A NEW PROOF OF VALIDITY OF BOUCHET’S CONJECTURE ON EULERIAN BIDIRECTED GRAPHS

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Abstract. Recently, E. Mácajová and M. Škoviera proved that every bidirected Eulerian graph which admits a nowhere zero flow, admits a nowhere zero 4-flow. This result shows the validity of Bouchet’s nowhere zero conjecture for Eulerian bidirected graphs. In this paper we prove the same theorem in a different terminology and with a short and simple proof. More precisely, we prove that every Eulerian undirected graph which admits a zero-sum flow, admits a zero-sum 4-flow. As a conclusion we obtain a shorter proof for the previously mentioned result of Mácajová and Škoviera.

1. Introduction

Let \( G = (V, E) \) be a simple graph with vertex set \( V \) and edge set \( E \). The incidence matrix of \( G \) denoted by \( W(G) \) and is defined as follows:
\[
  w_{ij} = \begin{cases} 
  1 & \text{if } e_j \text{ is incident with } v_i, \\
  0 & \text{otherwise.}
\end{cases}
\]

The kernel of the incidence matrix of \( G \) is denoted by \( \ker(G) \). We call a real valued edge function \( f : E \to \mathbb{R} \) a flow on \( G \), if \( \sum_{e: v \in e} f(e) = 0 \), for every vertex \( v \in V \). A \( k \)-flow is a flow with values in the set \( \{0, \pm 1, \pm 2, \ldots, \pm (k - 1)\} \). A zero-sum \( k \)-flow, is a \( k \)-flow whose value is nonzero for every edge. In fact a flow corresponds to a vector in \( \ker(G) \) and a zero-sum flow corresponds to a vector in \( \ker(G) \) with no zero entry. We call an edge function \( f \) satisfies the zero-sum rule on the vertex \( v \) if
\[
  \sum_{u : uv \in E(G)} f(uv) = 0.
\]

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A bidirected edge is an edge consisting of two half-edges which receive separate orientations and a bidirected graph is a graph whose edges are bidirected. Thus every edge is oriented with one of the four possible orientations:

\[
\begin{array}{cccc}
  & & & \\
  & & & \\
  & & & \\
  & & & \\
\end{array}
\]

The first two edges are called ordinary edges and the next two edges are called out-edge and in-edge (or opposite edges), respectively.

An integer valued function \( f \) on the edges of a bidirected graph \( G \) with the edge set \( E(G) \) is a nowhere-zero bidirected \( k \)-flow if for every \( e \in E(G) \), \( 0 < |f(e)| < k \), and at every vertex \( v \), the sum of values on the half-edges directed to \( v \) equals the sum of values on the half-edges directed out of \( v \).

In the case of bidirected graph, Bouchet in [3] have the following conjecture:

**Bouchet’s Conjecture** Every bidirected graph which admits a nowhere-zero bidirected flow admits a nowhere-zero bidirected 6-flow.

This conjecture has been verified for by Zyka if 6 is replaced by 30, see [8] and recently has been proved by DeVos in [4], when 6 replaced with 12.

In [2] we study zero-sum flows in undirected graphs and mentioned the following conjecture:

**Zero-Sum Conjecture (ZSC).** If \( G \) is a graph with a zero-sum flow, then \( G \) has a zero-sum 6-flow.

We also found the necessary and sufficient condition for a graph to have a zero-sum flow. It is easy to see that ZSC is a special case of Bouchet’s conjecture. In [1], it is proved that ZSC is equivalent to Bouchet’s conjecture.

In this paper, firstly we prove that every Eulerian graph which admits a zero-sum flow admits a zero-sum 4-flow. As a conclusion we obtain a shorter proof for the result of Mácajová and Škoviera in [6], which state that every Eulerian bidirected graph having nowhere zero flow, admits a nowhere zero 4-flow.

Here is some notations and definitions most of which can be found in [7]. A walk in a graph \( G \) is a list \( v_0e_1v_1 \cdots v_ke_k \) of vertices and edges such that, for \( 1 \leq i \leq k \), the edge \( e_i \) has end points \( v_{i-1} \) and \( v_i \). A trail is a walk with no repeated edge. A \( u,v \)-trail is a trail with first vertex \( u \) and last vertex \( v \), and we denote a \( v,v \)-trail by \( v \)-trail. A trail is called a closed trail if its end points are the same. A graph is Eulerian if it has a closed trail containing all edges of the graph. We call a closed trail a circuit when we do not specify the first vertex but keep the list in the cyclic order. An Eulerian circuit or Eulerian trail in a graph is a circuit or trail containing all edges.

Let \( G \) be a bidirected graph then \( S(G) \) is an undirected graph obtained from \( G \) by removing all directions of edges of \( G \) and replacing the ordinary edges of \( G \) with a path of length 2.

**2. Zero-sum flows in undirected graphs**

The following Lemma shows the relation between nowhere zero \( k \)-flow in bidirected graphs and zero-sum \( k \)-flow in undirected graphs. For the proof see [1, Theorem 2.1].

**Lemma 2.1.** A bidirected graph \( G \) has nowhere zero \( k \)-flow if and only if the graph \( S(G) \) has zero-sum \( k \)-flow.
Definition 1. [5] An odd edge-bicycle is a union of two odd cycles joined by a path (the path may have length zero); the cycles and the path are required to be edge-disjoint (but may intersect at vertices). An edge-$K_4$ is a union of six paths as shown in Figure 1, which are again required to be edge-disjoint (but may intersect at vertices). If the cycles $A, B, C, D$ are all odd ($D$ is the boundary of the outer region), we have an odd edge-$K_4$.

Theorem 2.2. [5] The following statements are equivalent for a connected graph $G$.

(1) $G$ does not have an odd edge-bicycle or an odd edge-$K_4$;
(2) There is an edge $e$ such that $G \setminus e$ is bipartite.

Theorem 2.3. [2] Suppose $G$ is not a bipartite graph. Then $G$ has a zero-sum flow if and only if for any edge $e$ of $G$, $G \setminus \{e\}$ has no bipartite component.

Corollary 2.4. Let $G$ be a non-bipartite graph. If $G$ has a zero-sum flow, then $G$ has an odd edge-bicycle or an odd edge-$K_4$.

Lemma 2.5. Every odd edge-bicycle graph and every odd edge-$k_4$ graph admits zero-sum 3-flow.

Proof. For an odd edge-bicycle $G$ one can simply assign $\pm 1$ to the edges of the cycles, respectively. And assign $\pm 2$ to the edges of the connecting path, respectively in such a way that the result is a flow on $G$. So, every odd edge-bicycle graph has a zero-sum 3-flow.

An edge-$K_4$ graph is an odd edge-$K_4$, if and only if one of the following cases occur:

(1) All the six paths $ox, oy, oz, xy, xz, yz$ have odd lengths,
(2) Two of the boundary paths and the inner path intersect them (for example the paths $xy, ox, xz$) have even lengths and the other paths have odd length,
(3) One boundary path and one inner path which do not intersect (for example paths $ox, yz$) have odd lengths and the other have even lengths,
(4) The three boundary paths have odd length and three inner paths have even lengths.

For example, in Figure 2 it is shown that in case (1) the odd edge-$k_4$ have zero-sum 3-flow. The other three cases have zero-sum 3-flow, similarly.

\[ \square \]
Lemma 3.1. Let $G$ be an Eulerian connected graph with an even number of edges, then $G$ admits a zero-sum 2-flow.

Proof. Let $C$ be an Eulerian circuit of $G$. If we traverse the edges of $C$ according to the cyclic ordering and assign $\pm 1$ to the edges, respectively, we yield a zero-sum 2-flow for $G$. \qed

Theorem 3.2. Let $G$ be an Eulerian graph which admits a zero-sum flow. Then $G$ admits a zero-sum 4-flow.

Proof. If $G$ is an Eulerian graph with an even number of edges, then by Lemma 3.1, $G$ has a zero-sum 2-flow. Now, let $G$ be an Eulerian graph with odd number of edges which admits a zero-sum flow. Then $G$ contains an odd cycle. Otherwise, $G$ is bipartite with parts $X$ and $Y$. Hence, $|E(G)| = \sum_{v \in X} \deg(v)$, since each vertex has even degree, $E(G)$ will be even, which contradicts to the assumption. By Corollary 2.4, $G$ contains an odd edge-bicycle or an odd edge-$K_4$. Let $C$ be an odd cycle which contained in an odd edge-bicycle or an odd edge-$K_4$.

We delete the edges of $C$ from $G$ and denote the result graph by $H = G \setminus E(C)$. Clearly each component of $H$ is an Eulerian Graph. The graph $H$ has some components with even number of edges. We denote the union of these components by $K$. By lemma 3.1, $K$ has a zero-sum 2-flow, we call it $f_0$. $H$ may also have some components with odd number of edges, suppose $H$ has $m$ of such components and called them $G_1,G_2,\ldots,G_m$. Then $m$ should be an even number, otherwise $G$ will have an even number of edges which is a contradiction.

If $m = 0$, then by Lemma 2.5, there is a 3-flow $f_1$ whose values are nonzero on the edges of $C$. Hence, $f = 3f_0 + f_1$ is a zero-sum 4-flow for $G$.

Now, suppose that $m > 0$. Let $H_1 = C \cup G_m$. Then $H_1$ is an Eulerian graph with even number of edges, so it has a zero-sum 2-flow called $f_1$. Let $H_2 = C \cup G_1 \cup \cdots \cup G_{m-1}$. Since $m$ is an even number and the number of edges of each of the graphs $C,G_1,\ldots,G_{m-1}$ is an odd number, the number

![Figure 2. zero-sum 3-flow on Odd edge-$K_4$](image-url)
of edges of $H_2$ is an even number. Hence, by Lemma 3.1, $H_2$ has a zero-sum 2-flow called $f_2$. Now, let $f = 2f_2 + f_1 + f_0$ is a zero-sum 3-flow for $G$. □

**Theorem 3.3.** Every Eulerian bidirected graph $G$ which admits a nowhere zero $k$-flow, admits a nowhere zero 4-flow.

**Proof.** Let $G$ be an Eulerian bidirected graph which admits a nowhere zero $k$-flow, then $S(G)$ is an Eulerian undirected graph which admits a zero-sum $k$-flow. By Theorem 3.2, $S(G)$ admits a zero-sum 4-flow. Hence, by Lemma 2.1, the bidirected graph $G$ admits a nowhere zero 4-flow, as desired. □

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