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SOME TOPOLOGICAL INDICES AND GRAPH PROPERTIES

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ABSTRACT. In this paper, by using the degree sequences of graphs, we present sufficient conditions for a graph to be Hamiltonian, traceable, Hamilton-connected or k-connected in light of numerous topological indices such as the eccentric connectivity index, the eccentric distance sum, the connective eccentricity index.

1. Introduction

Let G be a connected graph with vertex set $V(G)$ and edge set $E(G)$ such that $|V| = n$ and $|E| = m$. Let $d(v)$ be the degree of a vertex v in G. Let $d(u, v)$ be the distance between two vertices u and v in G, that is, the length of the shortest path connecting u and v in G. The eccentricity $\varepsilon(v)$ of a vertex v is the maximum distance from v to any other vertex. Let K_n, S_n, P_n be a complete graph, a star and a path on n vertices, respectively.

A cycle C in a graph G is called a Hamiltonian cycle of G if C contains all the vertices of G. A graph G is called *Hamiltonian* if G has a Hamiltonian cycle. A path P in a graph G is called a *Hamiltonian path* of G if P contains all the vertices of G. A graph G is called *traceable* if G has a Hamiltonian path. A graph G is called *Hamilton-connected* if for each pair of vertices in G there is a Hamiltonian path between them. A graph G is said to be k*-connected* (or k-vertex connected) if there does not exist a set of $k - 1$ vertices whose removal disconnects the graph. If G and H are two vertex-disjoint graphs, we use $G \vee H$ to denote the *join* of G and H .

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The topological indices are widely used in organic chemistry and have been found to be useful in chemical documentation, isomer discrimination, structure-property relationships, structure-activity (SAR) relationships and pharmaceutical drug design [14, 23]. In past decades, plenty of mathematical properties of numerous topological indices are reported such as the the matching energy [5, 6], Randić index [24] and the Balaban index [7].

For a connected graph G, its Wiener index, denoted by $W(G)$, is defined as

$$W(G) = \sum_{\{u,v\}\subseteq V(G)} d(u,v) = \frac{1}{2}\sum_{v\in V(G)} D(v).$$

Here $D(v) = \sum_{u\in V(G)} d_G(u,v)$. It can be easily verified that $D(v) \geq d(v) + 2(n-1-d(v))$. The Wiener index and its modifications are well studied in the past years, see [9, 17, 21, 19, 20].

The *eccentric connectivity index* (ECI) [22] of a connected graph G, denoted by $\xi^c(G)$, is defined as

$$\xi^c(G) = \sum_{v\in V(G)} \varepsilon(v)d(v).$$

The *eccentric distance sum* (EDS) [11] of a connected graph G is defined as

$$\xi^d(G) = \sum_{v\in V(G)} \varepsilon(v)\cdot D(v).$$

The *connective eccentricity index* (CEI) [10] of a connected graph G is defined as

$$\xi^{ce}(G) = \sum_{v\in V(G)} \frac{d(v)}{\varepsilon(v)}.$$

The above three topological indices involving eccentricities are widely studied from mathematical view, see [13, 18, 26, 27, 28].

In [25], Yang presented a sufficient condition for a graph to be traceable by using Wiener index. In [12], Hua and Wang presented a sufficient condition for a graph to be traceable by using Harary index. Li [15, 16] presented sufficient conditions in terms of the Harary index and Wiener index for a graph to be Hamiltonian or Hamilton-connected using some proof ideas in [25].

In this paper, as a continuance of the above results, we further study the conditions for a graph to be Hamiltonian, traceable, Hamilton-connected or k-connected in light of numerous topological indices such as the ECI, EDS and CEI.

2. **Preliminaries**

We first present some lemmas that will be used later.

Lemma 2.1. [8] *Let G be a graph of order $n \geq 3$ with degree sequence $d_1 \leq d_2 \leq \cdots \leq d_n$. If $d_k \leq k < \frac{n}{2} \Rightarrow d_{n-k} \geq n-k$, then G is Hamiltonian.*

Lemma 2.2. [1] *Let G be a nontrivial graph of order $n \geq 4$ with degree sequence $d_1 \leq d_2 \leq \cdots \leq d_n$. If $d_k + 1 \leq k < \frac{n+1}{2} \Rightarrow d_{n-k+1} \leq n-k-1$, then G is traceable.*

Lemma 2.3. [3] *Let G be a graph of order $n \geq 4$ with degree sequence $d_1 \leq d_2 \leq \cdots \leq d_n$. If $d_i \leq i + k - 2 \Rightarrow d_{n-k+1} \geq n - i$, for $1 \leq i \leq \frac{1}{2}(n - k + 1)$, then G is k-connected.*

Lemma 2.4. [2] *Let G be a graph of order $n \geq 4$ with degree sequence $d_1 \leq d_2 \leq \cdots \leq d_n$. If $2 \leq k \leq \frac{n}{2}, d_{k-1} \leq k \Rightarrow d_{n-k} \geq n - k + 1$, then G is Hamilton-connected.*

Lemma 2.5. [2, Page 210, Corollary 5] *Let $G = (X, Y; E)$ be a bipartite graph such that $X = \{x_1, x_2, \ldots, x_n\}$, $Y = \{y_1, y_2, \ldots, y_n\}$, $n \geq 2$, and $d(x_1) \leq d(x_2) \leq \cdots \leq d(x_n)$, $d(y_1) \leq d(y_2) \leq \cdots \leq d(y_n)$. If $d(x_k) \leq k < n \Rightarrow d(y_{n-k}) \geq n - k + 1$, then G is Hamiltonian.*

Lemma 2.6. [4] *Let G be a 2-connected graph of order $n \geq 12$. If $m \geq \binom{n-2}{2} + 4$, then G is Hamiltonian or $G = K_2 \vee ((2K_1) \cup K_{n-4})$.*

Lemma 2.7. [4] *Let G be a 3-connected graph of order $n \geq 18$. If $m \geq \binom{n-3}{2} + 9$, then G is Hamiltonian or $G = K_3 \vee ((3K_1) \cup K_{n-6})$.*

Lemma 2.8. [4] *Let G be a k-connected graph of order n. If $m \geq \binom{n}{2} - (k+1)(n-k-1)/2 + 1$, then G is Hamiltonian.*

3. Main Results

Theorem 3.1. *Let G be a connected graph of order $n \geq 6$.*

(1) *If $\xi^c(G) \geq n^3 - 3n^2 + 4n - \frac{4m^2}{n} > 0$, then G is Hamiltonian.*
(2) *If $\xi^d(G) \leq \frac{1}{n(n-1)}(n^2 + n - 4)^2$, then G is Hamiltonian.*
(3) *If $\xi^{ce}(G) \geq (n-1)\frac{n^2-3n+5}{n}$, then G is Hamiltonian.*

Proof. Suppose that G is not Hamiltonian, then from Lemma 2.1, there exists an integer $1 \leq k \leq \frac{n-1}{2}$ such that $d_k \leq k$ and $d_{n-k} \leq n - k - 1$.

(1) We consider $\xi^c(G)$. Since $\varepsilon(v) \leq n - d(v)$, from the definition, we have

$$\begin{aligned}
\xi^c(G) &= \sum_{v \in V(G)} \varepsilon(v) d(v) \leq \sum_{v \in V(G)} (n - d(v)) d(v) \\
&= n \left(\sum_{v \in V(G)} d(v) \right) - \sum_{v \in V(G)} d^2(v) \\
&\leq n \left(\sum_{v \in V(G)} d(v) \right) - \frac{1}{n} \left(\sum_{v \in V(G)} d(v) \right)^2 \\
&= n \left(\sum_{v \in V(G)} d(v) \right) - \frac{4m^2}{n} \\
&\leq n \left[k^2 + (n - 2k)(n - k - 1) + k(n - 1) \right] - \frac{4m^2}{n} \\
&= n^2(n-1) + n \left[3k^2 - (2n-1)k \right] - \frac{4m^2}{n}.
\end{aligned}$$

Suppose $f(x) = 3x^2 - (2n-1)x$ with $1 \leq x \leq \frac{1}{2}(n-1)$. It is easy to see that $f_{\max}(x) = \max\{f(1), f(\frac{1}{2}(n-1))\}$. As $f(1) = 4-2n$, $f(\frac{1}{2}(n-1)) = \frac{1}{4}(1-n^2)$, $f(\frac{n-1}{2})-f(1) = -\frac{1}{4}(n-5)(n-3) < 0$, so we have $f_{\max}(x) = f(1)$. Thus, $\xi^c(G) \leq n^2(n-1) + n(4-2n) - \frac{4m^2}{n} = n^3 - 3n^2 + 4n - \frac{4m^2}{n}$, so we get the result.

If $\xi^c(G) = n^3 - 3n^2 + 4n - \frac{4m^2}{n}$, then all the inequalities in the proof should be equalities, so $k = 1$, and hence $d_1 = 1$, $d_2 = d_3 = \cdots = d_{n-1} = n-2$, $d_n = n-1$. Thus $G = K_1 \vee (K_1 \cup K_{n-2})$, which is not Hamiltonian as stated in [1]. But this graph dose not satisfy $\sum_{v\in V(G)} d^2(v) = \frac{1}{n}\left(\sum_{v\in V(G)} d(v)\right)^2$, thus the equality can not hold.

(2) We consider $\xi^d(G)$. Since $\varepsilon(v) \geq \frac{D(v)}{n-1}$, $D(v) \geq d(v) + 2(n-1-d(v))$, from the definition, we have

$$\begin{aligned}
\xi^d(G) &= \sum_{v\in V(G)} \varepsilon(v)\cdot D(v) \geq \sum_{v\in V(G)} \frac{D(v)}{n-1}\cdot D(v) \\
&= \frac{1}{n-1}\sum_{v\in V(G)} (D(v))^2 \\
&\geq \frac{1}{n-1}\sum_{v\in V(G)} \left[4(n-1)^2 - 4(n-1)d(v) + (d(v))^2\right] \\
&= 4n(n-1) - 4\sum_{v\in V(G)} d(v) + \frac{1}{n-1}\sum_{v\in V(G)} (d(v))^2 \\
&\geq 4n(n-1) - 4\sum_{v\in V(G)} d(v) + \frac{1}{n-1}\cdot\frac{1}{n}\left(\sum_{v\in V(G)} d(v)\right)^2 \\
&= \frac{1}{n(n-1)}\left[\left(\sum_{v\in V(G)} d(v)\right)^2 - 4n(n-1)\sum_{v\in V(G)} d(v) + 4n^2(n-1)^2\right] \\
&= \frac{1}{n(n-1)}\left[\sum_{v\in V(G)} d(v) - 2n(n-1)\right]^2.
\end{aligned}$$

As $\sum_{v\in V(G)} d(v) < 2n(n-1)$, we have

$$\begin{aligned}
\xi^d(G) &\geq \frac{1}{n(n-1)}\left[k^2 + (n-2k)(n-k-1) + k(n-1) - 2n(n-1)\right]^2 \\
&= \frac{1}{n(n-1)}\{2n(n-1) - \left[k^2 + (n-2k)(n-k-1) + k(n-1)\right]\}^2 \\
&= \frac{1}{n(n-1)}\left[-3k^2 + (2n-1)k + n^2 - n\right]^2.
\end{aligned}$$

Suppose $f(x) = -3x^2 + (2n-1)x + n^2 - n$ with $1 \leq x \leq \frac{1}{2}(n-1)$. As $f(1) = n^2+n-4$, $f(\frac{n-1}{2}) = \frac{1}{4}(n-1)(5n+1)$, $f(\frac{n-1}{2}) - f(1) = \frac{1}{4}(n-3)(n-5) > 0$, so we have $f_{\min}(x) = \min\{f(1), f(\frac{1}{2}(n-1))\} = f(1)$. Thus $\xi^d(G) \geq \frac{1}{n(n-1)}(n^2+n-4)^2$, and we get the result.

If $\xi^d(G) = \frac{1}{n(n-1)}(n^2+n-4)^2$, then $k = 1$, the remaining is as in the previous proof.

(3) We consider $\xi^{ce}(G)$. From the definition, we have

$$\begin{aligned}\xi^{ce}(G) &= \sum_{v\in V(G)} \frac{d(v)}{\varepsilon(v)}\\ &\le \sum_{v\in V(G)} \frac{n-1}{D(v)}\cdot d(v)\\ &\le (n-1)\sum_{v\in V(G)} \frac{d(v)}{2(n-1)-d(v)}.\end{aligned}$$

Suppose $f(x)=\frac{x}{2(n-1)-x}$, then we have $f'(x)=\frac{2(n-1)}{[2(n-1)-x]^2}>0$, and thus $f(x)$ is strictly increasing, therefore

$$\begin{aligned}\xi^{ce}(G) &\le (n-1)\left[\frac{k^2}{2(n-1)-k}+\frac{(n-2k)(n-k-1)}{2(n-1)-(n-k-1)}+\frac{k(n-1)}{2(n-1)-(n-1)}\right]\\ &= (n-1)\left[\frac{k^2}{2n-k-2}+\frac{(n-2k)(n-k-1)}{n+k-1}+k\right].\end{aligned}$$

Since $1\le k\le \frac{n-1}{2}$, then $2n-k-2-(n+k-1)=n-2k-1\ge 0$, so $\frac{k^2}{2n-k-2}\le\frac{k^2}{n+k-1}$. Further, $\frac{(n-2k)(n-k-1)}{n+k-1}=\frac{(n-2k)(n+k-1-2k)}{n+k-1}=n-2k-\frac{2k(n-2k)}{n+k-1}$. Therefore,

$$\begin{aligned}\xi^{ce}(G) &\le (n-1)\left[\frac{k^2}{n+k-1}-\frac{2k(n-2k)}{n+k-1}-k+n\right]\\ &= (n-1)\left[\frac{k^2-2k(n-2k)}{n+k-1}-k+n\right]\\ &= (n-1)\left\{\frac{k\left[4k-(3n-1)\right]}{n+k-1}+n\right\}.\end{aligned}$$

Suppose $f(x)=\frac{x[4x-(3n-1)]}{n+x-1}$ with $1\le x\le\frac{1}{2}(n-1)$. As $f(1)=\frac{5-3n}{n}$, $f(\frac{n-1}{2})=-\frac{1}{3}(n+1)$, $f(\frac{n-1}{2})-f(1)=-\frac{1}{3n}(n-3)(n-5)<0$, so we have $f_{\max}(x)=\max\{f(1),f(\frac{1}{2}(n-1))\}=f(1)$. Thus $\xi^{ce}(G)\le(n-1)\frac{n^2-3n+5}{n}$, and we get the result.

If $\xi^{ce}(G)=(n-1)\frac{n^2-3n+5}{n}$, then all the inequalities in the proof should be equalities, so $k=1$, and hence $d_1=1$, $d_2=d_3=\cdots=d_{n-1}=n-2$, $d_n=n-1$. Thus $G=K_1\vee(K_1\cup K_{n-2})$, which is not Hamiltonian as stated in [1].

On the other hand, $\varepsilon(v)\ge\frac{D(v)}{n-1}$, with equality if and only if $d(v,u)$ (for fixed $v\in V(G)$) is a constant for all $u\in V(G)$ with $v\ne u$. Thus, $G=K_1\vee(K_1\cup K_{n-2})$ can not satisfy it, the equality can not hold. □

Theorem 3.2. *Let G be a connected graph of order $n\ge 11$.*

(1) *If $\xi^c(G)\ge n^3-5n^2+10n-\frac{4m^2}{n}>0$, then G is traceable.*
(2) *If $\xi^d(G)\le\frac{1}{n(n-1)}(n^2+3n-10)^2$, then G is traceable.*
(3) *If $\xi^{ce}(G)\ge(n-1)\frac{n^2-5n+12}{n+1}$, then G is traceable.*

Proof. Suppose that G is not traceable, then by Lemma 2.2, there is an integer $k\le\frac{n}{2}$ such that $d_k\le k-1$ and $d_{n-k+1}\le n-k-1$. Since G is connected and $d_k\le k-1$, we have $k\ge 2$.

(1) We consider $\xi^c(G)$. As in Theorem 3.1, we have

$$\begin{aligned}
\xi^c(G) &\leq n\left(\sum_{v\in V(G)} d(v)\right) - \frac{4m^2}{n}\\
&\leq n\left[k(k-1)+(n-2k+1)(n-k-1)+(k-1)(n-1)\right] - \frac{4m^2}{n}\\
&= n^2(n-1) - \frac{4m^2}{n} + n\left[3k^2-(2n+1)k\right].
\end{aligned}$$

Suppose $f(x) = 3x^2-(2n+1)x$ with $2 \leq x \leq \frac{n}{2}$. As $f(2) = 10-4n$, $f(\frac{n}{2}) = -\frac{1}{4}n(n+2)$, $f(\frac{n}{2}) - f(2) = -\frac{1}{4}(n-10)(n-4) < 0$, so we have $f_{\max}(x) = \max\{f(2), f(\frac{n}{2})\} = f(2)$. Thus $\xi^c(G) \leq n^2(n-1) - \frac{4m^2}{n} + n(10-4n) = n^3-5n^2+10n-\frac{4m^2}{n}$, so we get the result.

If $\xi^c(G) = n^3-5n^2+10n-\frac{4m^2}{n}$, then $k=2$, and hence $d_1 = d_2 = 1$, $d_3 = \cdots = d_{n-1} = n-3$, $d_n = n-1$. Thus $G = K_1 \vee (K_{n-3} \cup 2K_1)$, which is not traceable. But this graph does not satisfy $\sum_{v\in V(G)} d^2(v) = \frac{1}{n}\left(\sum_{v\in V(G)} d(v)\right)^2$, thus the equality can not hold.

(2) We consider $\xi^d(G)$, as in Theorem 3.1, we have

$$\xi^d(G) \geq \frac{1}{n(n-1)}\left[\sum_{v\in V(G)} d(v) - 2n(n-1)\right]^2.$$

Since $2n(n-1) - \sum_{v\in V(G)} d(v) > 0$, we have

$$\begin{aligned}
\xi^d(G) &\geq \frac{1}{n(n-1)}\left[k(k-1)+(n-2k+1)(n-k-1)+(k-1)(n-1)-2n(n-1)\right]^2\\
&= \frac{1}{n(n-1)}\{2n(n-1)-[k(k-1)+(n-2k+1)(n-k-1)+(k-1)(n-1)]\}^2\\
&= \frac{1}{n(n-1)}\left[-3k^2+(2n+1)k+n(n-1)\right]^2.
\end{aligned}$$

Suppose $f(x) = -3x^2+(2n+1)x+n^2-n$ with $2 \leq x \leq \frac{n}{2}$. As $f(2) = n^2+3n-10$, $f(\frac{n}{2}) = \frac{1}{4}n(5n-2)$, $f(\frac{n}{2}) - f(2) = \frac{1}{4}(n-4)(n-10) \geq 0$, so we have $f_{\min}(x) = \min\{f(2), f(\frac{n}{2})\} = f(2)$. Thus $\xi^d(G) \geq \frac{1}{n(n-1)}(n^2+3n-10)^2$, so we get the result.

If $\xi^d(G) = \frac{1}{n(n-1)}(n^2+3n-10)^2$, then $k=2$, the remaining is as in the previous proof.

(3) We consider $\xi^{ce}(G)$. As in Theorem 3.1,

$$\xi^{ce}(G) \leq (n-1)\sum_{v\in V(G)} \frac{d(v)}{2(n-1)-d(v)}.$$

Suppose $f(x) = \frac{x}{2(n-1)-x}$, then $f'(x) = \frac{2(n-1)}{[2(n-1)-x]^2} > 0$, so

$$\begin{aligned}
\xi^{ce}(G) &\leq (n-1)\left[\frac{k(k-1)}{2(n-1)-(k-1)} + \frac{(n-2k+1)(n-k-1)}{2(n-1)-(n-k-1)} + \frac{(k-1)(n-1)}{2(n-1)-(n-1)}\right]\\
&= (n-1)\left[\frac{k(k-1)}{2n-k-1} + \frac{(n-2k+1)(n-k-1)}{n+k-1} + k - 1\right].
\end{aligned}$$

Since $2 \le k \le \frac{n}{2}$, then $2n-k-1-(n+k-1) = n-2k \ge 0$. As $\frac{(n-2k+1)(n-k-1)}{n+k-1} = \frac{(n-2k+1)(n+k-1-2k)}{n+k-1} = n-2k+1-\frac{2k(n-2k+1)}{n+k-1}$, therefore,

$$\begin{aligned} \xi^{ce}(G) &\le (n-1)\left[\frac{k(k-1)}{n+k-1} - \frac{2k(n-2k+1)}{n+k-1} - k + n\right] \\ &= (n-1)\left[\frac{k(k-1)-2k(n-2k+1)}{n+k-1} - k + n\right] \\ &= (n-1)\left\{\frac{k\left[4k-(3n+2)\right]}{n+k-1} + n\right\}. \end{aligned}$$

Suppose $f(x) = \frac{x[4x-(3n+2)]}{n+x-1}$ with $2 \le x \le \frac{n}{2}$. It is easy to see that $f_{\max}(x) = \max\{f(2), f(\frac{n}{2})\}$. As $f(2) = \frac{6(2-n)}{n+1}$, $f(\frac{n}{2}) = -\frac{n(n+2)}{3n-2}$, $f(\frac{n}{2}) - f(2) = -\frac{(n-4)(n^2-11n+6)}{(3n-2)(n+1)} < 0$, so we have $f_{\max}(x) = f(2)$. Thus $\xi^{ce}(G) \le (n-1)\frac{n^2-5n+12}{n+1}$, so we get the result.

If $\xi^{ce}(G) = (n-1)\frac{n^2-5n+12}{n+1}$, then $k=2$, and hence $d_1 = d_2 = 1$, $d_3 = \cdots = d_{n-1} = n-3$, $d_n = n-1$. Thus $G = K_1 \vee (K_{n-3} \cup 2K_1)$, which is not traceable.

On the other hand, $\varepsilon(v) \ge \frac{D(v)}{n-1}$, with equality if and only if $d(v,u)$ (for fixed $v \in V(G)$) is a constant for all $u \in V(G)$ with $v \ne u$. But $G = K_1 \vee (K_{n-3} \cup 2K_1)$ can not satisfy it, and the equality case can not occur. □

Theorem 3.3. *Let G be a connected graph of order $n \ge 2$.*

(1) *If $\xi^c(G) \ge n^3 - 3n^2 + 2kn - \frac{4m^2}{n} > 0$, then G is k-connected.*
(2) *If $\xi^d(G) \le \frac{1}{n(n-1)}(n^2+n-2k)^2$, then G is k-connected.*
(3) *If $\xi^{ce}(G) \ge (n-1)(\frac{3k-3n-1}{n} + n)$, then G is k-connected.*

Proof. Suppose that G is not k-connected, then from Lemma 2.3, there exists an integer $1 \le i \le \frac{n-k+1}{2}$ such that $d_i \le i+k-2$ and $d_{n-k+1} \le n-i-1$. Obviously, $1 \le k \le n-1$.

(1) We consider $\xi^c(G)$, as in Theorem 3.1, we have

$$\begin{aligned} \xi^c(G) &\le n\left(\sum_{v\in V(G)} d(v)\right) - \frac{4m^2}{n} \\ &\le n\left[i(i+k-2) + (n-k-i+1)(n-i-1) + (k-1)(n-1)\right] - \frac{4m^2}{n} \\ &= n^2(n-1) - \frac{4m^2}{n} + 2n\left[i^2 - (n-k+1)i\right]. \end{aligned}$$

Suppose $f(x) = x^2 - (n-k+1)x$ with $1 \le x \le \frac{n-k+1}{2}$, then $f(x) \le f(1) = k-n$. Thus $\xi^c(G) \le n^2(n-1) - \frac{4m^2}{n} + 2n(k-n) = n^3 - 3n^2 + 2kn - \frac{4m^2}{n}$, so we get the result.

If $\xi^c(G) = n^3 - 3n^2 + 2kn - \frac{4m^2}{n}$, then all the inequalities in the proof should be equalities, so $i = 1$, $d_1 = k-1$, $d_2 = \cdots = d_{n-k+1} = n-2$, $d_{n-k+2} = \cdots = d_n = n-1$. Thus $G = (K_1 \cup K_{n-k}) \vee K_{k-1}$, which is not k-connected. But it can not satisfy $\sum_{v\in V(G)} d^2(v) = \frac{1}{n}\left(\sum_{v\in V(G)} d(v)\right)^2$, thus the equality can not hold.

(2) We consider $\xi^d(G)$. As in Theorem 3.1,

$$\xi^d(G) \geq \frac{1}{n(n-1)}\left[\sum_{v\in V(G)} d(v) - 2n(n-1)\right]^2.$$

Since $2n(n-1) - \sum_{v\in V(G)} d(v) > 0$, then

$$\begin{aligned}
\xi^d(G) &\geq \frac{1}{n(n-1)}\left[i(i+k-2)+(n-k-i+1)(n-i-1)+(k-1)(n-1)-2n(n-1)\right]^2 \\
&= \frac{1}{n(n-1)}\left\{2n(n-1)-[i(i+k-2)+(n-k-i+1)(n-i-1)+(k-1)(n-1)]\right\}^2 \\
&= \frac{1}{n(n-1)}\left[-2i^2-2i(-1+k-n)+n(n-1)\right]^2.
\end{aligned}$$

Suppose $f(x) = -2x^2 - 2x(-1+k-n) + n(n-1)$ with $1 \leq x \leq \frac{n-k+1}{2}$, $f(1) \leq f(x) \leq f(\frac{n-k+1}{2})$, $f(1) = n(n+1) - 2k$. Thus $\xi^d(G) \geq \frac{1}{n(n-1)}(n(n+1)-2k)^2$, so we get the result.

If $\xi^d(G) = \frac{1}{n(n-1)}(n^2+n-2k)^2$, then all the inequalities in the proof should be equalities, so $i = 1$, the remaining is as in the previous proof.

(3) We consider $\xi^{ce}(G)$, as in Theorem 3.1, we have

$$\xi^{ce}(G) \leq (n-1)\sum_{v\in V(G)} \frac{d(v)}{2(n-1)-d(v)}.$$

Suppose $f(x) = \frac{x}{2(n-1)-x}$, $f'(x) = \frac{2(n-1)}{[2(n-1)-x]^2} > 0$, so

$$\begin{aligned}
\xi^{ce}(G) &\leq (n-1)\Big[\frac{i(i+k-2)}{2(n-1)-(i+k-2)} + \frac{(n-k-i+1)(n-i-1)}{2(n-1)-(n-i-1)} \\
&\quad + \frac{(k-1)(n-1)}{2(n-1)-(n-1)}\Big] \\
&= (n-1)\left[\frac{i(i+k-2)}{2n-k-i} + \frac{(n-k-i+1)(n-i-1)}{n+i-1} + k - 1\right].
\end{aligned}$$

Since $1 \leq i \leq \frac{n-k+1}{2}$, then $2n-k-i-(n+i-1) = n-k-2i+1 \geq 0$. Further, $\frac{(n-k-i+1)(n-i-1)}{n+i-1} = \frac{(n-k-i+1)(n+i-1-2i)}{n+i-1} = n-k-i+1-\frac{2i(n-k-i+1)}{n+i-1}$.

Therefore,

$$\begin{aligned}
\xi^{ce}(G) &\leq (n-1)\left[\frac{i(i+k-2)}{n+i-1} - \frac{2i(n-k-i+1)}{n+i-1} - i + n\right] \\
&= (n-1)\left[\frac{i(i+k-2)-2i(n-k-i+1)}{n+i-1} - i + n\right] \\
&= (n-1)\left[\frac{i(2i+3k-3n-3)}{n+i-1} + n\right].
\end{aligned}$$

Suppose $f(x) = \frac{x(2x+3k-3n-3)}{n+x-1}$ with $1 \leq x \leq \frac{n-k+1}{2}$, we can easily compute that $f_{\max}(x) = f(1) = \frac{3k-3n-1}{n}$. Thus $\xi^{ce}(G) \leq (n-1)(\frac{3k-3n-1}{n} + n)$, so we get the result.

If $\xi^{ce}(G) = (n-1)(\frac{3k-3n-1}{n} + n)$, then $i = 1$, $d_1 = k-1$, $d_2 = \cdots = d_{n-k+1} = n-2$, $d_{n-k+2} = \cdots = d_n = n-1$. Thus $G = (K_1 \cup K_{n-k}) \vee K_{k-1}$, which is not k-connected.

On the other hand, $\varepsilon(v) \geq \frac{D(v)}{n-1}$, with equality if and only if $d(v,u)$ (for fixed $v \in V(G)$) is a constant for all $u \in V(G)$ with $v \neq u$. But $G = (K_1 \cup K_{n-k}) \vee K_{k-1}$ can not satisfy it, the equality can not hold. □

Theorem 3.4. *Let G be a connected graph of order $n \geq 7$.*

(1) *If $\xi^c(G) \geq n^3 - 3n^2 + 6n - \frac{4m^2}{n} > 0$, then G is Hamilton-connected.*
(2) *If $\xi^d(G) \leq \frac{1}{n(n-1)}(n^2+n-6)^2$, then G is Hamilton-connected.*
(3) *If $\xi^{ce}(G) \geq (n-1)(\frac{8}{n}+n-3)$, then G is Hamilton-connected.*

Proof. Suppose that G is not Hamilton-connected, then from Lemma 2.4, there exists an integer $2 \leq k \leq \frac{n}{2}$ such that $d_{k-1} \leq k$ and $d_{n-k} \leq n-k$.

(1) We consider $\xi^c(G)$, as in Theorem 3.1, we have

$$\begin{aligned}
\xi^c(G) &\leq n\left(\sum_{v\in V(G)} d(v)\right) - \frac{4m^2}{n} \\
&\leq n\left[(k-1)k + (n-2k+1)(n-k) + k(n-1)\right] - \frac{4m^2}{n} \\
&= n^2(n+1) - \frac{4m^2}{n} + n\left[3k^2 - (2n+3)k\right].
\end{aligned}$$

Suppose $f(x) = 3x^2 - (2n+3)x$ with $2 \leq x \leq \frac{n}{2}$. As $f(2) = 6-4n$, $f(\frac{n}{2}) = -\frac{1}{4}n(n+6)$, $f(\frac{n}{2}) - f(2) = -\frac{1}{4}(n-6)(n-4) < 0$, so we have $f_{\max}(x) = \max\{f(2), f(\frac{n}{2})\} = f(2)$. Thus, $\xi^c(G) \leq n^2(n+1) - \frac{4m^2}{n} + n(6-4n) = n^3 - 3n^2 + 6n - \frac{4m^2}{n}$, so we get the result.

If $\xi^c(G) = n^3 - 3n^2 + 6n - \frac{4m^2}{n}$, then $k = 2$, $d_1 = 2$, $d_2 = d_3 = \cdots = d_{n-2} = n-2$, $d_{n-1} = d_n = n-1$. Thus $G = K_2 \vee (K_1 \cup K_{n-3})$, which is not Hamilton-connected. But it can not satisfy $\sum_{v\in V(G)} d^2(v) = \frac{1}{n}\left(\sum_{v\in V(G)} d(v)\right)^2$, thus the equality can not hold.

(2) We consider $\xi^d(G)$, as in Theorem 3.1, we have

$$\xi^d(G) \geq \frac{1}{n(n-1)}\left[\sum_{v\in V(G)} d(v) - 2n(n-1)\right]^2.$$

Since $2n(n-1) - \sum_{v\in V(G)} d(v) > 0$, then

$$\begin{aligned}
\xi^d(G) &\geq \frac{1}{n(n-1)}\left[k(k-1) + (n-2k+1)(n-k) + k(n-1) - 2n(n-1)\right]^2 \\
&= \frac{1}{n(n-1)}\{2n(n-1) - [k(k-1) + (n-2k+1)(n-k) + k(n-1)]\}^2 \\
&= \frac{1}{n(n-1)}\left[-3k^2 + (2n+3)k + n(n-3)\right]^2.
\end{aligned}$$

Suppose $f(x) = -3x^2 + (2n+3)x + n(n-3)$ with $2 \leq x \leq \frac{n}{2}$. As $f(2) = n^2+n-6$, $f(\frac{n}{2}) = \frac{1}{4}n(5n-6)$, $f(\frac{n}{2}) - f(2) = \frac{1}{4}(n-4)(n-6) > 0$, so we have $f_{\min}(x) = \min\{f(2), f(\frac{n}{2})\} = f(2)$. Thus $\xi^d(G) \geq \frac{1}{n(n-1)}(n^2+n-6)^2$, so we get the result.

If $\xi^d(G) = \frac{1}{n(n-1)}(n^2+n-6)^2$, then $k = 2$, the remaining is as in the previous proof.

(3) We consider $\xi^{ce}(G)$, as in Theorem 3.1, we have

$$\xi^{ce}(G) \leq (n-1)\sum_{v\in V(G)}\frac{d(v)}{2(n-1)-d(v)}.$$

Suppose $f(x)=\frac{x}{2(n-1)-x}$, $f'(x)=\frac{2(n-1)}{[2(n-1)-x]^2}>0$, so

$$\begin{aligned}\xi^{ce}(G) &\leq (n-1)\left[\frac{k(k-1)}{2(n-1)-k}+\frac{(n-2k+1)(n-k)}{2(n-1)-(n-k)}+\frac{k(n-1)}{2(n-1)-(n-1)}\right]\\ &= (n-1)\left[\frac{k(k-1)}{2n-k-2}+\frac{(n-2k+1)(n-k)}{n+k-2}+k\right].\end{aligned}$$

Since $2\leq k\leq \frac{n}{2}$, then $2n-k-2-(n+k-2)=n-2k\geq 0$. Further, $\frac{(n-2k+1)(n-k)}{n+k-2}=\frac{(n-2k+1)(n+k-2-2k+2)}{n+k-2}=n-2k+1-\frac{(2k-2)(n-2k+1)}{n+k-2}$.

Therefore,

$$\begin{aligned}\xi^{ce}(G) &\leq (n-1)\left[\frac{k(k-1)}{n+k-2}-\frac{(2k-2)(n-2k+1)}{n+k-2}-k+n+1\right]\\ &= (n-1)\left[\frac{k(k-1)-(2k-2)(n-2k+1)}{n+k-2}-k+n+1\right]\\ &= (n-1)\left\{\frac{4k^2-(3n+5)k+2n+2}{n+k-2}+n+1\right\}.\end{aligned}$$

Suppose $f(x)=\frac{4x^2-(3n+5)x+2n+2}{n+x-2}$ with $2\leq x\leq\frac{n}{2}$. As $f(\frac{n}{2})=-\frac{n^2+n-4}{3n-4}$, $f(2)=\frac{8}{n}-4$, $f(\frac{n}{2})-f(2)=-\frac{(n-4)(n^2-7n+8)}{n(3n-4)}<0$. So we have $f_{\max}(x)=\max\{f(2),f(\frac{n}{2})\}=f(2)$. Thus $\xi^{ce}(G)\leq(n-1)\left[f(2)+n+1\right]=(n-1)(\frac{8}{n}+n-3)$, so we get the result.

If $\xi^{ce}(G)=(n-1)(\frac{8}{n}+n-3)$, then $k=2$, $d_1=2$, $d_2=d_3=\cdots=d_{n-2}=n-2$, $d_{n-1}=d_n=n-1$. Thus $G=K_2\vee(K_1\cup K_{n-3})$, which is not Hamilton-connected.

On the other hand, $\varepsilon(v)\geq\frac{D(v)}{n-1}$, with equality if and only if $d(v,u)$ (for fixed $v\in V(G)$) is a constant for all $u\in V(G)$ with $v\neq u$. Thus, $G=K_2\vee(K_1\cup K_{n-3})$ can not satisfy it, the equality can not hold. □

Theorem 3.5. *Let $G=(X,Y;E)$ be a connected bipartite graph with $X=\{x_1,x_2,\ldots,x_n\}$, $Y=\{y_1,y_2,\ldots,y_n\}$, and $n\geq 2$. Then we have:*

(1) *If $\xi^d(G)\leq\frac{1}{n(n-1)}(n^2+2n-4)^2+\frac{n(5n-2)^2}{n-1}$, then G is Hamiltonian.*

(2) *If $\xi^{ce}(G)\geq(n-1)\left[\frac{1}{6}(n^2-2n+2)+\frac{n^2}{3n-2}\right]$, then G is Hamiltonian.*

Proof. Suppose that G is not Hamiltonian, then from Lemma 2.5, there exists an integer $k<n$ such that $d(x_k)\leq k$ and $d(y_{n-k})\leq n-k$. Let $N(x_1):=\{z_1,z_2,\ldots z_s\}$ be the neighbors of x_1, where $s=d(x_1)$. Then $d(x_1,z_i)=1$ for each $z_i\in N(x_1)$, $d(x_1,x_i)\geq 2$ for each x_i with $2\leq i\leq n$, and $d(x_1,y_i)\geq 3$ for each $y_i\in Y-N(x_1)$.

(1) We consider $\xi^d(G)$. First, we have

$$D(x_1)\geq d(x_1)+2(n-1)+3(n-d(x_1))=5n-2-2d(x_1).$$

Similarly, for each i with $2 \leq i \leq n$ and each j with $1 \leq j \leq n$,

$$D(x_i) \geq d(x_i) + 2(n-1) + 3(n - d(x_i)) = 5n - 2 - 2d(x_i).$$

$$D(y_j) \geq d(y_j) + 2(n-1) + 3(n - d(y_j)) = 5n - 2 - 2d(y_j).$$

Therefore, we have

$$\begin{aligned}
\xi^d(G) &= \sum_{v \in V(G)} \varepsilon(v) \cdot D(v) \geq \sum_{v \in V(G)} \frac{D(v)}{n-1} \cdot D(v) = \frac{1}{n-1} \sum_{v \in V(G)} (D(v))^2 \\
&= \frac{1}{n-1} \left[\sum_{x_i \in X} (D(v_i)^2) + \sum_{y_j \in Y} (D(v_j)^2) \right] \\
&\geq \frac{1}{n-1} \Big\{ \sum_{x_i \in X} \left[(5n-2)^2 - 4(5n-2)d(x_i) + 4(d(x_i))^2 \right] \\
&\quad + \sum_{y_j \in Y} \left[(5n-2)^2 - 4(5n-2)d(y_j) + 4(d(y_j))^2 \right] \Big\} \\
&= \frac{1}{n-1} \left[2n(5n-2)^2 - 4(5n-2) \sum_{v \in V(G)} d(v) + 4 \sum_{v \in V(G)} (d(v))^2 \right] \\
&\geq \frac{1}{n-1} \left[2n(5n-2)^2 - 4(5n-2) \sum_{v \in V(G)} d(v) + \frac{4}{n} (\sum_{v \in V(G)} d(v))^2 \right] \\
&= \frac{1}{n(n-1)} \left[2n^2(5n-2)^2 - 4n(5n-2) \sum_{v \in V(G)} d(v) + 4(\sum_{v \in V(G)} d(v))^2 \right] \\
&= \frac{1}{n(n-1)} \left[n(5n-2) - 2 \sum_{v \in V(G)} d(v) \right]^2 + \frac{n(5n-2)^2}{n-1}.
\end{aligned}$$

Since

$$\begin{aligned}
2 \sum_{v \in V(G)} d(v) &\leq 2\left[k^2 + (n-k)n + (n-k)^2 + kn\right] \\
&< 2\left[kn + (n-k)n + (n-k)n + kn\right] \\
&= 4n^2 \leq n(5n-2),
\end{aligned}$$

it follows that

$$\begin{aligned}
\xi^d(G) &\geq \frac{1}{n(n-1)} \left[n(5n-2) - 2(k^2 + (n-k)n + (n-k)^2 + nk) \right]^2 + \frac{n(5n-2)^2}{n-1} \\
&= \frac{1}{n(n-1)} \left[-4k^2 + 4nk + n(n-2) \right]^2 + \frac{n(5n-2)^2}{n-1}.
\end{aligned}$$

Suppose $f(x) = -4x^2 + 4nx + n(n-2)$ with $1 \leq x \leq n-1$. It is easy to see that $f_{\min}(x) = \min\{f(1), f(n-1)\}$. As $f(1) = f(n-1)$, $f_{\min}(x) = f(1) = n^2 + 2n - 4$, thus $\xi^d(G) \geq \frac{1}{n(n-1)}(n^2 + 2n - 4)^2 + \frac{n(5n-2)^2}{n-1}$.

If $\xi^d(G) = \frac{1}{n(n-1)}(n^2+2n-4)^2 + \frac{n(5n-2)^2}{n-1}$, then $k=1$, $d(x_1)=1$, $d(x_2)=\cdots=d(x_n)=n$, $d(y_1)=d(y_2)=\cdots=d(y_{n-1})=n-1$, $d(y_n)=n$. Thus $G=K_{n,n}-K_{1,n-1}$, which is not Hamiltonian.

On the other hand, $\varepsilon(v) \geq \frac{D(v)}{n-1}$, with equality if and only if $d(v,u)$ (for fixed $v \in V(G)$) is a constant for all $u \in V(G)$ with $v \neq u$. However, $G=K_{n,n}-K_{1,n-1}$ can not satisfy it, the equality can not hold.

(2) We consider $\xi^{ce}(G)$.

$$D(x_1) \geq d(x_1) + 2(n-1) + 3(n-d(x_1)) = 5n-2-2d(x_1).$$

Similarly, for each i with $2 \leq i \leq n$ and each j with $1 \leq j \leq n$,

$$D(x_i) \geq d(x_i) + 2(n-1) + 3(n-d(x_i)) = 5n-2-2d(x_i).$$

$$D(y_j) \geq d(y_j) + 2(n-1) + 3(n-d(y_j)) = 5n-2-2d(y_j).$$

Therefore,

$$\begin{aligned}
\xi^{ce}(G) &= \sum_{v\in V(G)} \frac{d(v)}{\varepsilon(v)} \\
&\leq \sum_{x_i\in X} \frac{n-1}{D(x_i)} \cdot d(x_i) + \sum_{y_j\in X} \frac{n-1}{D(y_j)} \cdot d(y_j) \\
&\leq (n-1)\left[\sum_{x_i\in X} \frac{d(x_i)}{5n-2-2d(x_i)} + \sum_{y_j\in X} \frac{d(y_j)}{5n-2-2d(y_j)}\right].
\end{aligned}$$

Suppose $f(x) = \frac{x}{5n-2-2x}$, then we have $f'(x) = \frac{5n-2}{(5n-2-2x)^2} > 0$, so

$$\begin{aligned}
\xi^{ce}(G) &\leq (n-1)\left[\frac{k^2}{5n-2-2k} + \frac{(n-k)n}{5n-2-2n} + \frac{(n-k)^2}{5n-2-2(n-k)} + \frac{kn}{5n-2-2n}\right] \\
&= (n-1)\left[\frac{k^2}{5n-2-2k} + \frac{(n-k)^2}{3n+2k-2} + \frac{n^2}{3n-2}\right].
\end{aligned}$$

Since $1 \leq k \leq n-1$, then,

$$\begin{aligned}
\xi^{ce}(G) &\leq (n-1)\left[\frac{k^2}{5(k+1)-2-2k} + \frac{(n-k)^2}{3(k+1)+2k-2} + \frac{n^2}{3n-2}\right] \\
&= (n-1)\left[\frac{k^2}{3k+3} + \frac{(n-k)^2}{5k+1} + \frac{n^2}{3n-2}\right] \\
&\leq (n-1)\left[\frac{k^2}{3k+3} + \frac{(n-k)^2}{3k+3} + \frac{n^2}{3n-2}\right] \\
&= (n-1)\left[\frac{k^2+(n-k)^2}{3k+3} + \frac{n^2}{3n-2}\right].
\end{aligned}$$

Suppose $f(x) = \frac{x^2+(n-x)^2}{3x+3}$ with $1 \leq x \leq (n-1)$. It is easy to see that $f_{\max}(x) = \max\{f(1), f(n-1)\}$. As $f(n-1) = \frac{1}{3n}(n^2-2n+2)$, $f(1) = \frac{1}{6}(n^2-2n+2)$, $f(n-1)-f(1) = -\frac{(n-2)(n^2-2n+2)}{6n} < 0$, so we have $f_{\max}(x) = f(1)$.

If $\xi^{ce}(G) = (n-1)\left[\frac{1}{6}(n^2-2n+2)+\frac{n^2}{3n-2}\right]$, then $k = 1$, the remaining is as in the previous proof. □

Theorem 3.6. *Let G be a 2-connected graph of order $n \geq 12$. If $\xi^d(G) \leq \frac{1}{n(n-1)}\left[n^2+3n-12\right]^2$, then G is Hamiltonian.*

Proof. Suppose that G is not Hamiltonian and G is not $K_2 \vee (2K_1 \cup K_{n-4})$, then from Lemma 2.6, we have that $m \leq \binom{n-2}{2}+3$. As in Theorem 3.1,

$$\xi^d(G) \geq \frac{1}{n(n-1)}\left[\sum_{v\in V(G)} d(v) - 2n(n-1)\right]^2.$$

Since $2n(n-1) - \sum_{v\in V(G)} d(v) > 0$, then

$$\begin{aligned}\xi^d(G) &\geq \frac{1}{n(n-1)}\left[2n(n-1)-2m\right]^2\\ &\geq \frac{1}{n(n-1)}\left[n^2+3n-12\right]^2.\end{aligned}$$

On the other hand, $\varepsilon(v) \geq \frac{D(v)}{n-1}$, with equality if and only if $d(v,u)$ (for fixed $v \in V(G)$) is a constant for all $u \in V(G)$ with $v \neq u$. Thus, $K_2 \vee (2K_1 \cup K_{n-4})$ can not satisfy it, the equality can not hold. □

Theorem 3.7. *Let G be a 3-connected graph of order $n \geq 18$. If $\xi^d(G) \leq \frac{1}{n(n-1)}\left[n^2+5n-28\right]^2$, then G is Hamiltonian.*

Proof. Suppose that G is not Hamiltonian and G is not $K_3 \vee (3K_1 \cup K_{n-6})$. Then from Lemma 2.7, we have that $m \leq \binom{n-3}{2}+8$. Therefore,

$$\begin{aligned}\xi^d(G) &\geq \frac{1}{n(n-1)}\left[\sum_{v\in V(G)} d(v) - 2n(n-1)\right]^2\\ &\geq \frac{1}{n(n-1)}\left[2n(n-1)-2m\right]^2\\ &\geq \frac{1}{n(n-1)}\left[n^2+5n-28\right]^2.\end{aligned}$$

On the other hand, $\varepsilon(v) \geq \frac{D(v)}{n-1}$, with equality if and only if $d(v,u)$ (for fixed $v \in V(G)$) is a constant for all $u \in V(G)$ with $v \neq u$. Thus, $K_3 \vee (3K_1 \cup K_{n-6})$ can not satisfy it, the equality can not hold. □

Theorem 3.8. *Let G be a k-connected graph of order $n \geq 18$. If*

$$\xi^d(G) \leq \frac{1}{n(n-1)}\left[n(n-1)+(k+1)(n-k-1)\right]^2,$$

then G is Hamiltonian.

Proof. Suppose that G is not Hamiltonian, then from Lemma 2.8, we have that $m \leq \binom{n}{2} - (k+1)(n-k-1)/2$. Therefore,

$$\begin{aligned}
\xi^d(G) &\geq \frac{1}{n(n-1)}\left[\sum_{v\in V(G)} d(v) - 2n(n-1)\right]^2 \\
&= \frac{1}{n(n-1)}\left[2n(n-1) - 2m\right]^2 \\
&\geq \frac{1}{n(n-1)}\left[n(n-1) + (k+1)(n-k-1)\right]^2,
\end{aligned}$$

which is a contradiction. This completes the proof. □

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References

[1] D. Bauer, H. J. Broersma, J. van den Heuvel, N. Kahl, A. Nevo, E. Schmeichel, D. R. Woodall and M. Yatauro, Best monotone degree conditions for graph properties: a survey, *Graphs Combin.*, **31** (2015) 1–22.

[2] C. Berge, *Graphs and Hypergraphs*, North-Holland, Amsterdam, 1973.

[3] J. A. Bondy, Properties of graphs with constraints on degrees, *Studia Sci. Math. Hungar.*, **4** (1969) 473–475.

[4] W. Byer and D. Smeltzer, Edge bounds in nonhamiltonian k-connected graphs, *Discrete Math.*, **307** (2007) 1572–1579.

[5] L. Chen, J. Liu and Y. Shi, Matching energy of unicyclic and bicyclic graphs with a given diameter, *Complexity*, **21** (2015) 224–238.

[6] L. Chen and Y. T. Shi, The maximal matching energy of tricyclic graphs, *MATCH Commun. Math. Comput. Chem.*, **73** (2015) 105–120.

[7] Z. Chen, M. Dehmer, Y. Shi and H. Yang, Sharp upper bounds for the Balaban index of bicyclic graphs, *MATCH Commun. Math. Comput. Chem.*, **75** (2016) 105–128.

[8] V. Chvátal, On Hamiltons ideals, *J. Combin. Theory Ser. B*, **12** (1972) 163–168.

[9] A. Dobrynin, R. Entringerand I. Gutman, Wiener index of trees: theory and applications, *Acta Appl. Math.*, **66** (2001) 211–249.

[10] S. Gupta, M. Singh and A. K. Madan, Connective eccentricity index: a novel topological descriptor for predicting biological activity, *J. Mol. Graph. Model.*, **18** (2000) 18–25.

[11] S. Gupta, M. Singh and A. K. Madan, Eccentric distance sum: A novel graph invariant for predicting biological and physical properties, *J. Math. Anal. Appl.*, **275** (2002) 386–401.

[12] H. Hua and M. Wang, On Harary index and traceable graphs, *MATCH Commun. Math. Comput. Chem.*, **70** (2013) 297–300.

[13] A. Ilić, G. H. Yu and L. H. Feng, On the eccentric distance sum of graphs, *J. Math. Anal. Appl.*, **381** (2011) 590–600.

[14] O. Ivanciuc, QSAR comparative study of Wiener descriptors for weighted molecular graphs, *J. Chem. Inf. Comput. Sci.*, **40** (2000) 1412–1422.

[15] R. Li, Harary index and some Hamiltonian properties of graphs, *AKCE Inter. J. of Graphs and Comb.*, **12** (2015) 64–69.

[16] R. Li, Wiener index and some Hamiltonian properties of graphs, *Inter. J. of Math. and Soft Computing*, **5** (2015) 11–16.

[17] S. Li and Y. Song, On the sum of all distances in bipartite graphs, *Discrete Appl. Math.*, **169** (2014) 176–185.

[18] S. C. Li, M. Zhang, G. H. Yu and L. H. Feng, On the extremal values of the eccentric distance sum of trees, *J. Math. Anal. Appl.*, **390** (2012) 99–112.

[19] J. Ma, Y. Shi, Z. Wang and J. Yue, On Wiener polarity index of bicyclic networks, *Sci. Rep.*, **6** (2016) 19066. doi: 10.1038/srep19066.

[20] J. Ma, Y.bT. Shi and Y. Yue, The Wiener polarity index of graph products, *Ars Combin.*, **116** (2014) 235–244.

[21] J. Plesník, On the sum of all distances in a graph or diagraph, *J. Graph Theory*, **8** (1984) 1–21.

[22] V. Sharma, R. Goswami, A. K. Madan, Eccentric connectivity index: A novel highly discriminating topological descriptor for structure property and structure activity studies, *J. Chem. Inf. Comput. Sci.* **37** (1997) 273–282.

[23] S. Sardana, A. K. Madan, Predicting anti-HIV activity of TIBO derivatives: A computational approach using a novel topological descriptor, *J. Mol. Model.*, **8** (2002) 258–265.

[24] Y. Shi, Note on two generalizations of the Randić index, *Appl. Math. Comput.*, **265** (2015) 1019–1025.

[25] L. Yang, Wiener index and traceable graphs, *Bull. Austral. Math. Soc.*, **88** (2013) 380–383.

[26] G.H. Yu, L.H. Feng, A. Ilić, On the eccentric distance sum of trees and unicyclic graphs, *J. Math. Anal. Appl.*, **375** (2011) 99–107.

[27] G.H. Yu, H. Qu, L. Tang, L.H. Feng, On the connective eccentricity index of trees and unicyclic graphs with given diameter, *J. Math. Aanl. Appl.*, **420** (2014) 1776–1786.

[28] B. Zhou, Z. Du, On eccentric connectivity index, *MATCH Commun. Math. Comput. Chem.* **63** (2010) 181–198.

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