG(R) \), which is a contradiction. Hence any vertex in \( X \setminus \{ x \} \) is adjacent to \( x \). Similarly any vertex in \( Y \setminus \{ x \} \) is adjacent to \( x \). Thus \( x \) is a vertex which is adjacent
to all vertices of $AG(R)$. Hence by propositions 2.2 and 2.3 either $R \cong \mathbb{Z}_2 \times \mathbb{F}$ where $\mathbb{F}$ is a finite field or $R \cong R_1 \times R_2 \times \cdots \times R_n$, where each $R_i$ are finite local ring but not field. If $R \cong R_1 \times R_2 \times \cdots \times R_n$ and if atleast one of $R_i$ is such that $|Z(R_i)| \geq 2$ then $AG(R)$ does not have a cut vertex, which is a contradiction. Hence for each $R_i$ we have $|Z(R_i)| = 1$. But when $|Z(R_i)| = 1$ we have either $R_i \cong \mathbb{Z}_4$ or $R_i \cong \mathbb{Z}_2[t]/(t^2)$ [1, Example 2.1(i)]. So $R \cong R_1 \times R_2 \times \cdots \times R_n$ where either $R_i \cong \mathbb{Z}_4$ or $R_i \cong \mathbb{Z}_2[t]/(t^2)$. Let $y = (y_1, y_2, \ldots, y_n)$ where $y_i = 2$ if $R_i \cong \mathbb{Z}_4$ and $y_i = t$ if $R_i \cong \mathbb{Z}_2[t]/(t^2)$. Here $y$ is adjacent to all vertices of $AG(R)$. Now let us consider the vertices $w = (0, y_2, \ldots, y_n)$ and $z = (y_1, \ldots, y_{n-1}, 0)$. Then the vertices which not adjacent to $z$ are the elements of the set $S = \{u = (u_1, u_2, \ldots, u_n) | u_i \in U(R_i)$ for $i = 1, 2, \ldots, n - 1$ and $u_n \in Z(R_n)\}$ and the vertices which are not adjacent to $w$ are the elements of the set $S' = \{v = (v_1, v_2, \ldots, v_n) | v_1 \in Z(R_1)$ and $v_i \in U(R_i)$ for $i = 2, \ldots, n\}$. But $z$ is adjacent to each element of $S'$ and similarly $w$ is adjacent to each element of $S$. So the subgraph of the annihilator graph whose set of vertices is $Z(R)^* \setminus \{y\}$ is still a connected graph which shows that $y$ is not a cut vertex of $AG(R)$. Hence $AG(R)$ does not have any cut vertex which is a contradiction. So $AG(R)$ has a cut vertex if $R \cong \mathbb{Z}_2 \times \mathbb{F}$, where $\mathbb{F} \not\cong \mathbb{Z}_2$, for if $\mathbb{F} \cong \mathbb{Z}_2$ then $AG(R)$ is complete graph and a complete graph does not have a cut vertex. \hfill \qed

In the following two propositions we find the independence number of $AG(R)$ graph for certain classes of finite rings.

**Proposition 2.9.** Let $R$ be a finite reduced ring not a field such that $R \cong \mathbb{F}_1 \times \mathbb{F}_2 \times \cdots \times \mathbb{F}_n$, where each $\mathbb{F}_i$ are finite field, such that $|\mathbb{F}_1| \geq |\mathbb{F}_2| \geq |\mathbb{F}_3| \geq \cdots \geq |\mathbb{F}_n|$ then $\alpha(AG(R)) = |\mathbb{F}_1^*| + |\mathbb{F}_2^*||\mathbb{F}_3^*| + \cdots + |\mathbb{F}_n^*||\mathbb{F}_{n-1}^*|.$

**Proof.** As $R$ is a finite reduced ring, $R \cong \mathbb{F}_1 \times \mathbb{F}_2 \times \cdots \times \mathbb{F}_n$. Consider the set $S_1 = \{(x_1, \ldots, x_n) | x_i = 0$ for all but one $i, 1 \leq i \leq n\}$. The independent subsets of $S_1$ are $S_{1i} = \{(x_1, 0, \ldots, 0) | x_1 \in \mathbb{F}_1^*\}, \ldots, S_{1n} = \{(0, 0, \ldots, x_n) | x_n \in \mathbb{F}_n^*\}$. Among these independent sets, the one with maximum number of elements is $S_{1i} = |\mathbb{F}_1| \geq |\mathbb{F}_i| \forall i, 1 \leq i \leq n$. Consider the set $S_2 = \{(x_1, \ldots, x_n) | x_i = 0$ for all but 2 $i, 1 \leq i \leq n\}$. The maximal independent subset of $S_2$ is $S_{12} = \{(x_1, x_2, 0, \ldots, 0) | x_i \in \mathbb{F}_i^*, i = 1, 2\}$. Continuing in this way we get the maximal independent subset of $S_{n-1}$ is $S_{1(n-1)}$. Let $S' = S_{11} \cup S_{12} \cup \ldots \cup S_{1(n-1)}$. Clearly each pair of elements in $S'$ are nonadjacent. Also for any element $x \in Z(R)^*$ either it belong to $S'$ or there exist an element $y \in S'$ such that $x$ is adjacent to $y$. Hence we have $\alpha(AG(R)) = |\mathbb{F}_1^*| + |\mathbb{F}_2^*||\mathbb{F}_3^*| + \cdots + |\mathbb{F}_n^*||\mathbb{F}_{n-1}^*|.$ \hfill \qed

**Proposition 2.10.** Let $R$ be a finite ring such that $R \cong R_1 \times R_2 \times \cdots \times R_n$ where each $R_i$ are local ring and $|U(R_1)| \geq |U(R_2)| \geq \cdots \geq |U(R_n)|$ then $\alpha(AG(R)) = |U(R_1)| + |U(R_1)||U(R_2)| + \cdots + |U(R_1)||U(R_2)||U(R_{n-1})| + 2.$

**Proof.** Let $S_1 = \{U(R_1) \times 0 \times \cdots \times 0, 0 \times U(R_2) \times 0 \times \cdots \times 0, 0 \times 0 \times \cdots \times 0 \times U(R_n)\}$. Then each element of $S_1$ form an independent set of $AG(R)$ and the maximal among these independent sets is $A_1 = U(R_1) \times 0 \times \cdots \times 0$. Also in the set $S_2 = \{U(R_1) \times U(R_2) \times 0 \times \cdots \times 0, U(R_1) \times 0 \times U(R_3) \times 0 \times \cdots \times 0, 0 \times 0 \times \cdots \times 0 \times U(R_{n-1}) \times U(R_n)\}$ each element is an independent set of $AG(R)$ and the maximal among these independent sets is $A_2 = U(R_1) \times U(R_2) \times 0 \times \cdots \times 0$ since $|U(R_1)| \geq |U(R_2)| \geq \cdots \geq |U(R_{n-1})| \geq \cdots \geq |U(R_n)|.$
be a set of vertices in $AG$ where $t = 1$. Then any two distinct elements which has the same number of non-zero entries but not identical are $S_j = \{x, y\}$ is a maximal independent set of $AG(R)$. For if $z = (z_1, z_2, \ldots, z_n) \in Z(R)^* \setminus H'$, then at least one of $z_i$ must belong to $Z(R_i)$ for some $1 \leq i \leq n$; if $z_n \in Z(R_n)^*$ then clearly $z$ is adjacent to $y$ and if $z_i \in Z(R_i)^*$ for $1 \leq i \leq n - 1$ then clearly $z$ is adjacent to $x$. Also $x$ and $y$ are not adjacent. So $H'$ is disjoint and $H'$ is maximal and hence $\alpha(AG(R)) = |U(R_1)| + |U(R_1)||U(R_2)| + \ldots + |U(R_1)||U(R_2)||U(R_n-1)| + 2$. \hfill $\square$

We now derive the following lemma which will be needed to find the clique number of $AG(R)$ graph in the next proposition.

**Lemma 2.11.** If $R$ is a non-local ring with $R \cong R_1 \times R_2 \times \cdots \times R_n$, where each $R_i$ are local rings, then any two distinct elements which has the same number of non-zero entries but not identical are adjacent in $AG(R)$.

**Proof.** Let $x, y \in Z(R)^*$ be non identical vertices having exactly $i$ number of non-zero entries with $1 \leq i \leq n - 1$. So there exist at least one entry in $x$, say $j^{th}$ with $1 \leq j \leq n$, which is non-zero in $x$ but zero in $y$. If $xy = 0$ then clearly there exist an edge between $x$ and $y$ as $\text{ann}(xy) = R \neq \text{ann}(x) \cup \text{ann}(y) \subseteq Z(R)$. So we assume that $xy \neq 0$. As total number of zero entries are equal in $x$ and $y$, there exists another entry, say $k^{th}$, which is zero in $x$ but not in $y$ where $1 \leq k \leq n$ and $k \neq j$. Then $xy$ has less number of non-zero entries than in $x$ and $y$ with $j^{th}$ and $k^{th}$ entry zero. Now we consider $z = (0, \ldots, 0, 1, \ldots, 1, 0, \ldots, 0)$ with $1$ in $j^{th}$ and $k^{th}$ entry and $0$ in the remaining entries. Then $z \in \text{ann}(xy)$ but $z \notin \text{ann}(x) \cup \text{ann}(y)$. This shows that $\text{ann}(xy) \neq \text{ann}(x) \cup \text{ann}(y)$. Hence $x$ is adjacent to $y$ in $AG(R)$. \hfill $\square$

**Proposition 2.12.** If $R \cong \mathbb{F}_1 \times \mathbb{F}_2 \times \cdots \times \mathbb{F}_n$, where each $\mathbb{F}_i$'s are finite field, $\omega(AG(R)) = \left(\frac{n}{2^t}\right)$ if $n$ is odd or $\left(\frac{n}{2}\right)$ if $n$ is even.

**Proof.** We'll prove it by induction on $n$. If $n = 2$, then clearly $AG(R) \cong K_{m,n}$ which is a complete bipartite graph. Hence $\omega(AG(R)) = 2$. So result is true for $n = 2$. Now let us assume that result hold for $k$ less than $n$. Assume that $R \cong \mathbb{F}_1 \times \mathbb{F}_2 \times \cdots \times \mathbb{F}_n$, where each $\mathbb{F}_i$'s are finite field. Let $R' \cong \mathbb{F}_1 \times \mathbb{F}_2 \times \cdots \times \mathbb{F}_{n-1}$. Then by induction hypothesis we have $\omega(AG(R')) = \left(\frac{n-1}{2}\right)$ if $n$ is odd and $\left(\frac{n-1}{2}\right)$ if $n$ is even. Let $S = \{(x_1, x_2, \ldots, x_t, 0, \ldots, 0), (x_1, x_2, \ldots, x_{t-1}, 0, x_{t+1}, 0, \ldots, 0), \ldots, (0, 0, \ldots, x_{n-1}, \ldots, x_{n-1})\}$ be a set of vertices in $AG(R')$. Then clearly by lemma 2.11, $S$ is a complete subgraph of $AG(R')$ and $|S| = \left(\frac{n-1}{t}\right)$ where $t = \frac{n}{2}$ when $n$ is even and $\frac{n-1}{2}$ when $n$ is odd. Hence $S$ is a maximal complete subgraph of $AG(R')$. Now we extend $S$ into $S'$ in $AG(R)$ by adding elements $x_n \in \mathbb{F}_n^*$ in the $n^{th}$ co-ordinate of each element of $S$. Then $S'$ is also a complete subgraph of $AG(R)$ and $|S'| = |S'| = \left(\frac{n}{t}\right)$ where $t = \frac{n}{2}$ if $n$ is even and $\frac{n-1}{2}$ if $n$ is odd. Now we take $T$ to be set of elements in $V(AG(R'))$ which has $t + 1$ non-zero component entries. Then $T$ is a complete subgraph of $AG(R')$. Again we extend $T$ to $T'$ by adding zero element of $\mathbb{F}_n$ in the $n^{th}$ coordinate of each element of $T$. Then $T'$ is a complete
Corollary 2.16. Let $R$ be a finite local ring with $|R| = 2^m$ for some $m \geq 3$ then $AG(R)$ is an Eulerian graph.
Now we show that $AG(R)$ is Hamiltonian if $R \cong A \times A$ where $A$ is a finite local ring with identity.

**Proposition 2.17.** Let $R$ be a finite ring such that $R \cong A \times A$ where $A$ is a finite local ring with identity. Then $AG(R)$ is Hamiltonian.

**Proof.** First we consider $A$ a local ring but not a field. Let us consider the sets $A^* \times 0$, $0 \times A^*$, $A \times Z(A)^*$, $Z(A)^* \times A$. Then any non-zero zero divisors of $R$ must belong to either one of these sets. First we show that $Z(A)^* \times A$ or $A \times Z(A)^*$ is a complete subgraph of $AG(R)$. Let $x, y \in Z(A)^* \times A$ such that $x \neq y$, $x = (x_1, x_2)$ and $y = (y_1, y_2)$. If $x_1 \neq y_1$ then as $A$ is a finite local ring so $ann(x_1y_1) \neq ann(x_1) \cup ann(y_1)$ which shows that $x$ is adjacent to $y$. If $x_1 = y_1$ then $x_1^2 \neq x_1$ as $A$ is a finite local ring and $ann(x_1^2) \neq ann(x_1)$ as $Nil(A) = Z(A)$. Hence $x$ is adjacent to $y$. Therefore $Z(A)^* \times A$ and similarly $A \times Z(A)^*$ is a complete subgraph of $AG(R)$. As we can form a complete bipartite graph from the set of vertices $A^* \times 0$ and $0 \times A^*$, so there exist a path from $(0, 1)$ to $(1, 0)$ which passes through all the vertices of $A^* \times 0$ and $0 \times A^*$ exactly once and also connect $(1, 0)$ to one vertex of $Z(A)^* \times (A \setminus Z(A))$, $(0, 1)$ to one vertex of $(A \setminus Z(A)) \times Z(A)^*$ as $Z(A)^* \times Z(A)^*$ is a complete subgraph of $AG(R)$. So we get a cycle which passes through all the vertices of $AG(R)$ exactly once. Hence $AG(R)$ is a Hamiltonian graph. If $A$ is a field then $AG(R) \cong \Gamma(R) \cong K_{|A| - 1, |A| - 1}$ which is clearly Hamiltonian. □

3. Planarity of $AG(R)$

In this section we characterize the finite commutative rings whose annihilator graph $AG(R)$ is planar.

**Theorem 3.1.** (Kuratowski) A graph is planar if and only if it contain no sub-division heomomorphic to $K_5$ or $K_{3,3}$.

**Proposition 3.2.** Let $R$ be a non-local ring then $AG(R)$ is planar if $R$ is isomorphic to one of the following ring $\mathbb{Z}_2 \times \mathbb{Z}_4$, $\mathbb{Z}_2 \times \mathbb{Z}_2[x]/(x^2)$, $\mathbb{Z}_2 \times \mathbb{F}$, $\mathbb{Z}_3 \times \mathbb{Z}_4$, $\mathbb{Z}_3 \times \mathbb{Z}_2[x]/(x^2)$, $\mathbb{Z}_4 \times \mathbb{F}$.

**Proof.** Case 1: If $R \cong R_1 \times R_2 \times \cdots \times R_n$ and $n \geq 4$ then as $\Gamma(R)$ is non planar by S.Akbari et al. [3], $AG(R)$ is also non-planar.

Case 2: If $R \cong R_1 \times R_2 \times R_3$ where one of $|R_i| = 4$, then $\Gamma(R)$ is non-planar by S. Akbari et al. [3] and so is $AG(R)$. So let $|R_i| \leq 3$ for $i = 1, 2, 3$. If $R \cong Z_3 \times Z_3 \times Z_3$ then the subgraph formed by the vertices $\{(2, 0, 2), (1, 2, 0), (2, 1, 0), (2, 2, 0), (0, 0, 1), (0, 0, 2)\}$ contain $K_{3,3}$ and therefore $AG(R)$ is non planar. If $R \cong Z_3 \times Z_3 \times Z_2$ then the subgraph formed by the vertices $\{(1, 2, 0), (2, 1, 0), (1, 1, 0), (0, 2, 1), (0, 1, 1), (0, 0, 1)\}$, where $X = \{(1, 2, 0), (2, 1, 0), (1, 1, 0)\}$ and $Y = \{(0, 2, 1), (0, 1, 1), (0, 0, 1)\}$, contain $K_{3,3}$ as a subgraph and therefore $AG(R)$ is non planar. If $R \cong Z_2 \times Z_2 \times Z_2$ then clearly $AG(R)$ is planar. If $R \cong Z_2 \times Z_2 \times Z_3$ then the subgraph formed by the vertices $\{(0, 0, 0), (0, 1, 0), (0, 1, 1), (0, 2, 0), (1, 0, 0), (1, 0, 1), (1, 0, 2)\}$, where $X = \{(0, 0, 0), (0, 1, 0), (0, 1, 1), (0, 2, 0)\}$ and $Y = \{(1, 0, 2), (1, 0, 1), (1, 0, 0)\}$, contain $K_{3,3}$ as a subgraph and hence $AG(R)$ is non-planar.

Case 3: If $n = 2$ then $R \cong R_1 \times R_2$. If both $|R_1|$ and $|R_2|$ are not less than 4 then $K_{3,3}$ is a subgraph of $\Gamma(R)$ and so $AG(R)$ is non planar. So let atleast one of $R_i$, say $|R_1| \leq 3$. If $R_2$ such that $|Z(R_2)^*|$
\[ \geq 4 \text{ then } K_5 \text{ is a subgraph of } AG(R). \text{ Hence } AG(R) \text{ is non-planar. So } |Z(R_2)^*| \leq 3. \]

**SubCase 3.1:** If \( R_1 \cong \mathbb{Z}_2 \) and \(|Z(R_2)^*| \leq 3\). When \(|Z(R_2)^*| = 3\) then \( \Gamma(R_2) \cong K_{1,2} \) or \( K_3 \). If \( \Gamma(R_2) \cong K_{1,2} \) then \( R_2 \cong \mathbb{Z}_8 \) or \( \mathbb{Z}_2[x]/(x^2) \) or \( \mathbb{Z}_4[x]/(2x, x^2 - 2) \). If \( R \cong \mathbb{Z}_2 \times \mathbb{Z}_8 \) then \( Z(R) = \{(0,0), (0,1), (0,2), (0,3), (0,4), (0,5), (0,6), (0,7), (1,0), (1,2), (1,4), (1,6)\}. \] Now let \( X = \{(0,1), (0,3), (0,5), (0,7)\} \) and \( Y = \{(1,4), (1,2), (1,6)\}. \) As \( deg_{\Gamma(R)}(x,y) = 3 \) for \( x \in X \) and \( y \in Y \), by [7, lemma 2.1(5)] \( deg_{AG(R)}(x,y) = 1 \) and so \( K_{4,3} \) is a subgraph of \( AG(R) \) showing that \( AG(R) \) is non-planar. Similarly if \( R \cong \mathbb{Z}_2 \times (\mathbb{Z}_2[x]/(x^3)) \), \( Z(\mathbb{Z}_2 \times (\mathbb{Z}_2[x]/(x^3))) = \{(0,0), (0,1), (0,2), (0,3), (0,4), (0,5), (0,6), (0,7), (1,0), (1,2), (1,4), (1,6)\}. \] Now let \( X = \{(0,1), (0,3), (0,1+x), (0,3+x), (1,0), (1,2), (1,2+x)\} \) and \( Y = \{(1,x,2), (1,2), (1,0)\}. \) As \( x \in X \) and \( y \in Y \) \( deg_{\Gamma(R)}(x,y) = 3 \), \( deg_{AG(R)}(x,y) = 1 \) so \( K_{4,3} \) is a subgraph of \( AG(\mathbb{Z}_2 \times (\mathbb{Z}_2[x]/(2x, x^2 - 2))) \). Hence \( AG(\mathbb{Z}_2 \times (\mathbb{Z}_2[x]/(2x, x^2 - 2))) \) is non-planar.

If \( R_2 \) is such that \( Z(R_2) = \{0, x, y, z\} \) and \( xy = yz = xz = 0 \) then \( K_{3,3} \) is a subgraph of \( AG(R) \). Hence \( AG(R) \) is non-planar. We consider \(|Z(R_2)| \leq 3\).

If \(|Z(R_2)^*| = 2\) then \( R_2 \cong \mathbb{Z}_9 \) or \( \mathbb{Z}_3[x]/(x^2) \), \( Z(\mathbb{Z}_2 \times \mathbb{Z}_9) = \{(0,0), (0,1), (0,2), (0,3), (0,4), (0,5), (0,6), (0,7), (0,8), (1,0), (1,3), (1,6)\}. \] Now let \( X = \{(0,1), (0,2), (0,4), (0,5), (0,7)\} \), and \( Y = \{(1,0), (1,3), (1,6)\}. \) As \( deg_{\Gamma(R)}(x,y) = 3 \) for \( x \in X \) and \( y \in Y \), by [7, lemma 2.1(5)] \( deg_{AG(R)}(x,y) = 1 \) so \( K_{6,3} \) is a subgraph of \( AG(\mathbb{Z}_2 \times \mathbb{Z}_9) \). Hence \( AG(R) \) is non-planar. Similarly for \( \mathbb{Z}_2 \times \mathbb{Z}_9[x]/(x^2) \), \( AG(R) \) is non-planar.

If \(|Z(R_2)^*| = 1\) then \( R_2 \cong \mathbb{Z}_4 \) or \( \mathbb{Z}_2[x]/(x^2) \) and \( AG(R) \) is clearly planar.

If \(|Z(R_2)^*| = 0\) then \( R_2 \) is a field or an infinite integral domain and clearly \( AG(\mathbb{Z}_2 \times \mathbb{Z}_2) \cong K_{1,n} \) or \( K_{1,\infty} \) and so \( AG(R) \) is planar.

**SubCase 3.2:** Consider \( R_1 \cong \mathbb{Z}_3 \). If \(|Z(R_2)^*| = 3\), then by subcase 3.1 \( \Gamma(R_2) \cong K_{1,2} \) or \( K_3 \). In both the cases as \( AG(\mathbb{Z}_4 \times \mathbb{Z}_2) \) is a subgraph of \( AG(\mathbb{Z}_3 \times \mathbb{Z}_2) \), \( AG(\mathbb{Z}_3 \times \mathbb{Z}_2) \) is non-planar. \( \text{If } |Z(R_2)^*| = 2, \ R_2 \cong \mathbb{Z}_9 \) or \( \mathbb{Z}_3[x]/(x^2) \), as \( AG(\mathbb{Z}_2 \times \mathbb{Z}_9) \) is a subgraph of \( AG(\mathbb{Z}_3 \times \mathbb{Z}_9) \), \( AG(\mathbb{Z}_3 \times \mathbb{Z}_9) \) is non-planar. Similarly, \( AG(\mathbb{Z}_3 \times (\mathbb{Z}_3[x]/(x^2))) \) is non-planar as \( AG(\mathbb{Z}_2 \times (\mathbb{Z}_3[x]/(x^2))) \) is a subgraph. \( \text{If } |Z(R_2)^*| = 1 \text{ then } R_2 \cong \mathbb{Z}_4 \) or \( \mathbb{Z}_2[x]/(x^2). \text{ If } R \cong \mathbb{Z}_3 \times \mathbb{Z}_4, \ Z(\mathbb{Z}_3 \times \mathbb{Z}_4) = \{(0,0), (0,1), (0,2), (0,3), (1,0), (1,2), (2,0), (2,2)\}. \) Then clearly \( AG(\mathbb{Z}_3 \times \mathbb{Z}_4) \) is planar and similarly for \( \mathbb{Z}_3 \times (\mathbb{Z}_2[x]/(x^2)), \ AG(\mathbb{Z}_3 \times (\mathbb{Z}_2[x]/(x^2))) \) is planar. \( \text{If } |Z(R_2)^*| = 0 \text{ then } R_2 \text{ is either a field or integral domain. } AG(\mathbb{Z}_2 \times \mathbb{Z}_2) \cong K_{2,n-1} \text{ or } K_{2,\infty} \text{ if } R_2 \text{ is a field, otherwise it is a doubled star graph. In both the cases } AG(R) \text{ is planar.} \)

**Proposition 3.3.** If \( R \) is a local ring such that \( AG(R) \) is planar then \( R \) is isomorphic to one of the following \( \mathbb{Z}_4, \mathbb{Z}_2[x]/(x^2), \mathbb{Z}_8, \mathbb{Z}_2[x]/(x^3), \mathbb{Z}_2[x,y]/(x,y)^2, \mathbb{Z}_2[x,y]/(xy,y^2-x), \mathbb{Z}_9, \mathbb{Z}_3[x]/(x^2), \mathbb{Z}_{25}, \mathbb{Z}_5[x]/(x^2). \)

**Proof.** If \( R \) is a local ring such that \(|Z(R)^*| \geq 5\) then we have \( AG(R) \) is a non-planar graph as \( K_5 \) is a subgraph of \( AG(R) \). Therefore for a local ring \( R \), \( AG(R) \) is planar if and only if \( 1 \leq |Z(R)^*| \leq 3 \).
4. So the local ring for which $AG(R)$ is planar are the following: $\mathbb{Z}_4$, $\mathbb{Z}_2[x]/(x^2)$, $\mathbb{Z}_8$, $\mathbb{Z}_2[x]/(x^3)$, $\mathbb{Z}_2[x,y]/(x,y)^2$, $\mathbb{Z}_2[x,y]/(xy,y^2-x)$, $\mathbb{Z}_9$, $\mathbb{Z}_3[x]/(x^2)$, $\mathbb{Z}_{25}$, $\mathbb{Z}_5[x]/(x^2)$. □

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Sanghita Dutta
Department of Mathematics, North Eastern Hill University, Shillong - 793022, India
Email: sanghita22@gmail.com

Chanlemki Lanong
Department of Mathematics, North Eastern Hill University, Shillong - 793022, India
Email: lanongc@gmail.com