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NEW CLASS OF INTEGRAL BIPARTITE GRAPHS WITH LARGE DIAMETER

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ABSTRACT. In this paper, we construct a new class of integral bipartite graphs (not necessarily trees) with large even diameters. In fact, for every finite set A of positive integers of size k we construct an integral bipartite graph G of diameter $2k$ such that the set of positive eigenvalues of G is exactly A . This class of integral bipartite graphs has never found before.

1. Introduction

Let G be a simple graph with the vertex set $\{v_1, v_2, \dots, v_n\}$. The adjacency matrix of G is an $n \times n$ matrix $A(G)$ whose (i, j) -entry is 1 if v_i is adjacent to v_j and 0, otherwise. The characteristic polynomial of G , denoted by $f_G(x)$, is the characteristic polynomial of $A(G)$. We will write it simply f_G when there is no confusion. The roots of f_G are called the eigenvalues of G and can be ordered as $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$. It is important to mention that the set of eigenvalues of a bipartite graph is symmetrical with respect to 0. An integral graph is a graph for which the eigenvalues of its adjacency matrix are all integers [5]. Many different classes of integral graphs have been constructed in the past decades. For a long time it has been an open question whether there exist integral trees of arbitrarily large diameter [6]. Csikvari in [2] constructed integral trees with arbitrary large even diameter. A very elegant and short proof of the integrality of Csikvari's trees can be found in page 90 of [1]. Ghorbani, Mohammadian, and Tayfeh-Rezaie constructed integral trees with arbitrary large odd diameter [4]. In this paper by a recursive method we construct new class of integral bipartite graphs (not necessarily trees) with arbitrary large even diameter. Let $A = \{n_1, n_2, \dots, n_k\}$ be a set of positive integers such

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that $n_1 < n_2 < \dots < n_k$. In Section 2 we give a recursive method to construct integral bipartite graph G with diameter $2k$ which the set of all positive eigenvalues of G is exactly A . If $n_1 > 1$, then the constructed bipartite graph will be not a tree. In Section 3 we prove the integrality of the constructed graph.

2. Recursive Method

We use the notation $K_{n,m}$ for complete bipartite graph with two partite sets having n and m vertices.

Definition 2.1. *Let G_1 and G_2 be rooted graphs with roots a and b , respectively. Then the graph $G_1 \sim G_2$ obtained from $G_1 \cup G_2$ by joining the vertices a and b . We obtain the $G_1 \sim mG_2$ by taking G_1 and m copies of G_2 and we join a to each copies of b . The root of G_1 is considered to be the root of $G_1 \sim G_2$.*

Let $G - a$ be the graph obtained from G by deleting the vertex a and all edges containing a . Then we have following lemma.

Lemma 2.2. [3] *Let G_1 and G_2 be rooted graphs with roots a and b , respectively, and $m \geq 1$. Then*

$$f_{G_1 \sim mG_2} = f_{G_2}^{m-1}(f_{G_1}f_{G_2} - mf_{G_1-a}f_{G_2-b}).$$

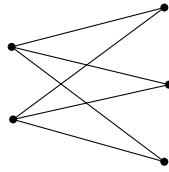
Let $A = \{n_1, n_2, \dots, n_k\}$ be a set of positive integers which $n_1 < n_2 < \dots < n_k$. Let a vertex in the part of size $n_1 + 1$ be the root of K_{n_1, n_1+1} and a vertex in the part of size 1 be the root of K_{1, n_1^2} . Define

$$\begin{aligned} G_1 &= K_{n_1, n_1+1}, \\ G_2 &= G_1 \sim (n_2^2 - n_1^2 - n_1)K_{1, n_1^2}, \\ G_3 &= K_{1, n_1^2} \sim (n_3^2 - n_2^2)G_2, \\ G_4 &= G_2 \sim (n_4^2 - n_3^2)G_3, \\ &\vdots \\ G_k &= G_{k-2} \sim (n_k^2 - n_{k-1}^2)G_{k-1}. \end{aligned}$$

It is easy to see that the diameter of G_k is $2k$. Clearly, if $n_1 = 1$, then all graphs constructed in the above are trees, and if $n_1 > 1$, then none of graphs constructed in the above is a tree. In Section 3 we prove that the set of positive eigenvalues of graph G_k is exactly A .

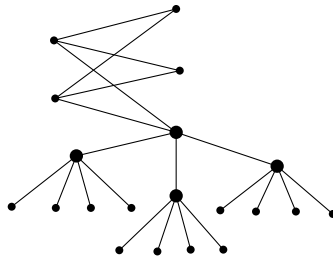
Example 2.3. *Let $A = \{2, 3, 4\}$. In the following we indicate the graphs G_1, G_2 , and G_3 constructed in the above manner.*

$$G_1 = K_{2,3}$$



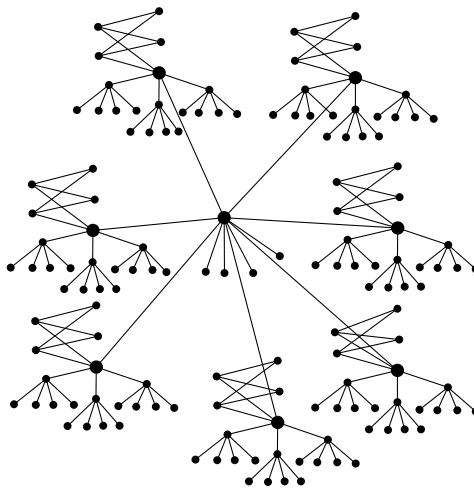
The set of the positive eigenvalues of G_1 is $\{\sqrt{6}\}$.

$$G_2 = G_1 \sim 3K_{1,4}$$



The set of the positive eigenvalues of G_2 is $\{2, 3\}$.

$$G_3 = K_{1,4} \sim 7G_2$$



The set of the positive eigenvalues of G_3 is $\{2, 3, 4\}$.

3. Proof Of The Integrality Of Graph G_k

Theorem 3.1. Let $A = \{n_1, n_2, \dots, n_k\}$ be a set of positive integers, $k \geq 2$, and $n_1 < n_2 < \dots < n_k$. Then the set of all positive eigenvalues of G_k is exactly A .

Proof. For convenience, we set $f_i(\lambda) = f_{G_i}(\lambda)$, $p_i = n_i^2 - n_{i-1}^2$ and $f'_i(\lambda) = f_{G_i - a_i}(\lambda)$ where a_i is the root of graph G_i for $i = 2, \dots, k$. By the recursive construction which is given in Section 2, it is easily seen that $f'_i = f'_{i-2} f_{i-1}^{p_{i-1}}$ for $i = 4, \dots, k - 1$. By Lemma 2.2 it follows that

$$\begin{aligned} f_k &= f_{k-1}^{p_{k-1}} (f_{k-1} f_{k-2} - p_k f'_{k-1} f'_{k-2}) \\ &= f_{k-1}^{p_{k-1}} \left(f_{k-2}^{p_{k-1}-1} (f_{k-2} f_{k-3} - p_{k-1} f'_{k-2} f'_{k-3}) f_{k-2} - p_k f'_{k-3} f_{k-2}^{p_{k-1}} f'_{k-2} \right) \\ &= f_{k-1}^{p_{k-1}} f_{k-2}^{p_{k-1}} (f_{k-2} f_{k-3} - (n_k^2 - n_{k-2}^2) f'_{k-2} f'_{k-3}) \\ &\vdots \\ &= f_{k-1}^{p_{k-1}} \prod_{i=2}^{k-3} f_{k-i}^{p_{k-i}+1} (f_3 f_2 - (n_k^2 - n_3^2) f'_3 f'_2). \end{aligned}$$

By Lemma 2.2 it is clear that

$$\begin{aligned} f_2 &= \lambda^{(n_1^2-1)(p_2-n_1-1)+n_1^2+2n_1-2} (\lambda^2 - n_1^2)^{p_2-n_1} (\lambda^2 - n_2^2), \\ f'_2 &= \lambda^{n_1^2 p_2 - n_1^3 - p_2 + 3n_1 - 1} (\lambda^2 - n_1^2)^{p_2-n_1+1}, \\ f_3 &= f_2^{p_3-1} (f_2 \lambda^{n_1^2-1} (\lambda^2 - n_1^2) - p_3 f'_2 \lambda^{n_1^2}), \\ f'_3 &= \lambda^{n_1^2} f_2^{p_3}. \end{aligned}$$

So we obtain that

$$\begin{aligned} f_k &= f_{k-1}^{p_{k-1}} \prod_{i=2}^{k-2} f_{k-i}^{p_{k-i}+1} \left(f_2 \lambda^{n_1^2-1} (\lambda^2 - n_1^2) - (n_k^2 - n_2^2) f'_2 \lambda^{n_1^2} \right) \\ &= f_{k-1}^{p_{k-1}} \prod_{i=2}^{k-2} f_{k-i}^{p_{k-i}+1} \left(\lambda^{n_1^2 p_2 - p_2 - n_1^3 + n_1^2 + 3n_1 - 2} (\lambda^2 - n_1^2)^{p_2-n_1+1} (\lambda^2 - n_k^2) \right). \end{aligned}$$

Since all the roots of polynomials f_2 and f_3 are integers, G_2 and G_3 are integral graphs. By the above relation, we obtain G_i is integral and its distinct positive eigenvalues are n_1, \dots, n_i for $i = 4, \dots, k$. This completes the proof. □

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