NEW CLASS OF INTEGRAL BIPARTITE GRAPHS WITH LARGE DIAMETER

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Abstract. In this paper, we construct a new class of integral bipartite graphs (not necessarily trees) with large even diameters. In fact, for every finite set $A$ of positive integers of size $k$ we construct an integral bipartite graph $G$ of diameter $2k$ such that the set of positive eigenvalues of $G$ is exactly $A$. This class of integral bipartite graphs has never found before.

1. Introduction

Let $G$ be a simple graph with the vertex set $\{v_1, v_2, \ldots, v_n\}$. The adjacency matrix of $G$ is an $n \times n$ matrix $A(G)$ whose $(i, j)$-entry is 1 if $v_i$ is adjacent to $v_j$ and 0, otherwise. The characteristic polynomial of $G$, denoted by $f_G(x)$, is the characteristic polynomial of $A(G)$. We will write it simply $f_G$ when there is no confusion. The roots of $f_G$ are called the eigenvalues of $G$ and can be ordered as $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. It is important to mention that the set of eigenvalues of a bipartite graph is symmetrical with respect to 0. An integral graph is a graph for which the eigenvalues of its adjacency matrix are all integers [5]. Many different classes of integral graphs have been constructed in the past decades. For a long time it has been an open question whether there exist integral trees of arbitrarily large diameter [6]. Csikvari in [2] constructed integral trees with arbitrary large even diameter. A very elegant and short proof of the integrality of Csikvari’s trees can be found in page 90 of [1]. Ghorbani, Mohammadian, and Tayfeh-Rezaie constructed integral trees with arbitrary large odd diameter [4]. In this paper by a recursive method we construct new class of integral bipartite graphs (not necessarily trees) with arbitrary large even diameter. Let $A = \{n_1, n_2, \ldots, n_k\}$ be a set of positive integers such as 

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that \( n_1 < n_2 < \cdots < n_k \). In Section 2 we give a recursive method to construct integral bipartite graph \( G \) with diameter \( 2k \) which the set of all positive eigenvalues of \( G \) is exactly \( A \). If \( n_1 > 1 \), then the constructed bipartite graph will be not a tree. In Section 3 we prove the integrality of the constructed graph.

### 2. Recursive Method

We use the notation \( K_{n,m} \) for complete bipartite graph with two partite sets having \( n \) and \( m \) vertices.

**Definition 2.1.** Let \( G_1 \) and \( G_2 \) be rooted graphs with roots \( a \) and \( b \), respectively. Then the graph \( G_1 \sim G_2 \) obtained from \( G_1 \cup G_2 \) by joining the vertices \( a \) and \( b \). We obtain the \( G_1 \sim mG_2 \) by taking \( G_1 \) and \( m \) copies of \( G_2 \) and we join \( a \) to each copies of \( b \). The root of \( G_1 \) is considered to be the root of \( G_1 \sim G_2 \).

Let \( G = a \) be the graph obtained from \( G \) by deleting the vertex \( a \) and all edges containing \( a \). Then we have following lemma.

**Lemma 2.2.** [3] Let \( G_1 \) and \( G_2 \) be rooted graphs with roots \( a \) and \( b \), respectively, and \( m \geq 1 \). Then

\[
  f_{G_1 \sim mG_2} = f_{G_2}^{m-1}(f_{G_1}f_{G_2} - mf_{G_1-a}f_{G_2-b}).
\]

Let \( A = \{n_1, n_2, \ldots, n_k\} \) be a set of positive integers which \( n_1 < n_2 < \cdots < n_k \). Let a vertex in the part of size \( n_1 + 1 \) be the root of \( K_{n_1,n_1+1} \) and a vertex in the part of size \( 1 \) be the root of \( K_{1,n_1} \). Define

\[
  G_1 = K_{n_1,n_1+1},
  G_2 = G_1 \sim (n_2^2 - n_1^2 - n_1)K_{1,n_1^2},
  G_3 = K_{1,n_1} \sim (n_3^2 - n_2^2)G_2,
  G_4 = G_2 \sim (n_4^2 - n_3^2)G_3,
  \vdots
  G_k = G_{k-2} \sim (n_k^2 - n_{k-1}^2)G_{k-1}.
\]

It is easy to see that the diameter of \( G_k \) is \( 2k \). Clearly, if \( n_1 = 1 \), then all graphs constructed in the above are trees, and if \( n_1 > 1 \), then none of graphs constructed in the above is a tree. In Section 3 we prove that the set of positive eigenvalues of graph \( G_k \) is exactly \( A \).

**Example 2.3.** Let \( A = \{2, 3, 4\} \). In the following we indicate the graphs \( G_1, G_2, \) and \( G_3 \) constructed in the above manner.

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The set of the positive eigenvalues of $G_1$ is $\{\sqrt{6}\}$.

The set of the positive eigenvalues of $G_2$ is $\{2, 3\}$.

The set of the positive eigenvalues of $G_3$ is $\{2, 3, 4\}$.

3. Proof Of The Integrality Of Graph $G_k$

**Theorem 3.1.** Let $A = \{n_1, n_2, \ldots, n_k\}$ be a set of positive integers, $k \geq 2$, and $n_1 < n_2 < \cdots < n_k$. Then the set of all positive eigenvalues of $G_k$ is exactly $A$.

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Proof. For convenience, we set \( f_i(\lambda) = f_{G_i}(\lambda) \), \( p_i = n_i^2 - n_{i-1}^2 \) and \( f_i' (\lambda) = f_{G_i-a_i}(\lambda) \) where \( a_i \) is the root of graph \( G_i \) for \( i = 2, \ldots, k \). By the recursive construction which is given in Section 2, it is easily seen that \( f_i' = f_{i-2}' f_{i-1}' \) for \( i = 4, \ldots, k - 1 \). By Lemma 2.2 it follows that

\[
\begin{align*}
  f_k &= f_{k-1}^{p_k} \left( f_{k-2} - p_k f_{k-1}' f_{k-2}' \right) \\
  &= f_{k-1}^{p_k-1} \left( f_{k-2} - \frac{1}{k} p_k f_{k-3}' f_{k-2}' \right) \\
  &= f_{k-1}^{p_k-1} f_{k-2}^{p_{k-1}} \left( f_{k-3} - \frac{p_k f_{k-3}' f_{k-2} f_{k-3}'}{k} \right) \\
  &= \ldots \\
  &= f_{k-1}^{p_k-1} \prod_{i=2}^{k-3} \left( f_{i} f_{i+1} - (n_k^2 - n_{k+1}^2) f_{i}' f_{i+1}' \right).
\end{align*}
\]

By Lemma 2.2 it is clear that

\[
\begin{align*}
  f_2 &= \lambda (n_1^2 - 1)(p_2 - n_1 - 1) + n_2^2 + 2n_1 - 2(\lambda^2 - n_1^2) p_2 - n_1 (\lambda^2 - n_2^2), \\
  f_3' &= \lambda n_1^2 p_3 - n_3^2 - 3n_1 - 1(\lambda^2 - n_1^2) p_2 + n_1 + 1, \\
  f_3 &= f_2^{p_3-1} (f_3 \lambda n_2^2 - 1(\lambda^2 - n_1^2) - p_3 f_3' \lambda n_1^2), \\
  f_3' &= \lambda n_1^2 f_2^{p_3}.
\end{align*}
\]

So we obtain that

\[
\begin{align*}
  f_k &= f_{k-1}^{p_k-1} \prod_{i=2}^{k-2} \left( f_{i} \lambda n_{i+1}^2 - 1(\lambda^2 - n_1^2) - (n_k^2 - n_{k+1}^2) f_{i}' \lambda n_i^2 \right) \\
  &= f_{k-1}^{p_k-1} \prod_{i=2}^{k-2} \left( \lambda n_1^2 p_2 - n_2^2 - n_3^2 + 3n_1 - 2(\lambda^2 - n_1^2) p_3 - n_1 + 1(\lambda^2 - n_k^2) \right).
\end{align*}
\]

Since all the roots of polynomials \( f_2 \) and \( f_3 \) are integers, \( G_2 \) and \( G_3 \) are integral graphs. By the above relation, we obtain \( G_i \) is integral and its distinct positive eigenvalues are \( n_1, \ldots, n_i \) for \( i = 4, \ldots, k \). This completes the proof.

\[\square\]

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