www.combinatorics.ir

Transactions on Combinatorics
ISSN (print): 2251-8657, ISSN (on-line): 2251-8665
Vol. 6 No. 1 (2017), pp. 47-54.

www.ui.ac.ir

ON THE SKEW SPECTRAL MOMENTS OF GRAPHS

FATEMEH TAGHVAEE AND GOLAMHOSSEIN FATH-TABAR*

Communicated by Dariush Kiani

Abstract. Let G be a simple graph, and G^σ be an oriented graph of G with the orientation σ and skew-adjacency matrix $S(G^\sigma)$. The $k-$th skew spectral moment of G^σ, denoted by $T_k(G^\sigma)$, is defined as $\sum_{i=1}^{n}(\lambda_i)^k$, where $\lambda_1, \lambda_2, \cdots, \lambda_n$ are the eigenvalues of G^σ. Suppose $G_1^{\sigma_1}$ and $G_2^{\sigma_2}$ are two digraphs. If there exists an integer k, $1 \le k \le n-1$, such that for each i, $0 \le i \le k-1$, $T_i(G_1^{\sigma_1}) = T_i(G_2^{\sigma_2})$ and $T_k(G_1^{\sigma_1}) < T_k(G_2^{\sigma_2})$ then we write $G_1^{\sigma_1} \prec_T G_2^{\sigma_2}$. In this paper, we determine some of the skew spectral moments of oriented graphs. Also we order some oriented unicyclic graphs with respect to skew spectral moment.

1. Introduction

Let G be a simple graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$, and let G^σ be an oriented graph of G with the orientation σ, which allocates to any edge of G a direction such that the induced graph G^σ becomes an oriented graph. Then G is called the underlying graph of G^σ. The skew-adjacency matrix of G^σ is the $n \times n$ matrix $S(G^\sigma) = [s_{ij}]$, where $s_{ij} = 1$ and $s_{ji} = -1$ if v_iv_j is an arc of G^σ, otherwise $s_{ij} = s_{ji} = 0$. Since S is skew-symmetric, iS is Hermitian and so all of the eigenvalues of iS are real. Thus the eigenvalues of S are 0 or pure imaginary and since characteristic polynomial of S has real coefficients, the eigenvalues occur in complex conjugate pairs, see [2] for details.

In [10] Shader et al. studied the relationship between the spectra of a graph G and the skew-spectra of an oriented graph G^σ of G. In [9] the authors characterized the coefficients of the characteristic polynomial of the skew-adjacency matrix of an oriented graph.

MSC(2010): Primary: 05C50; Secondary: 20D60.

Keywords: Oriented graph, skew spectral moment, skew eigenvalue, T-order, skew characteristic polynomial.

Received: 02 February 2016, Accepted: 08 September 2016.

*Corresponding author.

We use the notations P_n, C_n and S_n to denote a path, a cycle and a star with n vertices, respectively. Let F be a graph. An $F-$subgraph of G is a subgraph of G which is isomorphic to the graph F. Let $\varphi_G(F)$ be the number of all $F-$subgraphs of G.

Let G be a simple graph. A walk of length k in G is an alternating sequence $v_1, e_1, v_2, e_2, \ldots, v_k, e_k, v_{k+1}$ of vertices and edges such that for any $i = 1, 2, \ldots, k$ the vertices v_i and v_{i+1} are distinct end-vertices of the edge e_i. A closed walk is a walk in which the first and the last vertex are the same [5].

Let $S(G^\sigma) = [s_{ij}]$ be the skew adjacency matrix of an oriented graph G^σ and $W = v_1 v_2 \cdots v_k$(perhaps $v_i = v_j$ for $i \neq j$) be a walk from v_1 to v_k. The sign of W is defined as:

$$sgn(W) = \prod_{i=1}^{k-1} s_{v_i} s_{v_{i+1}}.$$

Let $W^{'}$ be the inverse walk of W obtained from W by replacing the ordering of vertices by its inverses, i.e., $W^{'} = v_k v_{k-1} \cdots v_1$. It is easy to check that $sgn(W^{'}) = sgn(W)$ if the length of such a walk is even and $sgn(W^{'}) = -sgn(W)$ otherwise.

In an oriented graph G^σ, an even cycle C is called evenly oriented if for either choice of direction of traversing arround C, the number of edges of C directed in the direction of traversal is even. Otherwise C is called oddly oriented. Throughout this paper we denote by C_n^+ an evenly oriented even cycle and C_n^- an oddly oriented even cycle with n vertices. Moreover, let $w_{ij}^+(k)$ and $w_{ij}^-(k)$ denote the number of all positive walks and negative walks starting from v_i and terminating at v_j with length k, respectively, see [3, 8] for more details.

Suppose that $A(G)$ is the adjacency matrix of graph G and $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of $A(G)$. For a simple graph G, the k-th spectral moment of G is defined as $\sum_{i=1}^{n} \lambda_i^k$, see [4, 14] for more information. In [11, 12, 13] the authors calculated the spectral moments of some graphs and they ordered these graphs with respect to the spectral moments. Also Wu and Liu [14] determined the last $\lfloor \frac{d}{2} \rfloor + 1$ graphs, among all n-vertex trees of diameter d $(4 \leq d \leq n-3)$ with respect to spectral moments. For more information on the spectral moments and additional results one can see [1, 6, 7].

In this paper we consider an orientation for the simple graph G and determine the skew spectral moments of oriented graph G^σ. Also we give an ordering for some oriented graphs with respect to the skew spectral moments.

Gong and Xu [8] obtained the following result on the relationship between the entries of S^k and the number of walks between any pair of ordered vertices.

Theorem 1.1. *Let S be the skew-adjacency matrix of an oriented graph G^σ and u and v be two arbitrary vertices of G^σ. Then*

$$(S^k)_{uv} = w_{uv}^+(k) - w_{uv}^-(k).$$

It is easy to see that for each odd k, $(S^k)_{uu} = 0$, see [8].

An oriented graph H is called a basic oriented graph if each component of H is K_2 or a cycle of even length. The characteristic polynomial of the oriented graph G^σ, which is called the skew characteristic polynomial of G^σ and denoted by $\chi(G^\sigma, \lambda)$, is defined as:

$$\chi(G^\sigma, \lambda) = det(\lambda I_n - S(G^\sigma)) = \sum_{i=0}^{n} (-1)^i s_i(G^\sigma) \lambda^{n-i}.$$

Hou and Lei in [9] proved the following result for the skew characteristic polynomial of the oriented graph G^σ.

Theorem 1.2. *Let G^σ be an oriented graph with n vertices and with skew characteristic polynomial*

$$\chi(G^\sigma,\lambda)=\sum_{i=0}^{n}(-1)^i s_i(G^\sigma)\lambda^{n-i}=\lambda^n-s_1\lambda^{n-1}+s_2\lambda^{n-2}+\cdots+(-1)^{n-1}s_{n-1}\lambda+(-1)^n s_n.$$

Then $s_i=0$ if i is odd and

$$s_i=\sum_{\mathcal{H}}(-1)^{c^+(\mathcal{H})}2^{c(\mathcal{H})},$$

if i if even, where the summation goes over all basic oriented graphs $\mathcal{H}$ of G^σ with i vertices and $c^+(\mathcal{H})$ and $c(\mathcal{H})$ are the number of evenly oriented even cycles and even cycles contained in $\mathcal{H}$, respectively.

In the following we express the Newtonś identity for the coefficients of characteristic polynomial of a graph G and at the end of this paper by using of this Theorem we determine the characteristic polynomial of an oriented graph G^σ.

Theorem 1.3. *(Newtonś identity) Let $\lambda_1,\ldots,\lambda_n$ be the roots of the polynomial*

$$\chi(G,\lambda)=\lambda^n-s_1\lambda^{n-1}+s_2\lambda^{n-2}+\cdots+(-1)^{n-1}s_{n-1}\lambda+(-1)^n s_n$$

with spectral moment T_k. Then the $k-$th coefficient of characteristic polynomial is equal to

$$s_k=\frac{-1}{k}(T_k+T_{k-1}s_1+\cdots+T_1 s_{k-1}).$$

2. Main Results

In this section, we first define the concept of the spectral moments of oriented graph G^σ and we call the skew spectral moment of G^σ. Then we determine the $k-$th skew spectral moments of oriented graphs for $k=0,1,2,3,4,5,6$. Also we consider an ordering of the oriented graphs and we give an ordering for the oriented unicyclic graphs. Let $\lambda_1,\lambda_2,\ldots,\lambda_n$ be the eigenvalues of S, where each of the λ_i, $(1\le i\le n)$, is 0 or pure imaginary. The number of $\sum_{i=1}^{n}(\lambda_i)^k$ is called the k-th skew spectral moment of G^σ, denoted by $T_k(G^\sigma)$.

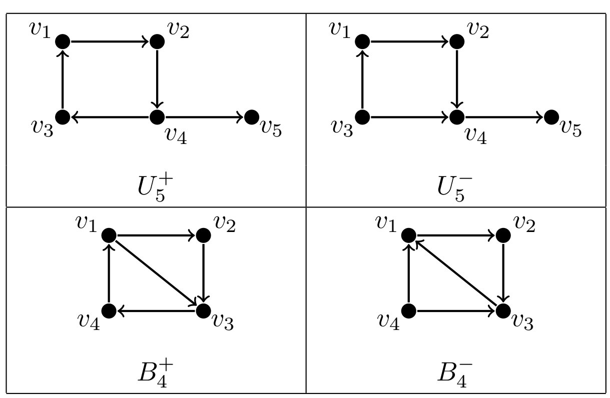

FIGURE 1. The oriented graphs U_5^σ and B_4^σ.

Lemma 2.1. *The $k-$th skew spectral moment of G^σ is the number of closed walks with positive sign of length k minus closed walks with negative sign of length k.*

Proof. Since iS is a Hermitian matrix and S is diagonalizable matrix over the complex filed $\mathbb{C}$, $tr(S^k)$ is equal to $k-$th skew spectral moment of G^σ. On the other hand by Theorem 1.1, $tr(S^k)$ is the number of closed walks with positive sign of length k minus closed walks with negative sign of length k. □

Let $T(G^\sigma) = (T_0(G^\sigma), T_1(G^\sigma), \ldots, T_{n-1}(G^\sigma))$ be the sequence of skew spectral moments of G^σ. For two oriented graphs $G_1^{\sigma_1}$ and $G_2^{\sigma_2}$ we shall write $G_1^{\sigma_1} =_T G_2^{\sigma_2}$ if $T_i(G_1^{\sigma_1}) = T_i(G_2^{\sigma_2})$ for $i = 0, 1, \ldots, n-1$. Similarly, we have $G_1^{\sigma_1} \prec_T G_2^{\sigma_2}$ if for some k, $k = 1, 2, \cdots, n-1$, $T_i(G_1^{\sigma_1}) = T_i(G_2^{\sigma_2})$ $(i = 0, 1, \ldots, k-1)$ and $T_k(G_1^{\sigma_1}) < T_k(G_2^{\sigma_2})$. We shall also write $G_1^{\sigma_1} \preceq_T G_2^{\sigma_2}$ if $G_1^{\sigma_1} \prec_T G_2^{\sigma_2}$ or $G_1^{\sigma_1} =_T G_2^{\sigma_2}$.

It is easy to see that $T_0(G^\sigma) = n$ and if k is odd then $T_k(G^\sigma) = 0$. In the following we compute the skew spectral moments of the oriented graph G^σ, $T_k(G^\sigma)$, for $k = 2, 4, 6$.

Theorem 2.2. *Let G^σ be an oriented graph. Then we have:*

(1) $T_2(G^\sigma) = -2m$,

(2) $T_4(G^\sigma) = 2\varphi(P_2) + 4\varphi(P_3) + 8(\varphi(C_4^+) - \varphi(C_4^-))$,

(3) $T_6(G^\sigma) = -2\varphi(P_2) - 12\varphi(P_3) - 6\varphi(P_4) - 12\varphi(K_{1,3}) + 12(\varphi(U_5^-) - \varphi(U_5^+)) + 12(\varphi(B_4^-) - \varphi(B_4^+)) - 12\varphi(C_3) + 48(\varphi(C_4^-) - \varphi(C_4^+)) + 12\varphi((C_6^+) - \varphi(C_6^-))$,

where the subgraphs B_4^+, B_4^-, U_5^+, U_5^- is shown in figure 1.

Proof. We have $T_2(G^\sigma) = tr(S^2) =$ the number of closed walks of length 2. For the any adjacent vertex to v_i, there is a closed walk of length 2. It is easy to see that all of such walks are negative and so $w_{ii}^+(2) = 0$. This implies that $T_2(G^\sigma) = -2m$.

Now we compute the 4-th skew spectral moment of G^σ. We have $T_4(G^\sigma) = tr(S^4) =$ the number of positive closed walks minus negative closed walks of length 4. Vertices that belong to a closed walk of length 4 induce in G a subgraph isomorphic to P_2, P_3 or C_4. For any edge $e = uv$ there are two closed walk with length 4 of the forms $u - v - u - v - u$ and $v - u - v - u - v$. It is easy to see that the sign of such walks is positive. Now we consider a cycle graph C_4. For any vertex of C_4 there are two closed walks of length 4. If C_4 is oddly oriented, then the sign of each of these closed walks is negative. While if C_4 is evenly oriented, then the sign of each of these closed walks is positive. Thus for any square the number of closed walks of length 4 is equal to $8(\varphi(C_4^+) - \varphi(C_4^-))$.

Now consider a path P_3. One can see that there are four closed walks with length 4 in P_3. It is easy to see that the sign such walks is positive, for any orientation of P_3. Thus the 4$-$th skew spectral moment of G^σ is equal to $2\varphi(P_2) + 4\varphi(P_3) + 8(\varphi(C_4^+) - \varphi(C_4^-))$.

The vertices that belong to a closed walk of length 6 induce in G a subgraph isomorphic to P_2, P_3, P_4, S_4, U_5, B_4, B_5, C_3, C_4 or C_6. With a simple check one can see that

$$\begin{aligned} T_6(G^\sigma) &= -2\varphi(P_2) - 12\varphi(P_3) - 6\varphi(P_4) - 12\varphi(K_{1,3}) + 12(\varphi(U_5^-) - \varphi(U_5^+)) + 12(\varphi(B_4^-) - \varphi(B_4^+)) \\ &- 12\varphi(C_3) + 48(\varphi(C_4^-) - \varphi(C_4^+)) + 12(\varphi(C_6^+) - \varphi(C_6^-)), \end{aligned}$$

where the oriented graphs U_5^σ and B_4^σ is depicted in figure 1. In the graph U_5^+, the cycle with length 4, is evenly oriented, while in the graph U_5^-, the cycle with length 4, is oddly oriented. Also in the graph B_4^+ (B_4^-) the cycle with length 4, is evenly (oddly) oriented and the orientation of the edge v_1v_3 is not effective. □

In the following we consider the set of oriented trees and the set of oriented unicyclic graphs with n vertices and we order these oriented graphs with respect to the skew spectral moments.

Corollary 2.3. *Suppose that T_n^σ is an oriented tree with n vertices. Then in an $T-$order we have:*

(1) $T_n^\sigma \preceq_T S_n^\sigma$ *and* $T_n^\sigma =_T S_n^\sigma$ *if and only if* $T_n^\sigma \cong S_n^\sigma$.
(2) $P_n^\sigma \preceq_T T_n^\sigma$ *and* $T_n^\sigma =_T P_n^\sigma$ *if and only if* $T_n^\sigma \cong P_n^\sigma$.

Proof. For $i = 0, 1, 2, 3$, we have $T_i(T_n^\sigma) = T_i(S_n^\sigma) = T_i(P_n^\sigma)$. Thus we consider the 4−th skew spectral moment. Theorem 2 of [4] states that, $\varphi(P_3)$ is minimal in P_n and maximal in S_n, and this completes the proof. □

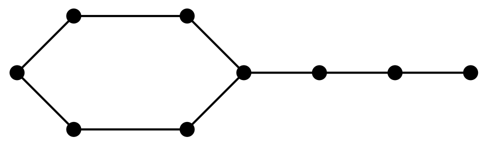

FIGURE 2. The lollipop graph $L(6, 3)$.

Now we consider the set of all unicyclic graphs with an orientation and we give an ordering for these graphs with respect to skew spectral moment. Let $U_{n,k}$ be the set of unicyclic graphs with n vertices ($n \geq 8$) and girth k, $k \geq 5$. Also let $L(k, f)$ be the graph obtained by the coalescence of a cycle C_k of length k with a path P_{f+1} at one of its end vertices. Let $F(k, f)$ be the graph obtained by the coalescence of C_k and a star $K_{1,f}$ at its central vertex. Then we have the following Theorem.

Theorem 2.4. *Let σ be an orientation for $U_{n,k}$. In an $T-$order of the set of $U^\sigma_{n,k}$, the first oriented graph is $L^\sigma(k, f)$ and the last oriented graph is $F^\sigma(k, f)$.*

Proof. It is easy to see that $T_i(L^\sigma(k, f)) = T_i(F^\sigma(k, f))$, for $i = 0, 1, 2, 3$. Thus we have to calculate the T_4. In $L^\sigma(k, f)$ we have:

$$\varphi(P_3) = \sum_{i=1}^{n} \binom{d_i}{2} = f + k + 1$$

and in $F^\sigma(k, f)$ we have:

$$\varphi(P_3) = \sum_{i=1}^{n} \binom{d_i}{2} = \frac{1}{2}(f^2 + 3f + 2k).$$

It is easy to prove that the minimal and the maximal value of P_3 occur in $L^\sigma(k, f)$ and $F^\sigma(k, f)$, respectively and this completes the proof. □

The graphs of the form $L(k, f)$ is called a lollipop graph, for example $L(6, 3)$ is shown in figure 2. In the following we consider an orientation σ for the set of all lollipop graphs with n vertices and we find the skew spectral moments of these oriented graphs. In a lollipop graph $L(k, f)$ with n vertices we have $f + k = n$. Since a lollipop graph is a unicyclic graph, then the number of edges in $L(k, f)$ is equal to $f + k = n$. Now let $\mathcal{L}^\sigma(k, f)$ be the set of all lollipop graphs with n vertices. Suppose that σ is an orientation for these graphs. Consider $\mathcal{L}^\sigma(k, f) = \{L^\sigma(k, f) : \; k, f \geqslant 3, \; f + k = n, \; n \geq 9\}$.

Now we give an ordering for all graphs of $\mathcal{L}^\sigma(k, f)$ with $f + k = n$ vertices.

Theorem 2.5. *For any oriented graph $G^\sigma \in \mathcal{L}^\sigma(k, f) \setminus L^\sigma(4, f)$ we have:*

$$L^-(4, f) \prec_T G^\sigma \prec_T L^+(4, f).$$

Proof. It is easy to see that for $i = 0, 1, 2, 3$, $T_i(L^-(4, f)) = T_i(G^\sigma) = T_i(L^+(4, f))$. Also by using of Theorem (2.2) we have:

$$\begin{aligned} T_4(L^-(4, f)) &= 2n + 4(3 + (n - 2)) - 8 = 6n - 4, \\ T_4(G^\sigma) &= 2n + 4(3 + (n - 2)) = 6n + 4, \\ T_4(L^+(4, f)) &= 2n + 4(3 + (n - 2)) + 8 = 6n + 12. \end{aligned}$$

Thus $T_4(L^-(4, f)) < T_4(G^\sigma) < T_4(L^+(4, f))$. Therefore

$$L^-(4, f) \prec_T G^\sigma \prec_T L^+(4, f).$$

□

Theorem 2.6. *For any oriented graph* $G^\sigma \in \mathcal{L}^\sigma(k, f) \setminus \{L^\sigma(4, f), L^\sigma(6, f), L^\sigma(3, f)\}$ *we have:*

$$L^-(6, f) \prec_T G^\sigma \prec_T L^\sigma(3, f) \prec_T L^+(6, f).$$

Proof. It is easy to see that $T_i(G^\sigma) = T_i(L^-(6, f)) = T_i(L^+(6, f)) = T_i(L^\sigma(3, f))$ for $i = 0, 1, 2, 3, 4, 5$. By Theorem (2.2) we have:

$$\begin{aligned} T_6(L^+(6, f)) &= -2n - 12(n + 1) - 6(n + 2) - 12 + 12 = -20n - 24, \\ T_6(L^-(6, f)) &= -2n - 12(n + 1) - 6(n + 2) - 12 - 12 = -20n - 48, \end{aligned}$$

and

$$T_6(L^\sigma(k, f)) = \begin{cases} -20n - 30 & k = 3 \\ -20n - 36 & \textit{Otherwise.} \end{cases}$$

Thus $T_6(L^-(6, f)) < T_6(L^\sigma(k, f)) < T_6(L^\sigma(3, f)) < T_6(L^+(6, f))$. Therefore

$$L^-(6, f) \prec_T L^\sigma(k, f) \prec_T L^\sigma(3, f) \prec_T L^+(6, f).$$

□

In the following we consider the skew characteristic polynomial of G^σ and we determine the coefficients of $\chi(G^\sigma, \lambda)$. It is clear that $s_k = 0$ if k is odd.

Theorem 2.7. *Let* G^σ *be an oriented graph with* n *vertices and* m *edges and skew characteristic polynomial* $\chi(G^\sigma, \lambda)$*. Then the* k*-th coefficient of* $\chi(G^\sigma, \lambda)$ *for* $k = 2, 4, 6$ *is as the following:*

(1) $s_2(G^\sigma) = m$,

(2) $s_4(G^\sigma) = \frac{1}{2}(m^2 - m) - \sum_{i=1}^{n} \binom{d_i}{2} - 2(\varphi(C_4^+) - \varphi(C_4^-))$,

(3) $s_6(G^\sigma) = \frac{1}{3}m - \frac{1}{2}m^2 + \frac{1}{6}m^3 - (m-2)\sum_{i=1}^{n} \binom{d_i}{2} + \varphi(P_4) + 2\sum_{i=1}^{n} \binom{d_i}{3} - 2(\varphi(U_5^-) - \varphi(U_5^+)) - 2(\varphi(B_4^-) - \varphi(B_4^+)) + 2\varphi(C_3) - (2m - 8)(\varphi(C_4^+) - \varphi(C_4^-)) - 2(\varphi(C_6^+) - \varphi(C_6^-))$.

Proof. By using of Theorem 1.3 we have

$$s_k = \frac{-1}{k}(T_k + T_{k-1}s_1 + \cdots + T_1 s_{k-1}),$$

where T_k is $k-$th skew spectral moment of G^σ. Notice that if $k = 2$, then

$$s_2(G^\sigma) = \frac{-1}{2}(T_2) = \frac{-1}{2}(-2m) = m.$$

If $k = 4$, then

$$\begin{aligned} s_4(G^\sigma) &= \frac{-1}{4}(T_4 + T_2 s_2) = \frac{-1}{4}(2m + 4\sum_{i=1}^{n}\binom{d_i}{2} - 2m^2 + 8(\varphi(C_4^+) - \varphi(C_4^-)) \\ &= \frac{1}{2}(m^2 - m) - \sum_{i=1}^{n}\binom{d_i}{2} - 2(\varphi(C_4^+) - \varphi(C_4^-)). \end{aligned}$$

Now assume that $k = 6$, then

$$\begin{aligned} s_6(G^\sigma) &= \frac{-1}{6}(T_6 + T_4 s_2 + T_2 s_4) = \frac{-1}{6}[-2\varphi(P_2) - 12\varphi(P_3) - 6\varphi(P_4) - 12\varphi(K_{1,3}) \\ &+ 12(\varphi(U_5^-) - \varphi(U_5^+)) + 12(\varphi(B_4^-) - \varphi(B_4^+)) - 12\varphi(C_3) + 48(\varphi(C_4^-) - \varphi(C_4^+)) \\ &+ 12(\varphi(C_6^+) - \varphi(C_6^-)) + m(2\varphi(P_2) + 4\varphi(P_3) + 8(\varphi(C_4^+) - \varphi(C_4^-))) - 2m(\frac{1}{2}(m^2 - m) \\ &- \sum_{i=1}^{n}\binom{d_i}{2} - 2(\varphi(C_4^+) - \varphi(C_4^-)))] \\ &= \frac{-1}{6}[-2m + (6m - 12)\sum_{i=1}^{n}\binom{d_i}{2} - 6\varphi(P_4) - 12\sum_{i=1}^{n}\binom{d_i}{3} \\ &+ 12(\varphi(U_5^-) - \varphi(U_5^+)) + 12(\varphi(B_4^-) - \varphi(B_4^+)) - 12\varphi(C_3) \\ &+ (12m - 48)(\varphi(C_4^+) - \varphi(C_4^-)) - 12(\varphi(C_6^+) - \varphi(C_6^-)) + 3m^2 - m^3] \\ &= \frac{1}{3}m - \frac{1}{2}m^2 + \frac{1}{6}m^3 - (m-2)\sum_{i=1}^{n}\binom{d_i}{2} + \varphi(P_4) + 2\sum_{i=1}^{n}\binom{d_i}{3} - 2(\varphi(U_5^-) - \varphi(U_5^+)) \\ &- 2(\varphi(B_4^-) - \varphi(B_4^+)) + 2\varphi(C_3) - (2m - 8)(\varphi(C_4^+) - \varphi(C_4^-)) - 2(\varphi(C_6^+) - \varphi(C_6^-)). \end{aligned}$$

□

Corollary 2.8. *Let G be an $r-$regular graph with n vertices and m edges and σ be an orientation for G. Then the coefficients of the skew characteristic polynomial of G^σ are:*

(1) $s_2(G^\sigma) = \frac{nr}{2}$,
(2) $s_4(G^\sigma) = \frac{1}{8}n^2r^2 + \frac{1}{4}nr - \frac{1}{2}nr^2 + 2(\varphi(C_4^+) - \varphi(C_4^-)))$,
(3) $s_6(G^\sigma) = \frac{1}{2}nr + \frac{1}{8}n^2r^2 + \frac{1}{48}n^3r^3 - \frac{1}{4}n^2r^3 - nr^2 + \frac{2}{3}nr^3 + \varphi(P_4) - 2(\varphi(U_5^-) - \varphi(U_5^+)) - 2(\varphi(B_4^-) - \varphi(B_4^+)) + 2\varphi(C_3) - (nr - 8)(\varphi(C_4^+) - \varphi(C_4^-)) - 2(\varphi(C_6^+) - \varphi(C_6^-))$.

Acknowledgement

The research of this paper is partially supported by the University of Kashan under grant no 504631/10.

References

[1] A. R. Ashrafi and G. H. Fath-Tabar, Bounds on the Estrada index of ISR (4, 6)-fullerenes, *Appl. Math. Lett.*, **24** (2011) 337–339.

[2] M. Cavers, S. M. Cioabă, S. Fallat, D. A. Gregory, W. H. Haemers, S. j. Kirkland, J. J. McDonald and M. Tsatsomeros, Skew adjacency matrices of graphs, *Linear Algebra Appl.*, **436** (2012) 5412–5429.

[3] X. Chen, X. Li and H. Lian, 4-Regular oriented graphs with optimum skew energy, *Linear Algebra Appl.*, **439** (2013) 2948–2960.

[4] D. Cvetković and P. Rowlinson, Spectra of unicyclic graphs, *Graphs Combin.*, **3** (1987) 7–23.

[5] D. Cvetković, M. Doob and H. Sachs, *Spectra of Graphs-Theory and Applications*, **87**, Academic Press, New York, 1980.

[6] G. H. Fath-Tabar, A. R. Ashrafi and I. Gutman, Note on Estrada and $L-$Estrada indices of graphs, *Bull. Cl. Sci. Math. Nat. Sci. Math.*, **139** (2009) 1–16.

[7] G. H. Fath-Tabar, A. R. Ashrafi and D. Stevanović, Spectral Properties of Fullerenes, *J. Comput. Theor. Nanosci.*, **9** (2012) 327–329.

[8] S. Gong and G. Xu, 3-Regular digraphs with optimum skew energy, *Linear Algebra Appl.*, **436** (2012) 465–471.

[9] Y. Hou and T. Lei, Characteristic polynomials of skew-adjacency matrices of oriented graphs, *Electron. J. Combin.*, **18** (2011) 156–167.

[10] B. Shader and W. So, Skew spectra of oriented graphs, *Electron. J. Combin.*, **16** (2009) 1–6.

[11] F. Taghvaee and A. R. Ashrafi, Comparing fullerenes by spectral moments, *J. Nanosci. Nanotechnol.*, **16** (2016) 1–4.

[12] F. Taghvaee and G. H. Fath-Tabar, Signless Laplacian spectral moments of graphs and ordering some graphs with respect to them, *Alg. Struc. Appl.*, **1** (2014) 133–141.

[13] F. Taghvaee and G. H. Fath-Tabar, Relationship between coefficients of characteristic polynomial and matching polynomial of regular graphs and its applications, *Iranian J. Math. Chem.*, **8** (2017) 7–24.

[14] Y. P. Wu and H. Q. Liu, Lexicographical ordering by spectral moments of trees with a prescribed diameter, *Linear Algebra Appl.*, **433** (2010) 1707–1713.

Fatemeh Taghvaee

Department of pure Mathematics, Faculty of Mathematical Sciences, University of Kashan, 87317-51167, Kashan, Iran

Email: taghvaei19@yahoo.com

Golamhossein Fath-Tabar

Department of pure Mathematics, Faculty of Mathematical Sciences, University of Kashan, 87317-51167, Kashan, Iran

Email: fathtabar@kashanu.ac.ir