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ANNIHILATING SUBMODULE GRAPH FOR MODULES

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Abstract. Let R be a commutative ring and M an R-module. In this article, we introduce a new generalization of the annihilating-ideal graph of commutative rings to modules. The annihilating submodule graph of M, denoted by $\mathbb{G}(M)$, is an undirected graph with vertex set $\mathbb{A}^*(M)$ and two distinct elements N and K of $\mathbb{A}^*(M)$ are adjacent if $N * K = 0$. In this paper we show that $\mathbb{G}(M)$ is a connected graph, $\mathrm{diam}(\mathbb{G}(M)) \leq 3$, and $\mathrm{gr}(\mathbb{G}(M)) \leq 4$ if $\mathbb{G}(M)$ contains a cycle. Moreover, $\mathbb{G}(M)$ is an empty graph if and only if $\mathrm{ann}(M)$ is a prime ideal of R and $\mathbb{A}^*(M) \neq \mathbb{S}(M) \setminus \{0\}$ if and only if M is a uniform R-module, $\mathrm{ann}(M)$ is a semi-prime ideal of R and $\mathbb{A}^*(M) \neq \mathbb{S}(M) \setminus \{0\}$. Furthermore, R is a field if and only if $\mathbb{G}(M)$ is a complete graph, for every $M \in R-\mathrm{Mod}$. If R is a domain, for every divisible module $M \in R-\mathrm{Mod}$, $\mathbb{G}(M)$ is a complete graph with $\mathbb{A}^*(M) = \mathbb{S}(M) \setminus \{0\}$. Among other things, the properties of a reduced R-module M are investigated when $\mathbb{G}(M)$ is a bipartite graph.

1. Introduction

Throughout this article, all rings are commutative with identity and all modules are right unitary modules. Let M be an R-module. For each subset X of M, $\mathrm{ann}(X) = \{r \in R |\ Xr = 0\}$. Moreover, for each submodule N of M, $(N : M) = \{r \in R : Mr \subseteq N\}$. In another conception of zero-divisor graph for modules, $\Gamma(M_R)$, has been considered; for a right R-module M and $x, y \in M$, it is said that $x * y = 0$ provided that either $x(yR : M) = 0$ or $y(xR : M) = 0$, see [18]. Also $Z(M) = \{x \in M : x * y = 0\ for\ some\ y \in M\}$ and $Z(M)^* = Z(M) \setminus \{0\}$. The zero-divisor graph of an R-module M, denoted by $\Gamma(M_R)$, is an undirected graph with vertex set $Z(M)^*$ and $x, y \in Z(M)^*$ are adjacent provided that $x * y = 0$. Let G be an undirected graph. We say that G is connected if there is a path between any two distinct vertices. For distinct vertices x and y in G, the distance between x and

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y, denoted by $d(x,y)$, is the length of a shortest path connecting x and y ($d(x,x) = 0$ and $d(x,y) = \infty$ if no such path exists). The diameter of G is

$$\mathrm{diam}(G) = sup\{d(x,y)| x \textit{ and } y \textit{ are vertices of } G\}.$$

A cycle of length n in G is a path of the form $x_1 - x_2 - x_3 \cdots - x_n - x_1$, where $x_i \neq x_j$ when $i \neq j$. We define the girth of G, denoted by $\mathrm{gr}(G)$, as the length of a shortest cycle in G, provided G contains a cycle; otherwise, $\mathrm{gr}(G) = \infty$. A graph is complete if any two distinct vertices are adjacent. A complete graph with n vertices is denoted by K_n. By a complete subgraph, we mean a subgraph which is complete as a graph. The graph G is called *bipartite* provided that the set of vertices of G is the union of two non-empty distinct subsets V_1 and V_2, such that no element of V_1 or V_2, is adjacent with another element of V_1 or V_2, respectively. Assume that $K_{m,n}$ denote the complete bipartite graph on two nonempty disjoint sets V_1 and V_2 with $|V_1| = m$ and $|V_2| = n$ (here m and n may be infinite cardinal numbers). A $K_{1,n}$ graph is often called a star graph. Any unexplained terminology, and all the basic results on rings, modules and graphs that are used in the sequel can be found in [1], [5], [20], [21] and [22]. In recent decades, the zero-divisor graphs of commutative rings have been extensively studied by many authors, see for example [3], [4], [6-11] and [17]. In [13] and [18] the authors have associated two different graphs to an R-module M and studied the zero-divisor graph of Abelian groups in [14]. In [9], [15], and [16], the graph of zero-divisors for commutative rings has been generalized to the annihilating-ideal graph of commutative rings (two ideals I and J are adjacent if $IJ = (0)$).

2. Annihilating Submodule Graph

Definition 2.1. *Let M be an R-module. The set of all submodules of M is denoted by $\mathbb{S}(M)$. For two elements N and K in $\mathbb{S}(M)$, we say that $N * K = 0$ provided that either $N(K : M) = 0$ or $K(N : M) = 0$. The submodule N of M is called annihilating submodule provided that $N * K = 0$, for some non-zero submodule K of M.*

Definition 2.2. *Let M be an R-module. Let $\mathbb{A}(M)$ be the set of all annihilating submodules of M. The annihilating submodule graph of M, denoted by $\mathbb{G}(M)$, is an undirected graph with vertex set $\mathbb{A}^*(M) = \mathbb{A}(M) \setminus \{0\}$ and $N, K \in \mathbb{A}^*(M)$ are adjacent if $N * K = 0$.*

If I is an ideal of a ring R, it is obvious that $(I : R) = I$. Then $\mathbb{G}(R)$ is precisely the annihilating-ideal graph of commutative ring R, see [15]. Let R be a commutative Noetherian ring. For $x, y \in R$, we say that $x \sim y$ if and only if $\mathrm{ann}(x) = \mathrm{ann}(y)$. The relation " $\sim$ " is an equivalence relation. The equivalence class of any element $x \in R$ is denoted by $[x]$. In [19] and [11], it was attributed a graph to R, denoted by $\Gamma_E(R)$, with vertex set $Z^*_E(R) = \{[x]| \ x \in R \textit{ and } \mathrm{ann}(x) \neq (0)\}$. Moreover, two distinct elements $[x], [y] \in Z^*_E(R)$ are adjacent if $xy = 0$. In the following proposition $\Gamma_E(\mathbb{Z}_n)$ is characterized, for each positive integer n.

Proposition 2.3. *For each positive integer n, $\Gamma_E(\mathbb{Z}_n) \cong \mathbb{G}(\mathbb{Z}_n)$.*

Proof. First we show that for each $\overline{x}, \overline{y} \in \mathbb{Z}_n$, $\text{ann}_{\mathbb{Z}_n}(\overline{x}) = \text{ann}_{\mathbb{Z}_n}(\overline{y})$ if and only if $\overline{x}\mathbb{Z} = \overline{y}\mathbb{Z}$. For this, the "only if " part is obvious. Conversely, since $\overline{x}\mathbb{Z}$ and $\overline{y}\mathbb{Z}$ are submodules of $\mathbb{Z}$-module $\mathbb{Z}_n$, there exist $\overline{a}, \overline{b} \in \mathbb{Z}_n$ such that a, b divide n, $\overline{x}\mathbb{Z} = \overline{a}\mathbb{Z}$ and $\overline{y}\mathbb{Z} = \overline{b}\mathbb{Z}$. Therefore there exist positive integers $s, t \in \mathbb{Z}$ such that $n = ta$ and $n = sb$. By hypothesis, since $\overline{t} \in \text{ann}_{\mathbb{Z}_n}(\overline{x})$, $\overline{t} \in \text{ann}_{\mathbb{Z}_n}(\overline{y}) = \text{ann}_{\mathbb{Z}_n}(\overline{b})$. Thus $n|tb$ and hence $tb = nk = sbk$, for some $k \in \mathbb{Z}$. So $s|t$. By similar role we can show that $t|s$ and hence $t = s$. It shows that $a = b$ and hence $\overline{x}\mathbb{Z} = \overline{y}\mathbb{Z}$. On the other hand it is easy to show that $(\overline{x}\mathbb{Z} : \mathbb{Z}_n) = d\mathbb{Z}$, where $(x, n) = d$. For this, assume that $c \in (\overline{x}\mathbb{Z} : \mathbb{Z}_n)$. Then $n|c - xb$, for some $b \in \mathbb{Z}$. Since both $d|n$ and $d|x$, $d|c$. Conversely, we know that if there exist $p, q \in \mathbb{Z}$ such that $xp + nq = d$, then $\overline{d} \in \overline{x}\mathbb{Z}$ and hence $d\mathbb{Z} \subseteq (\overline{x}\mathbb{Z} : \mathbb{Z}_n)$. By [18, Proposition 1.2], for each $\overline{x}, \overline{y} \in \mathbb{Z}_n$, $\overline{x}.\overline{y} = \overline{0}$ if and only if $\overline{x}(\overline{y}\mathbb{Z} : \mathbb{Z}_n) = \overline{0}$. Then the map $\phi : V(\Gamma_E(\mathbb{Z}_n)) \longrightarrow \mathbb{A}^*(\mathbb{Z}_n)$, by $\phi([\overline{x}]) = \overline{x}\mathbb{Z}$ is an isomorphism between $\Gamma_E(\mathbb{Z}_n)$ and $\mathbb{G}(\mathbb{Z}_n)$. □

Example. In [18, Proposition 1.2], the authors have showed that $\Gamma(\mathbb{Z}_n)$ as a ring is isomorphic to $\Gamma(\mathbb{Z}_n)$ as a $\mathbb{Z}$-module.

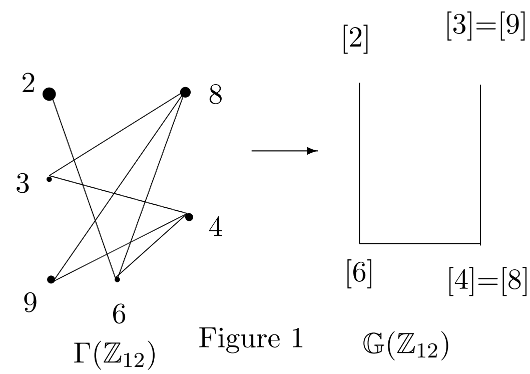

Figure 1

In the following we state a lemma which has important influence in this paper.

Lemma 2.4. *Let M be an R-module and $N, K \in \mathbb{A}^*(M)$. Then*

(1) *If N and K are adjacent in $\mathbb{G}(M)$, then for every non-zero submodule N' of N and every non-zero submodule K' of K, $N' * K' = 0$.*

(2) *If $N \bigcap K = 0$, then $N * K = 0$*

Proof. (1). Since $N * K = 0$, either $N(K : M) = 0$ or $K(N : M) = 0$. Suppose that $N(K : M) = 0$.Therefore

$$N'(K' : M) \subseteq N(K' : M) \subseteq N(K : M) = 0,$$

as desired.
(2). Since $N(K : M) \subseteq N \cap K$, the verification is immediate. □

In [18, Theorem 1.9], the authors have shown that $\Gamma(M)$ is an empty graph if and only if $\text{ann}(M)$ is a prime ideal of R and $Z^*(M) \neq M \setminus \{0\}$. In the following result we will see "When is $\mathbb{G}(M)$ an empty graph?".

Proposition 2.5. *Let M be an R-module. Then the following are equivalent.*

(1) $\mathbb{G}(M)$ *is an empty graph.*

(2) $\mathrm{ann}(M)$ *is a prime ideal of R and* $\mathbb{A}^*(M) \neq \mathbb{S}(M) \setminus \{0\}$.

(3) *M is a uniform R-module,* $\mathrm{ann}(M)$ *is a semiprime ideal of R and* $\mathbb{A}^*(M) \neq \mathbb{S}(M) \setminus \{0\}$.

Proof. $(1 \to 2)$ Assume that $\mathbb{G}(M)$ is an empty graph and $a, b \in R$ such that $a.b \in \mathrm{ann}(M)$ and neither $a \in \mathrm{ann}(M)$ nor $b \in \mathrm{ann}(M)$. Then Ma and Mb are non-zero submodules of M such that

$$Ma(Mb : M) = M(Mb : M)a \subseteq Mba = 0.$$

Therefore Ma and Mb are in $\mathbb{A}^*(M)$, a contradiction.
$(2 \to 1)$ Suppose that $N \in \mathbb{A}^*(M)$. Then there exists a non-zero submodule K of M such that $N * K = 0$. Hence either $N(K : M) = 0$ or $K(N : M) = 0$. Assume that $N(K : M) = 0$. Therefore $M(N : M)(K : M) = 0$. Since $\mathrm{ann}(M)$ is a prime ideal of R, either $(N : M) \subseteq \mathrm{ann}(M)$ or $(K : M) \subseteq \mathrm{ann}(M)$. Let $(N : M) \subseteq \mathrm{ann}(M)$. Then for each non-zero submodule E of M, we have $E(N : M) = 0$. Thus $E \in \mathbb{A}^*(M)$ and hence $\mathbb{A}^*(M) = \mathbb{S}(M) \setminus \{0\}$, a contradiction.
$(1 \to 3)$ It is sufficient to show that M_R is uniform. If N and K are non-zero submodules of M such that $N \cap K = 0$, then by Lemma 2.4, $N * K = 0$ and hence $N \in \mathbb{A}^*(M)$, a contradiction.
$(3 \to 1)$ To the contrary, assume that $N \in \mathbb{A}^*(M)$. There exists $K \in \mathbb{A}^*(M)$ such that $N * K = 0$. Since M_R is uniform, $N \cap K \neq 0$. By Lemma 2.4, $(N \cap K) * (N \cap K) = 0$. Therefore $(N \cap K)((N \cap K) : M) = 0$ and hence $M((N \cap K) : M)((N \cap K) : M) = 0$. Consequently $((N \cap K) : M)^2 \subseteq \mathrm{ann}(M)$. Since $\mathrm{ann}(M)$ is a semiprime ideal of R, we have $((N \cap K) : M) \subseteq \mathrm{ann}(M)$. This implies that for each non-zero submodule B of M, $B((N \cap K) : M) = 0$ which implies that $B * (N \cap K) = 0$. Therefore $\mathbb{A}^*(M) = \mathbb{S}(M) \setminus \{0\}$, a contradiction. □

When does $\mathbb{G}(M)$ contain a cycle? The next result gives a partial answer to this question. As we see, it happens when $\mathbb{G}(M)$ contains a path of length 4. In fact, when $\mathbb{G}(M)$ has a path of length 4, then $\mathrm{gr}(\mathbb{G}(M)) \leq 4$.

Proposition 2.6. *Let M be an R-module. If $\mathbb{G}(M)$ has a path of length 4, then $\mathbb{G}(M)$ contains a cycle of length less than or equal to 4.*

Proof. Let $N_1 - N_2 - N_3 - N_4 - N_5$ be a path in $\mathbb{G}(M)$. The following cases may happen.
(Case 1). If $N_2 \cap N_4 = 0$, then by Lemma 2.4(2), $N_2 * N_4 = 0$ and hence $N_2 - N_3 - N_4 - N_2$ is a cycle of length 3.
(Case 2). If $N_2 \cap N_4 \neq 0$ and $N_2 \cap N_4 \notin \{N_1, N_2, N_3, N_4, N_5\}$, then by Lemma 2.4(1), both $(N_2 \cap N_4) * N_1 = 0$ and $(N_2 \cap N_4) * N_3 = 0$ and hence $N_1 - N_2 - N_3 - N_2 \cap N_4 - N_1$ is a cycle of length 4.
(Case 3). If $N_2 \cap N_4 = N_1$, then by Lemma 2.4(1), since $N_2 * N_3 = 0$, $N_3 * (N_2 \cap N_4) = N_3 * N_1 = 0$ and hence $N_1 - N_2 - N_3 - N_1$ is a cycle of length 3.
(Case 4). If $N_2 \cap N_4 = N_2$, then by Lemma 2.4(1), since $N_5 * N_4 = 0$, $N_5 * (N_2 \cap N_4) = N_5 * N_2 = 0$ and hence $N_2 - N_3 - N_4 - N_5 - N_2$ is a cycle of length 4.
(Case 5). If $N_2 \cap N_4 = N_3$, then by Lemma 2.4(1), since $N_2 * N_1 = 0$, $N_1 * (N_2 \cap N_4) = N_1 * N_3 = 0$

and hence $N_1 - N_2 - N_3 - N_1$ is a cycle of length 3.
(Case 6). If $N_2 \cap N_4 = N_4$, then by Lemma 2.4(1), since $N_1 * N_2 = 0$, $N_1 * (N_2 \cap N_4) = N_1 * N_4 = 0$ and hence $N_1 - N_2 - N_3 - N_4 - N_1$ is a cycle of length 4.
(Case 7). If $N_2 \cap N_4 = N_5$, then by Lemma 2.4(1), since $N_3 * N_4 = 0$, $N_3 * (N_2 \cap N_4) = N_3 * N_5 = 0$ and hence $N_3 - N_4 - N_5 - N_3$ is a cycle of length 3. □

Proposition 2.7. *Let M be an R-module. Then*

(1) *$\mathbb{G}(M)$ is a connected graph.*
(2) *$diam(\mathbb{G}(M)) \leq 3$.*
(3) *If $\mathbb{G}(M)$ contains a cycle, then $\mathrm{gr}(\mathbb{G}(M)) \leq 4$.*

Proof. (1),(2). Let N and K be two distinct vertices of $\mathbb{G}(M)$. If $N * K = 0$, then N and K are adjacent in $\mathbb{G}(M)$. Suppose that $N * K \neq 0$. There exist non-zero submodules N' and K' of M such that both $N * N' = 0$ and $K * K' = 0$. Just the following cases may happen.
(Case 1). If $N' = K'$, then $N - N' - K$ is a path of length two between N and K.
(Case 2). If $N' * K' = 0$, then $N - N' - K' - K$ is a path of length three between N and K.
(Case 3). If $N' * K' \neq 0$, then by Lemma 2.4(2), $N' \cap K' \neq 0$. Again by Lemma 2.4(1), both $N * (N' \cap K') = 0$ and $K * (N' \cap K') = 0$. Therefore $N - N' \cap K' - K$ is a path of length two between N and K. Thus $\mathbb{G}(M)$ is connected and $\mathrm{diam}(\mathbb{G}(M)) \leq 3$.
(3). Assume that $N_1 - N_2 - \cdots N_k - N_1$ is a cycle in $\mathbb{G}(M)$. If $k \leq 4$, the proof is complete. If $k \geq 5$, then $N_1 - N_2 - N_3 - N_4 - N_5$ is a path of length 4 and hence by Proposition 2.6, $\mathbb{G}(M)$ contains a cycle of length less than or equal to 4. □

The rest of this section is devoted to the study of modules for which their annihilating submodule graphs are complete graphs. The first result shows that, when is the annihilating submodule graph of all R-modules complete.

Proposition 2.8. *Let R be a ring. Then R is a field if and only if for each R-module M, $\mathbb{G}(M)$ is a complete graph.*

Proof. First, for each non-zero element $r \in R$ we have $((\overline{0}, r)R : \frac{R}{M} \oplus R) \subseteq M$, then for each non-zero element $\overline{x} \in \frac{R}{M}$, $(\overline{x}, 0)((\overline{0}, r)R : \frac{R}{M} \oplus R) = 0$. Since $\mathbb{G}(\frac{R}{M} \oplus R)$ is a complete graph, for non-zero elements $r, s \in R$,

$$(\overline{0}, r)((\overline{0}, s)R : \frac{R}{M} \oplus R) = 0.$$

On the other hand for each $0, 1 \neq s \in R$, $Ms \subseteq ((\overline{0}, s)R : \frac{R}{M} \oplus R)$. Therefore $(\overline{0}, 1)Ms = 0$ and hence $Ms = 0$. Thus for each $0, 1 \neq s \in R$, $Ms = M(1 - s) = 0$ and hence $M = 0$, as desired. □

Proposition 2.9. *Let M be an R-module and $\{S_i\}_{i=1}^n$ a family of isomorphic simple R-submodules of M such that $M = \oplus_{i=1}^n S_i$. Then $\mathbb{A}^*(M) = \mathbb{S}(M) \setminus \{0\}$ and $\mathbb{G}(M)$ is a complete graph.*

Proof. Since for each $1 \leq i, j \leq n$, $S_i \cong S_j$, $\mathrm{ann}(S_i) = \mathrm{ann}(S_j)$. If N is a non-trivial submodule of M, then by [5, Lemma 9.1], there exists a subset I of $\{1, 2, \ldots, n\}$ such that $M = N \oplus (\oplus_{i \in I} S_i)$. Therefore

$$\mathrm{ann}(M) = \mathrm{ann}(S_1) \subseteq (N : M) = (N : N \oplus (\oplus_{i \in I} S_i)) = \mathrm{ann}(\oplus_{i \in I} S_i) = \mathrm{ann}(S_1).$$

Hence for each non-zero submodule K of M, $K(N : M) = 0$. Therefore any non-trivial submodule of M is adjacent to any non-zero submodule of M in $\mathbb{G}(M)$. □

Corollary 2.10. *Let M be a simple R-module. Then $\mathbb{G}(M \times M)$ is a complete graph.*

Proof. Put $M^2 = M \times M$, $M_1 = M \times \{0\}$ and $M_2 = \{0\} \times M$. Since M_1 and M_2 are isomorphic simple submodules of M^2 such that $M^2 = M_1 \oplus M_2$, by Proposition 2.9, $\mathbb{G}(M \times M)$ is a complete graph. □

By [21, 3.16], an R-module M is called *divisible* provided that for each $a \in R$ and $m \in M$, if $\text{ann}(a) \subseteq \text{ann}(m)$, there exists $n \in M$ such that $na = m$. If M is a divisible R-module and $a \in R$ such that $\text{ann}(a) = 0$, then for each $m \in M$, $\text{ann}(a) \subseteq \text{ann}(m)$ and hence m is divided by a. Therefore $Ma = M$.

Proposition 2.11. *Let R be a domain and M a divisible R-module. Then $\mathbb{G}(M)$ is a complete graph with $\mathbb{A}^*(M) = \mathbb{S}(M) \setminus \{0\}$.*

Proof. Since R is a domain, for each non-zero element $t \in R$, $\text{ann}(t) = 0$. Therefore divisibility of M implies that $Mt = M$. Hence for each proper submodule N of M, $(N : M) = 0$, for, $0 \neq t \in (N : M)$ implies that $Mt = M \subseteq N$, a contradiction. Therefore $\mathbb{A}^*(M) = \mathbb{S}(M) \setminus \{0\}$ and for each $K, N \in \mathbb{A}^*(M)$, $N * K = 0$. □

Proposition 2.12. *Let R be a domain which is not a field and M an R-module which has a non-zero divisible submodule D. Then $\mathbb{A}^*(M) = \mathbb{S}(M) \setminus \{0\}$.*

Proof. Assume that there exists $d \in D$ such that $D = dR$. If $0 \neq t \in \text{ann}(d)$, then $0 = dRt = Dt = D$, a contradiction. Then $\text{ann}(d) = 0$ and hence $D = dR \cong R$ as R-modules. Therefore R_R is a divisible module and hence for each non-zero element $a \in R$ there exists $b \in R$ such that $a.b = 1$. This implies that R is a field. It is a contradiction. Therefore D is not simple. Assume that N is a non-trivial submodule of D. If $0 \neq t \in (N : D)$, then $D = Dt \subseteq N$, a contradiction. On the other hand $(N : M) \subseteq (N : D) = 0$. Therefore for each non-zero submodule K of M, $K(N : M) = 0$. Hence $K \in \mathbb{A}^*(M)$. □

In Lemma 2.4, we have already observed that if $N \cap K = \{0\}$, then N is adjacent to K. In the sequel, we give a partial converse of the aforementioned observation. The reader is reminded that a homogeneous component of a semisimple module is the direct sum of all the simple isomorphic submodules.

Proposition 2.13. *Let M be a finitely generated semisimple R-module such that its homogenous components are simple. Then for each pair of non-trivial submodules N and K of M, $N * K = 0$ if and only if $N \cap K = 0$*

Proof. Let $M = \oplus_{i=1}^{n} S_i$, where S_i's are non-isomorphic simple submodules of M. By Lemma 2.4, the "only if" part is obvious. Conversely, assume that N and K are two non-zero submodules of M

such that $N * K = 0$. By [5, Lemma 9.1], there exist subsets I and J of $I_n = \{1, 2, \dots, n\}$ such that $M = N \oplus (\oplus_{i\in I} S_i)$ and $M = K \oplus (\oplus_{j\in J} S_J)$. Without loss of generality, suppose that $N(K : M) = 0$. Then

$$(K : M) = \bigcap_{j\in J} \operatorname{ann}(S_j) \subseteq \operatorname{ann}(N) = \bigcap_{i\in I_n\setminus I} \operatorname{ann}(S_i) \subseteq \operatorname{ann}(N \cap K).$$

If $N \cap K \neq 0$, then $N \cap K$ contains a simple submodule S which is isomorphic to S_t for some $t \in I_n \setminus J$. Therefore $\bigcap_{j\in J} \operatorname{ann}(S_j) \subseteq \operatorname{ann}(S) = \operatorname{ann}(S_t)$. Since $\operatorname{ann}(S_i)$'s are maximal ideals of R, $\operatorname{ann}(S_j) = \operatorname{ann}(S_t)$, for some $j \in J$ and hence $S_j \cong S_t$. Since the homogenous components of M are simple, $S_j = S_t = S$, a contradiction. □

Theorem 2.14. *Let S_1 and S_2 be two non-isomorphic simple submodules of M such that $M = S_1 \oplus S_2$. Then $\mathbb{G}(M)$ is a complete bipartite graph. Moreover, $V(\mathbb{G}(M)) = V_1 \bigcup V_2$, where for $i = 1, 2$, $V_i = \{K \leq M | K \cong S_i\}$.*

Proof. Since any non-trivial submodule of M is a summand, $\mathbb{S}(M) \setminus \{\{0\}, M\} \subseteq \mathbb{A}^*(M)$. On the other hand for each non-trivial submodule of M, say K, $K(M : M) \neq 0$. Moreover, if $M(K : M) = 0$, then $(K : M) \subseteq \operatorname{ann}(M) = \operatorname{ann}(S_1) \cap \operatorname{ann}(S_2)$. Since $(K : M) = \operatorname{ann}(S_1)$ or $(K : M) = \operatorname{ann}(S_2)$ and $\operatorname{ann}(S_i)$'s are maximal ideals of R, $\operatorname{ann}(S_1) = \operatorname{ann}(S_2)$ and hence $S_1 \cong S_2$, a contradiction. Thus $\mathbb{S}(M) \setminus \{\{0\}, M\} = \mathbb{A}^*(M)$. Now assume that $N, K \in \mathbb{A}^*(M)$ which are adjacent. By [5, Lemma 9.1], N and K are simple submodules of M. Therefore $N * K = 0$ implies that either $\operatorname{ann}(K^c) \subseteq \operatorname{ann}(N)$ or $\operatorname{ann}(N^c) \subseteq \operatorname{ann}(K)$ and hence N and K are not isomorphic. On the other hand if N and K are non-isomorphic simple submodules of M, then $N \cap K = 0$ and hence by Lemma 2.4, $N * K = 0$. □

3. Bipartite and Star Graphs

In this section we want to consider the modules for which their annihilating submodule graphs are bipartite. An R-module M is called reduced provided that for each $m \in M$ and $a \in R$, $ma^2 = 0$ implies that $ma = 0$. The ring R is reduced if R is reduced as an R-module. The following theorem plays a fundamental role in this section.

Theorem 3.1. *Let M be a reduced R-module such that $\mathbb{G}(M)$ is a bipartite graph. Then, one of the following cases may occur.*

1. U.dim$(M) = 2$.
2. Soc$(M) = S$ *where S is a simple essential submodule of M (hence M_R is uniform). Moreover, in both cases, $\mathbb{G}(M)$ is a complete bipartite graph.*

Proof. Assume that V_1 and V_2 are two non-empty subsets of $\mathbb{A}^*(M)$ such that $\mathbb{A}^*(M) = V_1 \cup V_2$, $V_1 \cap V_2 = \emptyset$ and no element of V_1 or V_2 is adjacent to another member of V_1 or V_2, respectively. Two cases my happen.

(Case 1). Assume that for each $A \in V_1$ and $B \in V_2$, $A \cap B = 0$. By Lemma 2.4, A and B are adjacent in $\mathbb{G}(M)$. Therefore $\mathbb{G}(M)$ is a complete bipartite graph. If C is a non-zero submodule of M such that $C \cap (A \oplus B) = 0$, then both $C \cap A = 0$ and $C \cap B = 0$. By Lemma 2.4(2), $C * A = 0$ and $C * B = 0$ and

hence $C \in V_1 \cap V_2$, a contradiction. Hence $A \oplus B \subseteq_{\text{ess}} M_R$. On the other hand, if A' and A'' are two non-zero submodules of A such that $A' \cap A'' = 0$, then by Lemma 2.4, $A' * B = A'' * B = A' * A'' = 0$ which implies that $A', A'' \in V_1$ that are adjacent, a contradiction. Consequently, A and B are uniform submodules of M such that $A \oplus B \subseteq_{\text{ess}} M_R$. This implies that $\text{U.dim}(M_R) = 2$.
(Case 2). Assume that there exist $N \in V_1$ and $K \in V_2$ such that $N \cap K \neq 0$. Put $S = N \cap K$. By proposition 2.7, $\mathbb{G}(M)$ is a connected graph and hence there exist $N' \in V_2$ and $K' \in V_1$ such that both $N * N' = 0$ and $K * K' = 0$. By Lemma 2.4(1), $S * N' = S * K' = 0$. If $S \neq N'$ and $S \neq K'$, then $S \in V_1 \cap V_2$, a contradiction. Assume that $S = K'$. Since $K * K' = 0$ and S is a submodule of K, by Lemma 2.4(1), $S * K' = S * S = 0$. Therefore $S(S : M) = 0$ and hence $M(S : M)(S : M) = 0$ and consequently $(S : M)^2 \subseteq \text{ann}(M)$. Since M_R is reduced, $(S : M) \subseteq \text{ann}(M)$. Consequently, for each non-zero submodule A of M, we have $A(S : M) = 0$ and hence $A * S = 0$. Hence $\mathbb{A}^*(M) = \mathbb{S}(M) \setminus \{0\}$. Since $\mathbb{G}(M)$ is bipartite and S is adjacent to any element of $\mathbb{A}^*(M)$, $V_1 = \{S\}$ and $V_2 = \mathbb{A}^*(M) \setminus \{S\} = \mathbb{S}(M) \setminus \{S, 0\}$ and $\mathbb{G}(M)$ is a complete bipartite graph. Since $N, K' \in V_1$, $N = K' = S$. On the other hand, by Lemma 2.4, for each non-zero submodule A of $N \cap K = S$, we have $A * K = 0$ since $K * K' = 0$ and hence $A \in V_1$. Therefore $A = S$. This implies that $N \cap K = S$ is a simple submodule of M. If there exists $T \in V_2$ such that $S \cap T = 0$, then with the same method that was described in the proof of first part, it can be shown that $\text{U.dim}(M_R) = 2$. Now, assume that for each $T \in V_2$, $S \cap T \neq 0$. Since S is simple and $V_2 = \mathbb{A}^*(M) \setminus \{S\} = \mathbb{S}(M) \setminus \{S, 0\}$, $\text{Soc}(M_R) = S$ and S is contained in any non-zero submodule of M. □

Corollary 3.2. (1) *Let M be a reduced R-module such that $\mathbb{G}(M)$ is a bipartite graph. Then* $\text{U.dim}(M) \leq 2$.

(2) *Let R be a reduced ring which is not a domain. If $\mathbb{G}(R)$ is a bipartite graph, then* $\text{U.dim}(R) = 2$.

(3) *Let M be a reduced R-module such that $\mathbb{G}(M)$ is a tree. Then $\mathbb{G}(M)$ is a star graph.*

Proof. (1) By Theorem 3.1, the verification is immediate.
(2). By part 1, $\text{U.dim}(R) = 1$ or $\text{U.dim}(R) = 2$. But $\text{U.dim}(R) = 1$ if and only if R is a domain. Therefore $\text{U.dim}(R) = 2$.
(3). Since each tree is a bipartite graph, $\mathbb{G}(M)$ is bipartite. By Theorem 3.1, $\mathbb{G}(M)$ is a complete bipartite graph. Since $\mathbb{G}(M)$ has no cycle, $\mathbb{G}(M)$ souled be a star graph. □

In [8, Theorem 2.2], it has been proved that for a reduced commutative ring R, $\text{gr}(\Gamma(R)) = 4$ if and only if $\Gamma(R) = K_{m,n}$ with $m, n \geq 2$. In [8, Theorem 2.4], it has been proved that for a reduced commutative ring R, $\Gamma(R)$ is nonempty with $\text{gr}(\Gamma(R)) = \infty$ if and only if $\Gamma(R) = K_{1,n}$ for some $n \geq 1$. Here we state and prove the analog of this result for $\mathbb{G}(M)$. We need an auxiliary lemma before giving the proof of our proposition.

Lemma 3.3. *Let M be an R-module. If $\mathbb{G}(M)$ contains a cycle of odd length, then* $\text{gr}(\mathbb{G}(M)) = 3$.

Proof. Assume that $K_1 - K_2 \cdots - K_{2n} - K_{2n+1} - K_1$ is a cycle of length $2n+1$ in $\mathbb{G}(M)$. We show that there exist a $k < n$ and a cycle of length $2k+1$ in $\mathbb{G}(M)$. Hence by induction the proof will be complete. If $K_1 \cap K_3 = 0$, then by Lemma 2.4(2), K_1 is adjacent to K_3 and hence $K_1 - K_2 - K_3 - K_1$

is a cycle of length 3 in $\mathbb{G}(M)$. Now assume that $K_1 \cap K_3 \neq 0$. The following cases may occur.
(Case 1). If $K_1 \cap K_3 = K_1$, then by Lemma 2.4(1), K_4 is adjacent to K_1 and hence $K_1 - K_4 - K_5 - \cdots - K_{2n+1} - K_1$ is a cycle of length $2(n-1)+1$ in $\mathbb{G}(M)$.
(Case 2). If $K_1 \cap K_3 = K_2$, then by Lemma 2.4(1), K_4 is adjacent to K_2 and hence $K_2 - K_3 - K_4 - K_2$ is a cycle of length 3 in $\mathbb{G}(M)$.
(Case 3). If $K_1 \cap K_3 = K_3$, then by Lemma 2.4(1), K_{2n+1} is adjacent to K_3 and hence $K_3 - K_4 - K_5 - \cdots - K_{2n+1} - K_3$ is a cycle of length $2(n-1)+1$ in $\mathbb{G}(M)$.
(Case 4). If $K_1 \cap K_3 = K_{2m}$, for some $2 \leq m \leq n$, then by Lemma 2.4(1), K_2 is adjacent to K_{2m} and hence $K_2 - K_3 - K_4 - \cdots - K_{2m} - K_2$ is a cycle of length $2m-1 = 2(m-1)+1$ in $\mathbb{G}(M)$.
(Case 5). If $K_1 \cap K_3 = K_{2m+1}$, for some $2 \leq m \leq n$, then by Lemma 2.4(1), K_2 is adjacent to K_{2m+1}. If $m = n$, then $K_1 - K_2 - K_{2n+1} - K_1$ is a cycle of length 3 in $\mathbb{G}(M)$. If $m < n$, then $K_2 - K_{2m+1} - K_{2m+2} - \cdots - K_{2n+1} - K_1 - K_2$ is a cycle of length $2(n-m+1)+1$ in $\mathbb{G}(M)$.
(Case 6). If $K_1 \cap K_3 \neq K_i$, for each $1 \leq i \leq 2n+1$, then by Lemma 2.4(1), K_4 and K_{2n+1} are adjacent to $K_1 \cap K_3$ and hence $K_{2n+1} - K_1 \cap K_3 - K_4 - \cdots - K_{2n} - K_{2n+1}$ is a cycle of length $2(n-1)+1$ in $\mathbb{G}(M)$. □

Proposition 3.4. *Let M be a reduced R-module such that $\mathbb{A}^*(M) \neq \mathbb{S}(M) \setminus \{0\}$. Then*

(1) $\mathrm{gr}(\mathbb{G}(M)) = 4$ *if and only if* $\mathbb{G}(M) = K_{m,n}$*, for some* $m, n \geq 2$.
(2) $\mathrm{gr}(\mathbb{G}(M)) = \infty$ *if and only if* $\mathbb{G}(M) = K_{1,n}$*, for some* $n \geq 1$.

Proof. (1). Assume that $\mathrm{gr}(\mathbb{G}(M)) = 4$. By Lemma 3.3, the length of any cycle in $\mathbb{G}(M)$ is even. Then $\mathbb{G}(M)$ is a bipartite graph. Suppose that the set of vertices of $\mathbb{G}(M)$ is the union of two distinct non-empty subsets V_1 and V_2. By Theorem 3.1, $\mathbb{G}(M)$ is complete bipartite. Since $\mathbb{G}(M)$ contains a cycle of length 4, both $|V_1|$ and $|V_2|$ must be greater than or equal to 2. The converse is trivial.
(2). Since $\mathbb{G}(M)$ is a connected graph which has no cycle, $\mathbb{G}(M)$ is a tree and hence is a (complete) bipartite graph. Since $\mathbb{G}(M)$ has no cycle, it must be a star graph. □

Corollary 3.5. *Let R be a reduced (semiprime) ring. Then*

(1) $\mathrm{gr}(\mathbb{G}(R)) = 4$ *if and only if* $\mathbb{G}(R) = K_{m,n}$*, for some* $m, n \geq 2$.
(2) $\mathrm{gr}(\mathbb{G}(R)) = \infty$ *if and only if* $\mathbb{G}(R) = K_{1,n}$*, for some* $n \geq 1$.

Proof. By Proposition 3.4, it is enough to show that $\mathbb{A}^*(R) \neq \mathbb{S}(R) \setminus \{0\}$. It is clear that for each ideal I of R, $(I : R) = I$. Therefore $R \notin \mathbb{A}^*(R)$. □

In the following we investigate R-modules such that for which their annihilating submodule graphs are star graphs.

Proposition 3.6. *Let M be a reduced R-module such that $\mathbb{G}(M)$ is a star graph. Then* $\mathrm{Soc}(M) \neq 0$*. Moreover, either there exist a simple submodule S and a uniform submodule U of M such that $S \oplus U \subseteq_{\mathrm{ess}} M$ or there exists a simple submodule S of M which is contained in any non-zero submodule of M.*

Proof. Since $\mathbb{G}(M)$ is a star graph, there exists $S \in \mathbb{A}^*(M)$ such that $\mathbb{A}^*(M) = V_1 \cup V_2$, where $V_1 = \{S\}$ and $V_2 = \mathbb{A}^*(M) \setminus \{S\}$. In addition, by Theorem 3.1, for each $N, K \in V_2$, $S * N = 0$ and $N * K \neq 0$. If S' is a non-zero submodule of S, then by Lemma 2.4, $S' * K = 0$ for each $K \in V_2$. Then $S' \in V_1 = \{S\}$. Therefore $S' = S$. It implies that S is a simple submodule of M. By Theorem 3.1, either $S \oplus U \subseteq_{\text{ess}} M$, for a uniform submodule U of M or S is contained in any non-zero submodule of M. □

The ring R is called a right V-ring provided that every simple right R-module is injective. Commutative ring R is a V-ring if and only if R is a Von-Neumann regular ring.

Corollary 3.7. *Let R be a* V*-ring and M a reduced R-module. If $\mathbb{G}(M)$ is a star graph, then* U.dim$(M) = 2$.

Proof. By Proposition 3.6, Soc$(M) \neq 0$. Let S be a simple submodule of M. Since R is a V-ring, S is injective and hence it has no proper essential extension. Since $\mathbb{G}(M)$ is a star graph, $S \neq M$. Then by Proposition 3.6, there exist a simple submodule S and uniform submodule U of M such that $S \oplus U \subseteq_{\text{ess}} M$. □

Proposition 3.8. *Let M be a finitely co-generated reduced R-module. If $\mathbb{G}(M)$ is a non-empty bipartite graph, then either $\mathbb{G}(M)$ is a star graph with $\mathbb{A}^*(M) = \mathbb{S}(M) \setminus \{0\}$ or there are simple submodules S_1 and S_2 of M such that $\mathbb{A}^*(M) = V_1 \cup V_2$, where for $i = 1, 2$, $V_i = \{N \in \mathbb{S}(M) \setminus \{0\} | N \cap S_i = \{0\}\}$.*

Proof. Assume that V_1 and V_2 are two non-empty subsets of $\mathbb{A}^*(M)$ such that $\mathbb{A}^*(M) = V_1 \cup V_2$, $V_1 \cap V_2 = \emptyset$ and no element of V_1 or V_2 is adjacent to another member of V_1 or V_2, respectively. Since M_R is finitely co-generated, Soc$(M) \subseteq_{\text{ess}} M$. On the other hand by Corollary 3.2, U.dim$(M) \leq 2$. Therefore either Soc$(M) = S$ for some simple submodule S of M or Soc$(M) = S_1 \oplus S_2$ for simple submodules S_1 and S_2 of M. Suppose that Soc$(M) = S$ and $N \in \mathbb{A}^*(M)$. There exists $K \in \mathbb{A}^*(M)$ such that $N * K = 0$. Since $S \subseteq_{\text{ess}} M$, both $S \subseteq N$ and $S \subseteq K$. By Lemma 2.4(1), $S * S = 0$. Therefore $S(S : M) = 0$ and hence $M(S : M)(S : M) = 0$. It implies that $(S : M)^2 \subseteq \text{ann}(M)$. Since M_R is reduced, $(S : M) \subseteq \text{ann}(M)$. Consequently, for each non-zero submodule A of M, we have $A(S : M) = 0$. Therefore $A * S = 0$. Hence $\mathbb{A}^*(M) = \mathbb{S}(M) \setminus \{0\}$.
Now, assume that Soc$(M) = S_1 \oplus S_2$ for simple submodules S_1 and S_2 of M. By Lemma 2.4, $S_1, S_2 \in \mathbb{A}^*(M)$ which are adjacent. Without loss of generality, suppose that $S_1 \in V_2$ and $S_2 \in V_1$. If $K \in V_1$ such that $K \cap S_1 \neq 0$, then $S_1 \subseteq K$. By Theorem 3.1, $\mathbb{G}(M)$ is a complete bipartite graph and $S_1 * K = 0$. By Lemma 2.4, $S_1 * S_1 = 0$ and as above $(S_1 : M) \subseteq \text{ann}(M)$. Therefore $\mathbb{G}(M)$ is a star graph with $V_2 = \{S_1\}$. Now suppose that for each $K \in V_1$ and $N \in V_2$, $K \cap S_2 = 0$ and $N \cap S_1 = 0$, respectively. By Lemma 2.4, for $i = 1, 2$, $V_i = \{N \in \mathbb{S}(M) \setminus \{0\} | N \cap S_i = \{0\}\}$, for $i = 1, 2$. □

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References

[1] M. F. Atiyah and I. G. Macdonald, *Introduction to Commutative Algebra*, University of Oxford, Addison-Wesley publishing company, 1969.

[2] G. Aalipour, S. Akbari, M. Behboodi, R. Nikandish, M. J. Nikmehr and F. Shaveisi, The classification of the annihilating-ideal graphs of commutative rings, *Algebra Colloq.*, **21** (2014) 249–256.

[3] S. Akbari and A. Mohammadian, On zero-divisor graphs of finite rings, *J. Algebra*, **314** (2007) 168–184.

[4] D. F. Anderson, M. C. Axtell and J. A. Stickles, *Zero-divisor graphs in commutative rings, in Commutative Algebra, Noetherian and Non-Noetherian Perspectives*, (M. Fontana, S.-E. Kabbaj, B. Olberding, I. Swanson, Eds.), Springer-Verlag, New York, 2011.

[5] F. W. Anderson and K. R. Fuller, *Ring and Category of Modules*, New York, Springer-Verlag, 1992.

[6] D. F. Anderson, A. Frazier, A. Lauve and P. S. Livingston, *The zero-divisor graph of a commutative ring II*, Ideal theoretic methods in commutative algebra (Columbia, MO, 1999), Lecture Notes in Pure and Appl. Math., **220**, Dekker, New York, 2001 61–72.

[7] D. F. Anderson and P. S. Livingston, The zero-divisor graph of a commutative ring, *J. Algebra*, **217** (1999) 434–447.

[8] D. F. Anderson and S. B. Mulay, On the diameter and girth of a zero-divisor graph, *J. Pure Appl. Algebra*, **210** (2007) 543–550.

[9] G. Aalipour, S. Akbari, M. Behboodi, R. Nikandish, M. J. Nikmehr and F. Shaveisi, The classification of the annihilating-ideal graphs of commutative rings, *Algebra Colloq.*, **21** (2014) 249–256.

[10] D. Lu and T. Wu, On bipartite zero-divisor graphs, *Discrete Math.*, **309** no. 4 (2009) 755–762.

[11] B. Allen, E. Martin, E. New and D. Skabelund, Diameter, girth and cut vertices of the graph of equivalence classes of zero-divisors, *Involve* **5** (2012) 51–60.

[12] I. Beck, Coloring of commutative rings, *J. Algebra*, **116** (1988) 208–226.

[13] M. Baziar, E. Momtahan and S. Safaeeyan, A Zero-divisor Graph for Module with Respect to Their (First) Dual, *J. Algebra Appl.*, **12** (2013) pp. 11.

[14] M. Baziar, E. Momtahan, S. Safaeeyan and N. Ranjbar, Zero-divisor graph of abelian groups, *J. Algebra Appl.*, **13** (2014) pp. 13.

[15] M. Behboodi and Z. Rakeei, The annihilating-ideal graph of commutative rings I, *J. Algebra Appl.*, **10** (2011) 727–739.

[16] M. Behboodi and Z. Rakeei, The annihilating-ideal graph of commutative rings II, *J. Algebra Appl.*, **10** (2011) 741–753.

[17] S. B. Mulay, Cycles and symmetries of zero-divisors, *Comm. Algebra*, **30** (2002) 3533–3558.

[18] S. Safaeeyan, M. Baziar and E. Momtahan, A generalization of the zero-divisor graph for modules, *J. Korean Math. Soc.*, **51** (2014) 87–98.

[19] S. Spiroff and C. Wickham, A Zero Divisor Graph Determined by Equivalenc Classes of Zero Divisors, *comm. Algebra*, **39** (2011) 2338–2348.

[20] T. Y. Lam, *A first Course in Noncommutative Rins*, Graduate Texts in Mathematics, **131**, New York, Berlin, Springer-Verlag, (1991).

[21] T. Y. Lam , *Lectures on Modules and Rings*, Graduate Texts in Mathematics, **139**, New York, Berlin, Springer-Verlag, (1998).

[22] D. B. West, *Introduction to Graph Theory*, 2nd ed., Prentice Hall, Upper Saddle River, 2001.

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