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A CLASS OF RAMSEY-EXTREMAL HYPERGRAPHS

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Abstract. In 1991, McKay and Radziszowski proved that, however each 3-subset of a 13-set is assigned one of two colours, there is some 4-subset whose four 3-subsets have the same colour. More than 25 years later, this remains the only non-trivial classical Ramsey number known for hypergraphs. In this article, we find all the extremal colourings of the 3-subsets of a 12-set and list some of their properties. We also provide an answer to a question of Dudek, La Fleur, Mubayi and Rödl about the size-Ramsey numbers of hypergraphs.

1. **Introduction**

A colouring of all the s-subsets of an n-set with two colours is called $R(j,k;s)$-*good* if there is no j-subset (of the n-set) containing only s-subsets of the first colour, and no k-subset containing only s-subsets of the second colour. (Note that it is the s-subsets receiving colours, not the elements of the n-set.) The *Ramsey number* $R(j,k;s)$ is defined to be the least n for which there is no $R(j,k;s)$-good colouring.

Although there are several known values of $R(j,k;2)$ [8], which is usually written as just $R(j,k)$, the only known non-trivial value of $R(j,k;s)$ for $s \geq 3$ is $R(4,4;3)=13$. As a lower bound, a suitable colouring of the 3-subsets of a 12-set was presented by Isbell in 1969 [2], and this was proved best possible by the present author and Radziszowski in 1991 [6]. During that project we found more than 200000 $R(4,4;3)$-good colourings for 12 points, but did not have the resources to compute them all. With the aid of an improved algorithm and the much greater computing resources available today, we can now show that the number of $R(4,4;3)$-good colourings for 12 points is precisely 434714. We also explore how close it is possible to get to a $R(4,4;3)$-good colouring for 13 points, and show that it is possible with two uncoloured 3-subsets but not with one uncoloured 3-subset.

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2. Method

We prefer to use slightly different terminology for this description. Suppose we have an $R(4,4;3)$-good colouring of the 3-subsets of an n-set V. We will call the 4-subsets of V *quadruples*.

If we choose just the 3-subsets of V having the first colour, we obtain a 3-uniform hypergraph on V with the property that every quadruple contains 1, 2 or 3 edges (the other possibilities 0 and 4 being forbidden). We will call this a $R(4,4;3)$-good hypergraph. Note that we could have chosen the other colour instead and would have obtained the complementary hypergraph. We can obviously recover the colouring from the hypergraph, so we lose nothing by continuing with hypergraph terminology.

Denote by $\mathcal{R}(n)$ the set of $R(4,4;3)$-good hypergraphs with n points. If we wish to emphasize the point set V, we may write $\mathcal{R}(V)$ instead. More generally, $\mathcal{R}(n,e)$ is the set of $R(4,4;3)$-good hypergraphs with n points and e edges, and notations like $\mathcal{R}(V,\leq 110)$ have their obvious meanings.

Our aim is to find $\mathcal{R}(12)$. By the remark just made, it will suffice to find $\mathcal{R}(12,\leq 110)$, since $110 = \frac{1}{2}\binom{12}{3}$ and the rest are complements. Given $G \in \mathcal{R}(V)$ and $v \in V$, define G_v to be the hypergraph with point set $V - v$ and all the edges of G that lie in $V - v$. Clearly $G_v \in \mathcal{R}(V - v)$. Since the points of $G \in \mathcal{R}(n,e)$ lie on average in $3e/n$ edges, we find that for $G \in \mathcal{R}(12,\leq 110)$ there is some v such that $G_v \in \mathcal{R}(11,\leq 82)$. Continuing such logic we find a construction path

$$\mathcal{R}(9,\leq 41) \to \mathcal{R}(10,\leq 59) \to \mathcal{R}(11,\leq 82) \to \mathcal{R}(12,\leq 110). \tag{2.1}$$

Each step in (2.1) involves adding one point and some edges that include the new point. Moreover, we can assume that the new point is in at least as many edges as any of the old points (after the new edges are added).

The programs developed for [6] are fast enough to find $\mathcal{R}(9,\leq 41)$ in a few hours. There are exactly 3030480232 such hypergraphs and these form our starting point. It would be convenient to perform each of the three steps of (2.1) separately, but it would be quite expensive. The number of hypergraphs in $\mathcal{R}(10)$ and $\mathcal{R}(11)$ is greater than 10^{11} and even the task of extending one hypergraph by one point requires solution of a large set of integer inequalities. We need a better way.

If S is a set and $B \subseteq T \subseteq S$, then the *interval* $[B,T]$ is $\{X \subseteq S \mid B \subseteq X \subseteq T\}$. The use of intervals for solving sets of inequalities efficiently was introduced in [7].

Define $V_9 = \{0,1,\ldots,8\}$ and $V_{10} = V_9 \cup \{a\}$. Consider extending $G_9 \in \mathcal{R}(V_9)$ to all possible $G_{10} \in \mathcal{R}(V_{10})$ by adding the point a and some edges that include a. The possible edges all have the form $\{i,j,a\}$ where $i,j \in V_9$; number these $e_0,e_1,\ldots,e_{35}$ in some order. Each solution for G_{10} corresponds to a subset of $\{e_0,e_1,\ldots,e_{35}\}$.

Now consider the constraints required for G_{10} to be $R(4,4;3)$-good. The quadruples within V_9 are fine already, since we are not adding any further edges inside V_9. So consider a quadruple $\{i,j,k,a\}$, where $i,j,k \in V_9$. If $\{i,j,k\}$ is an edge of G_9, we need that at least one of the edges $\{i,j,a\},\{i,k,a\},\{j,k,a\}$ is not selected, while if $\{i,j,k\}$ is not an edge of G_9, at least one of those three edges must be selected.

Now we can describe how intervals are used to process many cases simultaneously. Consider one interval $[B, T] \subseteq \{e_0, e_1, \ldots, e_{35}\}$ and one quadruple $\{i, j, k, a\}$. Define $X = \big\{\{i, j, a\}, \{i, k, a\}, \{j, k, a\}\big\}$. Now we apply the following *collapsing rules*:

$\{i,j,k\} \in G\,?$	$B \cap X$	$T \cap X$	replace $[B,T]$ by
NO	$\neq \emptyset$	any	$[B,T]$
	$\emptyset$	$\emptyset$	nothing
	$\emptyset$	$\{\alpha\}$	$[B{+}\alpha, T]$
	$\emptyset$	$\{\alpha, \beta\}$	$[B{+}\alpha, T], [B{+}\beta, T{-}\alpha]$
	$\emptyset$	$\{\alpha, \beta, \gamma\}$	$[B{+}\alpha, T], [B{+}\beta, T{-}\alpha], [B{+}\gamma, T{-}\alpha{-}\beta]$

$\{i,j,k\} \in G\,?$	$\bar{T} \cap X$	$\bar{B} \cap X$	replace $[B,T]$ by
YES	$\neq \emptyset$	any	$[B,T]$
	$\emptyset$	$\emptyset$	nothing
	$\emptyset$	$\{\alpha\}$	$[B, T{-}\alpha]$
	$\emptyset$	$\{\alpha, \beta\}$	$[B, T{-}\alpha], [B{+}\alpha, T{-}\beta]$
	$\emptyset$	$\{\alpha, \beta, \gamma\}$	$[B, T{-}\alpha], [B{+}\alpha, T{-}\beta], [B{+}\alpha{+}\beta, T{-}\gamma]$

FIGURE 1. Collapsing rules for an interval $[B, T]$ based on quadruple $\{i, j, k, a\}$.

By considering each case, we find that the effect of the collapsing rules is to replace $[B, T]$ by a set of disjoint intervals whose union is the set of all sets in $[B, T]$ that satisfy the quadruple $\{i, j, k, a\}$. For best practical performance, subsets of $\{e_0, e_1, \ldots, e_{35}\}$ can be represented by the bits in a single machine word, then the collapsing rules can be implemented in a few machine instructions each.

Starting with the interval $[\emptyset, \{e_0, e_1, \ldots, e_{35}\}]$ we apply the collapsing rules for each quadruple $\{i, j, k, a\}$. The result is a set of disjoint intervals (typically a few hundred) whose union gives exactly the set of all extensions of G_9 to $\mathcal{R}(10)$. The efficiency depends a lot on the order in which quadruples are processed; we found a good order by experiment.

Now consider further extension to $R(4,4;3)$-good hypergraphs on $V_{11} = \{0, \ldots, 8, a, b\}$. The edges we need to add in total to G_9 either have the form $\{i, j, a\}$ (already added in making G_{10}), $\{i, j, b\}$, or $\{i, a, b\}$, where in each case $i, j, k \in V_9$. Here we can make an observation that is key to the whole computation: *The sets of edges $\{i, j, b\}$ that satisfy quadruples of the form $\{i, j, k, b\}$ are the same as the sets of edges $\{i, j, a\}$ that satisfy quadruples of the form $\{i, j, k, a\}$, except that a is replaced by b.*

Given this observation, we make the possibilities for G_{11} as follows, given G_9, a set $\mathcal{I}$ of intervals describing the extensions of G_9 to $\mathcal{R}(10)$, and a particular extension G_{10}. The possible new edges are numbered $e_0, \ldots, e_{44}$, where $e_0, \ldots, e_{35}$ are edges of the form $\{i, j, b\}$ numbered in the same order as we numbered the edges $\{i, j, a\}$ in the previous step, and $e_{36}, \ldots, e_{44}$ are the edges of the form $\{i, a, b\}$ in any order. To find all solutions, instead of starting with the single interval $[\emptyset, \{e_0, e_1, \ldots, e_{44}\}]$ as in the previous step, we start with the set of intervals $[B, T \cup \{e_{36}, \ldots, e_{44}\}]$ for $[B, T] \in \mathcal{I}$. Then we avoid collapsing rules which are unnecessary for the stated reasons. This results in a massive speedup.

To complete the process by extending from 11 to 12 points, we use the same idea to begin with intervals obtained during the extension to 11 points. This phase is very fast as most intervals are destroyed very quickly and only a comparatively small number of solutions remain.

It would be possible to apply the general method of [4] to perform exhaustive isomorph reduction at each step in the computation, but the large number of intermediate hypergraphs makes that unwise. Instead, we applied a weaker filter. For a hypergraph with points V and point $v \in V$, define d_v to be the number of edges that include v. Also define $f_v = \sum_e d_v d_w d_x$, where the sum is over all edges $e = \{v, w, x\}$ that include v. Suppose we make $G \in G(V)$ by extending a smaller hypergraph, and that $v \in V$ is the last point added. The construction path (2.1) assumed that $d_v \geq d_w$ for all $w \in V$, so that is the first filter applied. If that doesn't eliminate G, we also require that f_v be maximum out of all $w \in V$ with maximum d_w. These rules eliminate most isomorphs and are fast to apply. When we finally have a collection of $R(4,4;3)$-good hypergraphs on 12 points, we perform complete isomorphism reduction using `nauty` [5].

3. Results

There are about 8.4×10^{11} $R(4,4;3)$-good hypergraphs altogether, including 434714 with 12 points. Table 1 details the numbers of $R(4,4;3)$-good hypergraphs in $\mathcal{R}(12)$ according to their automorphism groups, which are the groups of permutations of V which preserve the edge set. As shown in the table, most hypergraphs in $\mathcal{R}(12)$ have a trivial group and none have a transitive group. The unique hypergraph with $|\mathrm{Aut}(G)| = 60$, which has two orbits of size 6, is presented in Figure 2 using letters for elements of V. This hypergraph is one of the 1306 in $\mathcal{R}(12)$ that are self-complementary and is isomorphic to the one found by Isbell [2].

In Table 2 we give some counts of $R(4,4;3)$-good hypergraphs, including some families not required by (2.1). The totals for 10 and 11 points are estimates.

$\mathrm{Aut}(G)$	order	orbits	count
Z_1	1	1^{12}	432300
Z_2	2	2^6	18
		$1^2 2^5$	112
		$1^4 2^4$	1669
Z_3	3	3^4	529
Z_2^2	4	2^6	32
Z_6	6	6^2	20
S_3	6	3^4	17
D_{10}	10	$1^2 5^2$	1
A_4	12	6^2	15
A_5	60	6^2	1

TABLE 1. Counts of $\mathcal{R}(12)$ by automophism group.

Let $\Gamma = \langle (\mathtt{cd})(\mathtt{ef})(\mathtt{CD})(\mathtt{EF}), (\mathtt{bc})(\mathtt{de})(\mathtt{BC})(\mathtt{DE}), (\mathtt{ab})(\mathtt{ef})(\mathtt{AB})(\mathtt{EF}) \rangle$ be a permutation acting on the points `abcdefABCDEF`. It is isomorphic to the alternating group A_5 and acts 2-transitively on each of its orbits $\{\mathtt{a}, \ldots, \mathtt{f}\}$ and $\{\mathtt{A}, \ldots, \mathtt{F}\}$. Now construct a hypergraph by applying Γ to each of the starting edges $\{\mathtt{abe}, \mathtt{ABE}, \mathtt{abC}, \mathtt{aAB}, \mathtt{cAB}\}$. These provide 10, 10, 30, 30 and 30 edges, respectively. The hypergraph induced by each orbit is the same 2-(6,3,2) design. The relabelling `(aD)(bC)(cB)(dA)(eF)(fE)` takes the hypergraph onto its complement.

FIGURE 2. The unique hypergraph in $\mathcal{R}(12)$ with automorphism group of order 60.

None of the hypergraphs in $\mathcal{R}(12)$ extends to a hypergraph in $\mathcal{R}(13)$, consistently with the finding of [6] that $\mathcal{R}(13) = \emptyset$. This raises the question of how close we can get to a hypergraph in $\mathcal{R}(13)$; specifically, how many edges of the complete hypergraph $K_{13}^{(3)}$ can we colour without obtaining a monochromatic $K_4^{(3)}$? The generation method described in the previous section can be adapted to leave some edges uncoloured. If there is some vertex v included in all the uncoloured edges, we can label v last and then the problem is just that of extending good colourings of $K_{12}^{(3)}$ to a 13th vertex while ignoring the constraints that would normally be attributed to the quadruples that contain an uncoloured edge. Using this method we found that $K_{13}^{(3)}$ minus one edge cannot be coloured with two colours without creating a monochromatic $K_4^{(3)}$.

```
acd bcd abe ace bce cde adf cdf
def adg aeg beg ceg deg afg bfg
efg ach bch adh bdh aeh beh deh
afh efh bgh dgh fgh bdi cdi bei
cei afi bfi dfi efi bgi cgi dgi
ahi chi dhi ehi abj acj bcj cdj
aej dej bfj cfj agj bgj cgj dgj
bhj ehj fhj aij gij abk bck bdk
cek afk bfk cfk efk cgk fgk dhk
ghk aik bik eik hik bjk djk gjk
hjk ijk abl acl bcl adl bel afl
bfl cfl bgl dgl chl fhl ghl eil
fil gil hil ajl djl ejl fjl ijl
akl ckl dkl ekl hkl abm adm bdm
aem dem bfm cfm dfm cgm egm ahm
bhm chm ghm aim cim fim ejm fjm
hjm ijm akm dkm ekm fkm gkm jkm
blm clm dlm elm glm ilm
```

Omitted edges: `abc ade`

```
bcd cde acf bcf aef def adg bdg
cdg aeg beg deg bfg efg abh ach
bch adh bdh beh cfh egh fgh aci
aei bei cei dei afi dfi agi cgi
fgi bhi dhi fhi adj bdj aej cej
bfj cfj dfj agj bgj ahj bhj chj
ehj ghj bij dij eij abk ack bck
cdk aek bek cek cfk dfk efk bgk
cgk fgk ahk dhk fhk aik bik gik
hik ejk fjk gjk hjk ijk abl acl
bcl adl bdl afl bfl dfl efl cgl
dgl fgl chl dhl ehl ghl bil cil
eil fil hil ajl cjl ejl fjl gjl
bkl dkl gkl abm adm bdm cdm bem
cem afm bfm cfm dfm efm agm bgm
ahm ehm fhm ghm cim fim gim cjm
djm gjm ijm dkm ekm hkm jkm blm
elm ilm jlm klm
```

Omitted edges: `abc abd`

FIGURE 3. Two $R(4,4;3)$-good colourings of the complete hypergraph $K_{13}^{(3)}$ minus two edges. Edges not mentioned have the second colour.

n	e	count	n	e	count
3	0	1	9	33	2
total		2		34	204
4	1	1		35	22616
	2	1		36	774043
total		3		37	10877731
5	3	1		38	79336073
	4	3		39	341024774
	5	4		40	928650036
total		12		41	1669794753
6	6	1		42	2025923846
	7	5	total		8086884310
	8	22	10	50	13
	9	50		51	1810
	10	70		52	121356
total		226	$\cdots$		
7	12	1	total		$\approx 6.2\times 10^{11}$
	13	26	11	73	36
	14	338		74	4725
	15	1793		75	246299
	16	5055	$\cdots$		
	17	8317	total		$\approx 2.1\times 10^{11}$
total		31060	12	104	4
8	21	1		105	123
	22	278		106	1465
	23	9763		107	10235
	24	107241		108	41939
	25	573596		109	98235
	26	1764747		110	130712
	27	3380337	total		434714
	28	4182459			
total		15854385			

TABLE 2. The numbers of $R(4,4;3)$-good hypergraphs with n points and e edges. The totals include complements.

On the other hand, it is possible to colour $K_{13}^{(3)}$ minus two overlapping edges without creating a monochromatic $K_4^{(3)}$. Examples are given in Figure 3. As the description of the method in the previous paragraph suggests, it is not so easy to find a similar example when the uncoloured edges are disjoint; we found none in an incomplete search. We can report these partial results: there is no good colouring of $K_{13}^{(3)}$ minus the edges $\{1,2,3\}$ and $\{4,5,6\}$ such that a good colouring of $K_{12}^{(3)}$ can be obtained either by deleting point 1 and colouring edge $\{4,5,6\}$, or by deleting point 7 and colouring both edges $\{1,2,3\}$ and $\{4,5,6\}$. The remaining cases of two disjoint edges are computationally feasible but remain unresolved.

If H is a 3-uniform hypergraph, the *size-Ramsey number* $\hat{R}^{(3)}(H)$ is the least number m such that for some 3-uniform hypergraph G with m edges, every colouring of the edges of G with two colours includes a monochromatic copy of H. If $H = K_4^{(3)}$, then the value $R(4,4;3) = 13$ implies that $\hat{R}^{(3)}(H) \le \binom{13}{3} = 286$ since we can take $G = K_{13}^{(3)}$. Dudek, La Fleur, Mubayi and Rödl [1, Question 2.2] ask whether this bound is sharp. Since $K_{13}^{(3)}$ minus one edge cannot be coloured without creating a monochromatic $K_4^{(3)}$ we have $\hat{R}^{(3)}(H) \le 285$, which answers Dudek et al.'s question in the negative.

The extremal $R(4,4;3)$-good hypergraphs are available online [3]. The computation of $\mathcal{R}(12)$ required about 2 GHz-years of computation.

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