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COMMON EXTREMAL GRAPHS FOR THREE INEQUALITIES INVOLVING DOMINATION PARAMETERS

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Abstract. Let $\delta(G)$, $\Delta(G)$ and $\gamma(G)$ be the minimum degree, maximum degree and domination number of a graph $G = (V(G), E(G))$, respectively. A partition of $V(G)$, all of whose classes are dominating sets in G, is called a domatic partition of G. The maximum number of classes of a domatic partition of G is called the domatic number of G, denoted $d(G)$. It is well known that $d(G) \leq \delta(G)+1$, $d(G)\gamma(G) \leq |V(G)|$ [6], and $|V(G)| \leq (\Delta(G)+1)\gamma(G)$ [3]. In this paper, we investigate the graphs G for which all the above inequalities become simultaneously equalities.

1. **Introduction**

All graphs considered in this paper are finite, undirected, loopless, and without multiple edges. We refer the reader to the book [9] for graph theory notation and terminology not described here. We denote the vertex set and the edge set of a graph G by $V(G)$ and $E(G)$, respectively. For a subset $S \subseteq V(G)$ the *subgraph induced by* S is the graph $\langle S\rangle$ with vertex set S and edge set $\{xy \in E(G) \colon x, y \in S\}$. The *complement* $\overline{G}$ of G is the simple graph whose vertex set is V and whose edges are the pairs of nonadjacent vertices of G. The square G^2 of a graph G is another graph that has the same set of vertices, but in which two vertices are adjacent when their distance in G is at most 2. We write K_n for the *complete graph* of order n and C_r for the *cycle* of length r. For any vertex x of a graph G, $N_G(x)$ denotes the set of all neighbors of x in G, $N_G[x] = N_G(x) \cup \{x\}$ and the *degree* of x is $deg_G(x) = |N_G(x)|$. The minimum and maximum degree of a graph G are denoted by $\delta(G)$ and $\Delta(G)$, respectively.

A set of vertices D in a graph G is a dominating set if every vertex in $V(G) - D$ is adjacent to at least one vertex in D. The *domination number* of a graph G, denoted by $\gamma(G)$, is the minimum cardinality

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of a dominating set of G. A dominating set of G is a *γ-set*, if its cardinality is $\gamma(G)$. A graph G is said to be *excellent* if every vertex belongs to some γ-set. A dominating set D of a graph G is called an *efficient dominating set* (an ED-*set*) if the distance between any two vertices in D is at least three. Not all graphs have ED-sets. If G has an ED-set, then any ED-set is a γ-set of G [1]. A partition of a nonempty set X is a family of nonempty subsets of X such that every element x in X is in exactly one of these subsets. A *domatic partition* of a graph G is a partition of $V(G)$ into dominating sets. Since $V(G)$ is a dominating set of G, each graph has a domatic partition. The *domatic number* $d(G)$ of G is the maximum number of elements in a domatic partition of G. The concept of domatic number of a graph was introduced by Cockayne and Hedetniemi [6]. A domatic partition of order $d(G)$ is a *d-partition*. The problem of obtaining such a partition is known to be NP-complete even for circular arc graphs [2] but can be solved in linear time for interval graphs [14]. A graph G is called *uniquely domatic*, if G has exactly one d-partition. A graph G is called *domatically critical* if after deleting an arbitrary edge from G, a graph with a smaller domatic number than that of G is obtained [4]. The domatic partition problem arises in various situations of locating facilities in a network. Assume that a node in a network can access only resources located at neighboring nodes (or at itself). Then if there is an essential type of resource that must be accessible from every node (a hospital, a printer, a file, etc.), copies of the resource need to be distributed over a dominating set of the network. If there are several essential types of resources, each one of them occupies a dominating set. If each node has bounded capacity, there is a limit to the number of resources that can be supported. In particular, if each node can only serve a single resource, the maximum number of resources supportable equals the domatic number of the graph [8].

A set of vertices $I \subseteq V(G)$ is *independent* if no two vertices in I are adjacent. An *independent dominating set* in G is a set of vertices of G which is both independent and dominating. Partitions into independent dominating sets of $V(G)$ were first considered as particular domatic partitions under the term of *indominable partitions* [5] or *idomatic partitions* [18]. Now the term idomatic is more usual (cf. for instance [15]). Not each graph has an idomatic partition. When an idomatic partition exists on a graph G, then G is called *idomatic* and the *idomatic number* $id(G)$ equals the maximum number of elements in an idomatic partition.

To continue we need the following results.

Theorem A. *For any n-vertex graph G,*

(i) [6] $d(G) \leq \delta(G) + 1$ *and* $d(G)\gamma(G) \leq n$, *and*

(ii) [3] $n \leq (\Delta(G) + 1)\gamma(G)$.

In this paper, we mainly turn our attention to the graphs G for which all the inequalities in Theorem A become simultaneously equalities.

2. **Results**

A graph G without isolated vertices is said to be *δ-edge critical* if $\delta(G) > \delta(G - e)$ for each $e \in E(G)$ [10]. Clearly each k-regular graph, $k \geq 1$, is δ-edge critical. A graph G is said to be *domatically full* if $d(G) = \delta(G) + 1$. All elements of the following classes of graphs are domatically full: (a) trees with at least 2 vertices [6], (b) outerplanar graphs [6], (c) cycles on $3k$ vertices [6], and (c) strongly chordal graphs [7].

Theorem 2.1. *Let G be a δ-edge critical and domatically full graph. Then the following holds:*

(i) $d(G) = id(G)$ *and each domatic partition of order* $d(G)$ *is idomatic.*

(ii) *If $D_1, \dots, D_k$ is a domatic partition of G with $k = \delta(G) + 1$, then (a) each connected component of the graph $\langle D_i \cup D_j \rangle$ is a star, $i, j = 1, \dots, k$ and $i \neq j$, and (b) if $x \in D_i \cup D_j$ is of minimum degree in G, then x is a leaf of $\langle D_i \cup D_j \rangle$.*

(iii) *If G is regular and the partition $D_1, \dots, D_k$ is as in (ii), then $\langle D_i \cup D_j \rangle$ is 1-regular for all $i, j = 1, 2, \dots, k$, $i \neq j$.*

(iv) *G is domatically critical.*

Proof. Since G is δ-critical, the set M_δ consisting of all vertices of G having degree more then $\delta(G)$ is independent or empty. Consider any domatic partition $D_1, \dots, D_k$ of G with $k = d(G)$. Since G is domatically full, $k = \delta(G) + 1$. Let $x \in D_i$ and $deg(x) = \delta(G)$. Since each D_j, $j \neq i$, dominates x, each D_j contains exactly one vertex of $N(x)$. Hence x is a leaf of $\langle D_i \cup D_j \rangle$ and D_i is independent. But then $D_1, \dots, D_k$ is an idomatic partition of G and if a vertex $y \in V(\langle D_i \cup D_j \rangle)$ is in M_δ, then all its neighbors are leaves. Thus, (i) and (ii) are satisfied. Clearly, (iii) is an immediate consequence of (ii).

(iv) Since G is δ-edge critical, $\delta(G - e) = \delta(G) - 1$ for each edge e in G. Now by Theorem A, $d(G - e) \leq \delta(G - e) + 1 = \delta(G) \leq d(G) - 1$. Thus, G is domatically critical. □

An *efficient domination partition* (or an *ED-partition*) of a graph G is a partition of $V(G)$ into ED-sets. We say that a graph G is an *efficient domination partitionable* graph (or an EDP-*graph*) if G has an ED-partition. First results on the graphs whose vertex set has a partition in ED-sets are obtained by Mollard in [13]. Clearly, any ED-partition of an EDP-graph is both a domatic partition and an idomatic partition of order $d(G) = id(G)$. A graph G is a *uniquely efficient domination partitioned graph* (or a UEDP-*graph*) if it has only one ED-partition.

The next theorem shows that each regular domatically full graph is an EDP-graph, and vise versa.

Theorem 2.2. *Let G be a graph of order n. Then the following assertions are equivalent.*

(i) *G is an EDP-graph.*

(ii) *G is regular and domatically full.*

(iii) $n = \gamma(G)(\Delta(G) + 1)$ *and* $n = d(G)\gamma(G)$.

Proof. (i) ⇒ (ii): Any 2 vertices in the same closed neighborhood of a vertex of G belong to different ED-sets. Hence $d(G) \geq \Delta(G) + 1$. This and Theorem A(i) leads to $\delta(G) = \Delta(G)$ and $d(G) = \delta(G) + 1$. Thus G is regular and domatically full.

(ii) ⇒ (iii): By Theorem A, we know that $n \leq \gamma(G)(\Delta(G)+1)$ and $d(G)\gamma(G) \leq n$. Since $\delta(G) = \Delta(G)$ and $d(G) = \delta(G) + 1$, (iii) is clearly valid.

(iii) ⇒ (ii): By (iii), we immediately have $d(G) = \Delta(G)+1$. But $d(G) \leq \delta(G)+1$ (Theorem A). Hence (ii) hods.

(ii) $\Rightarrow$ (i): Let $D_1, \dots, D_k$ be a d-domatic partition of G. Hence $k = d(G) = \delta(G)+1$. By Theorem 2.1, this partition is idomatic and $\langle D_i \cup D_j \rangle$ is 1-regular for all $i, j = 1, 2, \dots, k$, $i \neq j$. Therefore $D_1, \dots, D_k$ is an ED-partition of G. □

Corollary 2.3. *Let G be a s-regular graph. If $s \in \{0, 1\}$ then G is an EDP-graph. If $s = 2$ then G is an EDP-graph if and only if the order of each connected component of G is divisible by three.*

Observation 2.4. *(Folklore) Let G be a graph of order n.*

(i) *If G has an ED-set D whose all vertices have a maximum degree then $n = (\Delta(G) + 1)\gamma(G)$.*

(ii) *Let $n = (\Delta(G)+1)\gamma(G)$. Then all γ-sets of G are efficient dominating, and each vertex belonging to some γ-set of G has maximum degree. If G is excellent then G is regular.*

Proof. (i) Obvious.

(ii) Let $D = \{x_1, \dots, x_r\}$ be a γ-set of G. Then

$$n = |\cup_{i=1}^r N[x_i]| \leq \sum_{i=1}^r (deg(x_i) + 1) \leq r(\Delta(G) + 1) = \gamma(G)(\Delta(G) + 1). \tag{2.1}$$

Suppose that $n = \gamma(G)(\Delta(G) + 1)$. Then the inequalities in (2.1) must be equalities. Therefore $N[x_k] \cap N[x_l]$ is empty and $deg(x_i) = \Delta(G)$ for all $k, l, i \in \{1, 2, \dots, r\}$ and $k \neq l$. Thus, D is an ED-set and since D was chosen arbitrarily, each γ-set is an ED-set. The rest is obvious. □

Proposition 2.5. *Let G be an EDP-graph.*

(i) [16] *Then G is domatically critical.*

(ii) [16] *Any domatic partition $D_1, \dots, D_k$ of G, where $k = \delta(G) + 1$, is an ED-partition of G and the graph $\langle D_i \cup D_j \rangle$ is 1-regular, $i, j = 1, \dots, k$ and $i \neq j$.*

(iii) *Each γ-set of G is efficient dominating.*

(iv) *If G is an UEDP-graph, then G is uniquely domatic.*

Proof. (i) By Theorem 2.2, G is regular and domatically full. Now Theorem 2.1(iv) implies G is domatically critical.

(ii) Immediately by Theorem 2.1(iii).

(iii) Theorem 2.2 and Observation 2.4(ii) together leads to the required.

(iv) Immediately by (iii). □

Proposition 2.6. *Let G be an n-order r-regular graph and $n = \gamma(G)(r + 1) = \gamma(\overline{G})(\Delta(\overline{G}) + 1)$. Then $G \in \{K_n, \overline{K}_n\}$.*

Proof. Clearly $\overline{G}$ is a $(n - r - 1)$-regular graph. If $\gamma(G) = 1$ then $G = K_n$. If $\gamma(G) = n$ then $G = \overline{K_n}$. So, let $n > \gamma(G) \geq 2$. By Observation 2.4, any γ-set of G is effcient dominating. Hence any two vertices in a γ-set of G are at distance at least 3 and must form a γ-set of $\overline{G}$. Then $n = \gamma(G)(r + 1) = 2(n - r)$, which implies $n = 2r$ and $\gamma(G) = 2 - 2/(r + 1)$, a contradiction. □

Corollary 2.7. *The graphs G and $\overline{G}$ are both EDP-graphs if and only if one of them is complete.*

3. Examples

In this section we present some examples of EDP-graphs. A *crown graph* $H_{n,n}$, is a graph obtained from the complete bipartite graph $K_{n,n}$ by removing a perfect matching.

Example 3.1. *Let* $G = H_{n,n}$, $n \geq 3$, $V(G) = \{v_i, u_i \mid i = 1, 2, \ldots, n\}$ *and* $E(G) = \{v_i u_j \mid i \neq j\}$. *Then all* γ*-sets of* G *are* $\{u_i, v_i\}$, $i = 1, 2, \ldots, n$. *Obviously, they form the unique ED-partition of* G. *Thus,* G *is an UEDP-graph.*

Denote by $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$ the additive group of order n. Let S be a subset of $\mathbb{Z}_n$ such that $0 \notin S$ and $x \in S$ implies $-x \in S$. The *circulant graph* with distance set S is the graph $C(n; S)$ with vertex set $\mathbb{Z}_n$ and vertex x adjacent to vertex y if and only if $x - y \in S$. It is clear from the definition that $C(n; S)$ is vertex-transitive and regular of degree $|S|$.

Example 3.2. *Let* $G = C(n = (2k+1)t; \{1, \ldots, k\} \cup \{n-1, \ldots, n-k\})$, *where* $k, t \geq 1$. *Denote by* D_r *the set all elements of which are* $r, r + (2k+1), \cdots, r + (2k+1)(t-1)$, *where addition is taken mod* $(2k+1)t$ *and* $r \in \{0, \ldots, (2k+1)t - 1\}$. *Clearly all* D_r*'s are ED-sets, and* $D_0, D_1, \ldots, D_{2k}$ *is the unique ED-partition of* G. *Thus,* G *is an UEDP-graph.*

Example 3.3. *Let* $G = C(n; \{\pm 1, \pm s\})$ *where* $2 \leq s \leq n-2$ *and* $s \neq n/2$. *Then* G *has an ED-set if and only if* $5|n$ *and* $s \equiv \pm 2 \pmod 5$*; in addition, all ED-sets in* G *have the form* $D_i = \{v \in V(G) \mid v \equiv i \pmod 5\}$ [11]. *Thus (a)* G *is an EDP-graph if and only if* $5|n$ *and* $s \equiv \pm 2 \pmod 5$, *and (b) if* G *is an EDP-graph then* G *is an UEDP-graph.*

Let $n \geq 3$ and $k \in \mathbb{Z}_n - \{0\}$. The *generalized Petersen graph* $P(n,k)$ is the graph on the vertex-set $\{x_i, y_i \mid i \in \mathbb{Z}_n\}$ with adjacencies $x_i x_{i+1}, x_i y_i$, and $y_i y_{i+k}$ for all i. The graph $P(n,1)$ is equivalent to the *n-prism.*

Example 3.4. *A graph* $P(n,k)$ *is an EDP-graph if and only if* $n \equiv 0 \pmod 4$ *and* k *is odd. If* $P(n,k)$ *is an EDP-graph, then* $P(n,k)$ *is an UEDP-graph. In particular, an* n*-prism is an UEDP-graph if and only if* $n \equiv 0 \pmod 4$.

Proof. A generalized Petersen graph $P(n,k)$ has an ED-set if and only if $n \equiv 0 \pmod 4$ and k is odd (Theorem 1 [12]). Moreover, by the proof of this theorem it follows that each vertex of $P(n,k)$ belongs to exactly one ED-set. Let $n = 4r$ and k odd. We construct the only ED-partition D_0, D_1, D_2, D_3 of $P(n,k)$ as follows: $D_s = \{u_{4i+1+s} \mid 0 \leq i \leq r-1\} \cup \{v_{4i+3+s} \mid 0 \leq i \leq r-1\}$, $s = 0, 1, 2, 3$. Thus $P(n,k)$ is an UEDP-graph. The rest is obvious. $\square$

For two graphs G_1 and G_2, the *Cartesian product* $G_1 \Box G_2$ is the graph with vertex set $V(G_1) \times V(G_2)$ and $(x_1, x_2)(y_1, y_2) \in E(G_1 \Box G_2)$ if and only if $x_1 y_1 \in E(G_1)$ and $x_2 = y_2$ or $x_2 y_2 \in E(G_2)$ and $x_1 = y_1$. The *Cartesian power* $G^{\Box n}$ of a graph G is the graph recursively defined by $G^{\Box 1} = G$, and $G^{\Box n} = G^{\Box(n-1)} \Box G$ for $d > 1$. The *hypercube of dimension* n is the graph $Q_n = K_2^{\Box n}$.

Example 3.5. *The following result (reformulated in our present terminology) is from* [13].

(i) *Let G and H be two n-regular EDP-graphs and let H be bipartite. Then $G\Box H\Box P_2$ is an EDP-graph.*

(ii) *Let G be a bipartite EDP-graph. Then $G^{\Box 2^k}\Box Q_{2^k-1}$ is an EDP-graph, $k \geq 1$.*

The *wreath product* of graphs G and H is the graph, $G\,wr\,H$ with vertex set $V(G\,wr\,H) = V(G)\times V(H)$ and edge set $E(G\,wr\,H) = \{(x,y)(v,w) \mid xv \in E(G), \text{ or } x = v \text{ and } yw \in E(H)\}$. Informally, $G\,wr\,H$ is the graph obtained by replacing each vertex of G by a copy of H and putting all possible edges between copies of H that replaced adjacent vertices of G.

Example 3.6. *If G is an EDP-graph, then $G\,wr\,K_r$ is also an EDP-graph.*

Proof. Let $(D_i = \{x_i^1, x_i^2, \dots, x_i^s\})_{i=1}^k$ be an ED-partition of G, and let $V(K_r) = \{y_1, \dots, y_r\}$. Then $(U_i^j = \{(x_i^1, y_j), (x_i^2, y_j), \dots, (x_i^s, y_j)\})_{i=1}^k{}_{j=1}^r$ is an ED-partition of $G\,wr\,K_r$. □

4. EDP-graphs of order at most 10

We say that the partitions $(D_i)_{i=1}^r$ and $(U_j)_{j=1}^s$ of a set X are *orthogonal* whenever $|D_i \cap U_j| = 1$ for all $i = 1, 2, \dots, r$ and $j = 1, 2, \dots, s$.

The next observation is obvious but useful in the sequel.

Observation 4.1. *Let $\pi = (D_i = \{x_i^1, x_i^2, \dots, x_i^s\})_{i=1}^k$ be an ED-partition of an EDP-graph G.*

(A_1) *Then $\sigma_i = (N_G[x_i^j])_{j=1}^s$ is a partition of $V(G)$ which is orthogonal to π, $i = 1, 2, \dots, k$. If $k \geq 2$ then $G - D_i$ is an EDP-graph, $\pi - D_i$ is an ED-partition of $V(G - D_i)$, and $(N_G(x_i^j))_{j=1}^s$ is a partition of $V(G - D_i)$ which is orthogonal to $\pi - D_i$.*

(A_2) *Let $\tau = (U_j)_{j=1}^s$ be a partition of $V(G)$ which is orthogonal to π. Define the graph G' as obtained from G by adding s new vertices $u_1, u_2, \dots, u_s$ and $s(\Delta(G) + 1)$ new edges, so that G is an induced subgraph of G' and $N_{G'}(u_j) = U_j$. Then G' is an EDP-graph, $\pi' = \pi \cup \{u_1, u_2, \dots, u_s\}$ is an ED-partition of G', and $\tau' = (U_j \cup \{u_j\})_{j=1}^s$ is a partition of G' which is orthogonal to π'.*

(A_3) *$G_r = \langle \cup_{i=1}^r D_i\rangle$ is an EDP-graph for all $r = 1, \dots, k$.*

We next give some applications of Observation 4.1. Denote by $\mathtt{EDP}_n$ the set of all n-vertex mutually non-isomorphic EDP-graphs.

Theorem 4.2. *Let n be a positive integer.*

(i) *$\overline{K_n}, K_n \in \mathtt{EDP}_n$ and whenever n is prime, $\mathtt{EDP}_n = \{\overline{K_n}, K_n\}$.*

(ii) $\mathtt{EDP}_4 = \{\overline{K_4}, 2K_2, K_4\}$.

(iii) $\mathtt{EDP}_6 = \{\overline{K_6}, 3K_2, 2K_3, C_6, K_6\}$.

(iv) *$\mathtt{EDP}_8$ consists of $\overline{K_8}, 4K_2, 2K_4, H_{4,4} = Q_3, K_8$ and the 3-regular connected graph G_0 obtained from 2 disjoint copies of $K_4 - e$ by adding 2 edges.*

(v) $\mathtt{EDP}_9 = \{\overline{K_9}, 3K_3, K_3 \cup C_6, C_9, K_9\}$.

(vi) *$\mathtt{EDP}_{10}$ consists of $\overline{K_{10}}, 5K_2, 2K_5, H_{5,5} = C(10; \{\pm 1, \pm 3\}), K_{10}$ and the graphs $G_1, \dots, G_5$ depicted in Fig. 1.*

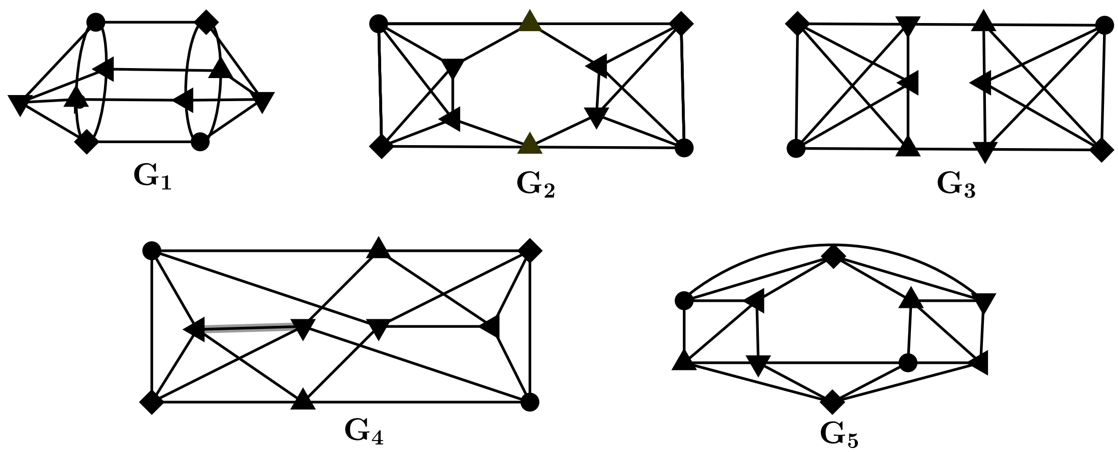

FIGURE 1. Graphs $G_1, \ldots, G_5$.

Proof. First note that all graphs mentioned in (i)–(vi) are clearly EDP-graphs. Consider any $G \in \mathtt{EDP}_n$. We know by Theorem 2.2 that G is regular and $n = \gamma(G)d(G) = \gamma(G)(\Delta(G) + 1)$. Hence if n is prime, then either $\Delta(G) = 0$ and $G = \overline{K_n}$, or $\Delta(G) = n - 1$ and $G = K_n$. Thus, (i) holds.

In what follows, let $n \in \{4, 6, 8, 9, 10\}$. If $\Delta(G) \in \{0, 1, 2\}$ then $G \in \{\overline{K_4}, 2K_2, \overline{K_6}, 3K_2, 2K_3, C_6, \overline{K_8}, 4K_2, \overline{K_9}, 3K_3, K_3 \cup C_6, C_9\}$, because of Corollary 2.3. If $\Delta(G) = n - 1$ then clearly $G = K_n$. So, let $3 \leq \Delta(G) \leq n - 2$. Since $\Delta(G) + 1$ is a divisor of n, only the following 2 cases are possible: (a) $n = 8$ and $\Delta(G) = 3$, and (b) $n = 10$ and $\Delta(G) = 4$.

Case 1: $n = 8$ and $\Delta(G) = 3$. Hence $\gamma(G) = 2$ and $d(G) = 4$. Let D_1, D_2, D_3, D_4 be any ED-partition of G. By Observation 4.1, $G - D_4$ is a 2-regular 6-vertex EDP-graph; hence by (iii), $G - D_4 \in \{2K_3, C_6\}$. It is easy to see that $G \in \{2K_4, G_0\}$ when $G - D_4 = 2K_3$, and $G \in \{G_0, Q_3\}$ when $G - D_4 = C_6$. Thus, (iv) holds.

Case 2: $n = 10$ and $\Delta(G) = 4$. Hence $\gamma(G) = 2$ and $d(G) = 5$. Consider any ED-partition $D_1, \ldots, D_5$ of G. Note that $G - D_5$ is a 3-regular 8-vertex EDP-graph because of Observation 4.1. Now by (iv), $G - D_5 \in \{2K_4, Q_3, G_0\}$. It is not hard to see that (a) $G \in \{2K_5, G_2, G_3\}$ when $G - D_5 = 2K_4$, (b) $G \in \{G_1, \ldots, G_5\}$ when $G - D_5 = G_0$, and (c) $G \in \{G_1, G_4, H_{5,5}\}$ when $G - D_5 = Q_3$. Thus, (vi) is valid. □

Proposition 4.3. *Let G be a s-regular EDP-graph of order n, where $n - 1 > s \geq 3$.*

(i) *Then $s \notin \{n - 3, n - 2\}$.*
(ii) *$s = n - 4$ if and only if $G = C_6$.*
(iii) *$s = n - 5$ if and only if $G \in \{\overline{K_5}, 3K_2, 2K_4, Q_3, G_0\}$.*
(iv) *$s = n - 6$ if and only if $G \in \{\overline{K_6}, 2K_5, H_{5,5}, G_1, \ldots, G_5\}$.*
(v) *If $n - s - 1$ is a prime, then $n = 2s + 2$ and $\gamma(G) = 2$.*

Proof. Obviously $n \geq 5$. By Theorem 2.2, $n = \gamma(G)(s + 1) = d(G)\gamma(G)$ and $d(G) = s + 1$. Hence $n/(s + 1)$ is an integer. But then (i) holds. Note now that all graphs mentioned in (ii)–(vi) are clearly EDP-graphs.

(ii) Since $n/(n-3)$ is an integer and $n \geq 5$, we have $n = 6$ and $s = 2$. Hence $G = C_6$.

(iii) $n/(n-4)$ is an integer implies that either ($n = 5$, $s = 0$ and then $G = \overline{K_5}$), or ($n = 6$, $s = 1$ and $G = 3K_2$), or ($s = 3$ and $G \in \mathtt{EDP}_8$). The result follows by Theorem 4.2(iv).

(iv) As $n/(n-5)$ is an integer, either $G = \overline{K_6}$ or ($s = 4$ and $G \in \mathtt{EDP}_{10}$). By Theorem 4.2(vi) we immediately obtain the required.

(v) Since $p = n - s - 1$ is a prime and $\gamma(G) = n/(s+1) = 1 + p/(s+1)$, it follows that either G is edgeless or ($p = s + 1$, $\gamma(G) = 2$ and $n = 2p$). □

5. Uniquely colorable graphs

The *chromatic number* $\chi(G)$ of a graph G is the minimum number of independent subsets that partition the vertex set of G. Any such minimum partition is called a *χ-partition* of $V(G)$. A graph G is called *uniquely $\chi(G)$-colorable* if G has exactly one χ-partition. Each member of the only χ-partition of a uniquely $\chi(G)$-colorable graph G is an independent dominating set of G; hence G is idomatic [5]. We need the following result, due to Zelinka [17].

Theorem B. [17] *Let G be a regular and domatically full graph. Then $d(G) = \chi(G^2)$ and each χ-partition of G^2 is a domatic partition of G with $d(G)$ members and vice versa. Furthermore, G is uniquely domatic if and only if G^2 is uniquely $\chi(G^2)$-colorable.*

Theorem 5.1. *Let G be a graph for which one of the following holds.*

(i) $G = H_{n,n}$, $n \geq 3$.

(ii) $G = C(n = (2k+1)t; \{1, \ldots, k\} \cup \{n-1, \ldots, n-k\})$, *where* $1 \leq k \leq (n-1)/2$ *and* $t \geq 1$.

(iii) *Let* $G = C(n; \{\pm 1, \pm s\})$, *where* $5|n$, $2 \leq s \leq n-2$, $s \equiv \pm 2 \pmod 5$ *and* $s \neq n/2$.

(iv) $G = P(n, k)$, *where* $n \equiv 0 \pmod 4$ *and* k *is odd.*

Then G^2 is uniquely $\chi(G^2)$-colorable and $d(G) = \chi(G^2)$.

Proof. Examples 3.1–3.4 show that all graphs in (i)–(iv) are UEDP-graphs. By Proposition 2.5(iv) all these graphs are uniquely domatic. The result now immediately follows by Theorem B. □

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References

[1] D. W. Bange, A. E. Barkauskas and P. J. Slater, Efficient Dominating Sets in Graphs, SIAM, Philadelphia, 1988, 189–199.

[2] M. A. Bonucelli, Dominating sets and domatic number of circular arc graphs, *Discrete Appl. Math.*, **12** (1985) 203–213.

[3] C. Berge, *Graphs and Hypergraphs*, North-Holland, Amsterdam, 1973.

[4] E. J. Cockayne, *Domination in undirected graphs - a survey*, Springer-Verlag, Berlin, 1978, 141–147.

[5] E. J. Cockayne and S. T. Hedetniemi, Disjoint independent dominating sets in graphs, *Discrete Math.*, **15** (1976) 213–222.

[6] E. J. Cockayne and S. T. Hedetniemi, Towards a Theory of Domination in Graphs, *Networks*, **7** (1977) 247–261

[7] M. Farber, Domination, idependent domination, and duality in strongly chordal graph, *Discrere Appl. Math.*, **7**(1984) 115–130

[8] S. Fujita, M. Yamashita and T. Kameda, A study on r-configurations – a resource assignment problem on graphs, *SIAM J. Disc. Math.*, **13** (2000) 227–254.

[9] T. W. Haynes, S. T. Hedetniemi and P. J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker, New York, 1998.

[10] T. W. Haynes, L. M. Lawson, R. C. Brigham and R. D. Dutton, Changing and unchanging of the graphical invariants: minimum and maximum degree, maximum clique sizes, node independence number and edge independence number, *Congr. numer.*, **72** (1990) 239–252

[11] J. Huang, J.-M. Xu The bondage numbers and efficient dominations of vertex-transitive graphs, *Discrete Math.*, **308** (2008) 571–582

[12] B. J. Ebrahimi, N. Jahanbakht, E. S. Mahmoodian, Vertex domination of generalized Petersen graphs, *Discrete Math.*, **309** (2009) 4355–4361

[13] M. Mollard, On perfect codes in Cartesian products of graphs, *Europ. J. Combin.*, **32** (2011) 398–403

[14] A. Srinivas Rao and C. Pandu Rangan, A linear algorithm for domatic number of interval graphs, *Inform. Process. Lett.*, **33** (1989-90) 29–33.

[15] M. Valencia-Pabon, Idomatic partitions of direct products of complete graphs, *Discrete Math.*, **310** (2010) 1118–1122.

[16] B. Zelinka, Domatically critical graphs, *Czechoslovak Math. J.*, **30** (1980) 486–489.

[17] B. Zelinka, On two problems of the graph theory, *Časopis pro pěstováni matematiky*, **107** (1982) 109–113

[18] B. Zelinka, Adomatic and idomatic numbers of graphs, *Math. Slovaca*, **33** (1983) 99–103

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