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NEW CLASS OF INTEGRAL BIPARTITE GRAPHS WITH LARGE DIAMETER

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Abstract. In this paper, we construct a new class of integral bipartite graphs (not necessarily trees) with large even diameters. In fact, for every finite set A of positive integers of size k we construct an integral bipartite graph G of diameter $2k$ such that the set of positive eigenvalues of G is exactly A. This class of integral bipartite graphs has never found before.

1. **Introduction**

Let G be a simple graph with the vertex set $\{v_1, v_2, \ldots, v_n\}$. The adjacency matrix of G is an $n \times n$ matrix $A(G)$ whose (i, j)-entry is 1 if v_i is adjacent to v_j and 0, otherwise. The characteristic polynomial of G, denoted by $f_G(x)$, is the characteristic polynomial of $A(G)$. We will write it simply f_G when there is no confusion. The roots of f_G are called the eigenvalues of G and can be ordered as $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. It is important to mention that the set of eigenvalues of a bipartite graph is symmetrical with respect to 0. An integral graph is a graph for which the eigenvalues of its adjacency matrix are all integers [5]. Many different classes of integral graphs have been constructed in the past decades. For a long time it has been an open question whether there exist integral trees of arbitrarily large diameter [6]. Csikvari in [2] constructed integral trees with arbitrary large even diameter. A very elegant and short proof of the integrality of Csikvari's trees can be found in page 90 of [1]. Ghorbani, Mohammadian, and Tayfeh-Rezaie constructed integral trees with arbitrary large odd diameter [4]. In this paper by a recursive method we construct new class of integral bipartite graphs (not necessarily trees) with arbitrary large even diameter. Let $A = \{n_1, n_2, \ldots, n_k\}$ be a set of positive integers such that $n_1 < n_2 < \cdots < n_k$. In Section 2 we give a recursive method to construct integral bipartite graph

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G with diameter $2k$ which the set of all positive eigenvalues of G is exactly A. If $n_1 > 1$, then the constructed bipartite graph will be not a tree. In Section 3 we prove the integrality of the constructed graph.

2. Recursive Method

We use the notation $K_{n,m}$ for complete bipartite graph with two partite sets having n and m vertices.

Definition 2.1. *Let G_1 and G_2 be rooted graphs with roots a and b, respectively. Then the graph $G_1 \sim G_2$ obtained from $G_1 \cup G_2$ by joining the vertices a and b. We obtain the $G_1 \sim mG_2$ by taking G_1 and m copies of G_2 and we join a to each copies of b. The root of G_1 is considered to be the root of $G_1 \sim G_2$.*

Let $G - a$ be the graph obtained from G by deleting the vertex a and all edges containing a. Then we have following lemma.

Lemma 2.2. [3] *Let G_1 and G_2 be rooted graphs with roots a and b, respectively, and $m \geq 1$. Then*

$$f_{G_1 \sim mG_2} = f_{G_2}^{m-1}(f_{G_1} f_{G_2} - m f_{G_1 - a} f_{G_2 - b}).$$

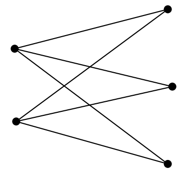
Let $A = \{n_1, n_2, \ldots, n_k\}$ be a set of positive integers which $n_1 < n_2 < \cdots < n_k$. Let a vertex in the part of size $n_1 + 1$ be the root of K_{n_1, n_1+1} and a vertex in the part of size 1 be the root of K_{1,n_1^2}. Define

$$\begin{aligned}
G_1 &= K_{n_1, n_1+1}, \\
G_2 &= G_1 \sim (n_2^2 - n_1^2 - n_1) K_{1,n_1^2}, \\
G_3 &= K_{1,n_1^2} \sim (n_3^2 - n_2^2) G_2, \\
G_4 &= G_2 \sim (n_4^2 - n_3^2) G_3, \\
&\vdots \\
G_k &= G_{k-2} \sim (n_k^2 - n_{k-1}^2) G_{k-1}.
\end{aligned}$$

It is esay to see that the diameter of G_k is $2k$. Clearly, if $n_1 = 1$, then all graphs constructed in the above are trees, and if $n_1 > 1$, then none of graphs constructed in the above is a tree. In Section 3 we prove that the set of positive eigenvalues of graph G_k is exactly A.

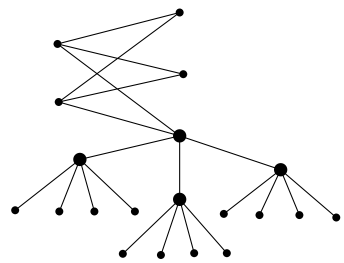
Example 2.3. *Let $A = \{2,\ 3,\ 4\}$. In the following we indicate the graphs G_1, G_2, and G_3 constructed in the above manner.*

$G_1 = K_{2,3}$



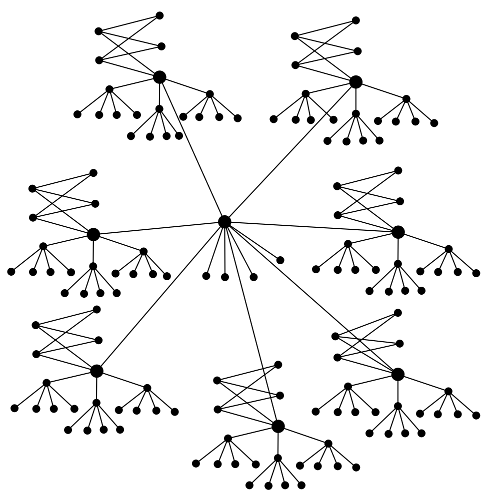
The set of the positive eigenvalues of G_1 *is* $\{\sqrt{6}\}$.

$G_2 = G_1 \sim 3K_{1,4}$



The set of the positive eigenvalues of G_2 *is* $\{2, 3\}$.

$G_3 = K_{1,4} \sim 7G_2$



The set of the positive eigenvalues of G_3 *is* $\{2, 3, 4\}$.

3. Proof Of The Integrality Of Graph G_k

Theorem 3.1. *Let* $A = \{n_1, n_2, \ldots, n_k\}$ *be a set of positive integers,* $k \geq 2$*, and* $n_1 < n_2 < \cdots < n_k$*. Then the set of all positive eigenvalues of* G_k *is exactly* A.

Proof. For convenience, we set $f_i(\lambda) = f_{G_i}(\lambda)$, $p_i = n_i^2 - n_{i-1}^2$ and $f_i{}'(\lambda) = f_{G_i - a_i}(\lambda)$ where a_i is the root of graph G_i for $i = 2, \ldots, k$. By the recursive construction which is given in Section 2, it is easily seen that $f_i' = f_{i-2}' f_{i-1}^{p_i}$ for $i = 4, \ldots, k-1$. By Lamma 2.2 it follows that

$$
\begin{aligned}
f_k &= f_{k-1}^{p_k-1}\left(f_{k-1}f_{k-2} - p_k f'_{k-1} f'_{k-2}\right)\\
&= f_{k-1}^{p_k-1}\left(f_{k-2}^{p_{k-1}-1}\left(f_{k-2}f_{k-3} - p_{k-1} f'_{k-2} f'_{k-3}\right) f_{k-2} - p_k f'_{k-3} f_{k-2}^{p_{k-1}} f'_{k-2}\right)\\
&= f_{k-1}^{p_k-1} f_{k-2}^{p_{k-1}}\left(f_{k-2}f_{k-3} - (n_k^2 - n_{k-2}^2) f'_{k-2} f'_{k-3}\right)\\
&\ \vdots\\
&= f_{k-1}^{p_k-1}\prod_{i=2}^{k-3} f_{k-i}^{p_{k-i+1}}\left(f_3 f_2 - (n_k^2 - n_3^2) f'_3 f'_2\right).
\end{aligned}
$$

By Lemma 2.2 it is clear that

$$
\begin{aligned}
f_2 &= \lambda^{(n_1^2-1)(p_2-n_1-1)+n_1^2+2n_1-2}(\lambda^2 - n_1^2)^{p_2-n_1}(\lambda^2 - n_2^2),\\
f'_2 &= \lambda^{n_1^2 p_2 - n_1^3 - p_2 + 3n_1 - 1}(\lambda^2 - n_1^2)^{p_2 - n_1 + 1},\\
f_3 &= f_2^{p_3-1}(f_2 \lambda^{n_1^2-1}(\lambda^2 - n_1^2) - p_3 f'_2 \lambda^{n_1^2}),\\
f'_3 &= \lambda^{n_1^2} f_2^{p_3}.
\end{aligned}
$$

So we obtain that

$$
\begin{aligned}
f_k &= f_{k-1}^{p_k-1}\prod_{i=2}^{k-2} f_{k-i}^{p_{k-i+1}}\left(f_2\lambda^{n_1^2-1}(\lambda^2 - n_1^2) - (n_k^2 - n_2^2) f'_2 \lambda^{n_1^2}\right)\\
&= f_{k-1}^{p_k-1}\prod_{i=2}^{k-2} f_{k-i}^{p_{k-i+1}}\left(\lambda^{n_1^2 p_2 - p_2 - n_1^3 + n_1^2 + 3n_1 - 2}(\lambda^2 - n_1^2)^{p_2 - n_1 + 1}(\lambda^2 - n_k^2)\right).
\end{aligned}
$$

Since all the roots of polynomials f_2 and f_3 are integers, G_2 and G_3 are integral graphs. By the above relation, we obtain G_i is integral and its distinct positive eigenvalues are $n_1, \ldots, n_i$ for $i = 4, \ldots, k$. This completes the proof. □

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