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MAJORIZATION AND THE NUMBER OF BIPARTITE GRAPHS FOR GIVEN VERTEX DEGREES

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Abstract. The *bipartite realisation problem* asks for a pair of non-negative, non-increasing integer lists $a := (a_1, \dots, a_n)$ and $b := (b_1, \dots, b_{n'})$ if there is a labeled bipartite graph $G(U, V, E)$ (no loops or multiple edges) such that each vertex $u_i \in U$ has degree a_i and each vertex $v_i \in V$ degree b_i. The Gale-Ryser theorem provides characterisations for the existence of a 'realisation' $G(U, V, E)$ that are strongly related to the concept of *majorisation*. We prove a generalisation; list pair (a, b) has more realisations than (a', b), if a' majorises a. Furthermore, we give explicitly list pairs which possess the largest number of realisations under all (a, b) with fixed n, n' and $m := \sum_{i=1}^{n} a_i$. We introduce the notion *minconvex list pairs* for them. If n and n' divide m, minconvex list pairs turn in the special case of two constant lists $a = (\frac{m}{n}, \dots, \frac{m}{n})$ and $b = (\frac{m}{n'}, \dots, \frac{m}{n'})$.

1. Introduction

The *bipartite realisation problem* asks for a pair of non-negative, non-increasing integer lists $(a, b) := ((a_1, \dots, a_n), (b_1, \dots, b_{n'}))$ with $\sum_{i=1}^{n} a_i = \sum_{i=1}^{n'} b_i$ whether there exists a labeled bipartite graph $G(U, V, E)$ (without loops and multiple edges) such that $d_G(u_i) = a_i$ for $u_1, \dots, u_n \in U$ and $d_G(v_i) = b_i$ for $v_1, \dots, v_{n'} \in V$ where $d_G : U \cup V \mapsto \{0, \dots, \max\{n, n'\}\}$ denotes the vertex degrees in G. We call list pair (a, b) *bigraphic* and G *realisation* if the decision problem can be answered positively. In this case, it must be fulfilled that $0 \leq a_i \leq n'$ and $0 \leq b_i \leq n$. Hence, it is a necessary condition for (a, b) to

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be bigraphic. For simplicity we use the term *list pair* (a, b) in the whole paper as a pair of non-negative, non-increasing integer lists a and b with $0 \leq a_i \leq n'$ and $0 \leq b_i \leq n$ and $\sum_{i=1}^{n} a_i = \sum_{i=1}^{n'} b_i$.

The problem occurs in many applied fields under several terms like *matrices with fixed column and row sums*, or *contingency tables with fixed margins*. Indeed, it is possible to formulate the same problem as the question; if there is a binary $n \times n'$-matrix with row sums $(b_1, \ldots, b_{n'})$ and column sums $(a_1, \ldots, a_n)$. One only needs to consider the bi-adjacency matrix of a bipartite graph. On the other hand, such a matrix can also be considered as the adjacency matrix of a digraph (without multiple arcs), where at most one loop per vertex is allowed. In our paper, we use the term 'realisation' synonymously for the bipartite graph and its bi-adjacency matrix.

The classical literature and some new developments are focused on approaches to solve and understand structural properties of this problem [16, 28, 18, 17, 8, 12, 14]. Another question is the corresponding enumeration problem. To the best of our knowledge, the complexity status for the problem of counting the realisations for a given list pair is open. In contrast, for a related problem – *the bipartite multigraph realisation problem*, i.e. counting the number of bipartite multigraphs for a given list, is known to be $\sharp P$-hard [13]. On the other hand, there has been a lot of work on this topic which either use 'closed formula approaches' and generating functions, estimated formulas, or approximation algorithms. For 'formula approaches', lower bounds and estimates which are based on structural matrix insights, we recommend an overview in Brualdi's book [9, Chapter 4]. An improved result can be found in [27]. The problem with these formulas is often that the calculation requires exponentially many operations. Other authors are more focused on estimating formulas [5, 25, 7, 10], or focus on approximation algorithms [22, 4, 6, 2, 26]. We want to highlight the survey of Barvinok [3] giving an efficient approximation algorithm in solving a convex optimisation problem.

In our work, we focus more on understanding how the number of realisations is related to the list pair itself. How many realisations exist is strongly dependent on the structure of a list pair. Roughly spoken, some list pairs only possess one unique realisation, whereas others, posses exponentially many. Bipartite graphs which only possess one unique realisation were first denoted by 'difference graphs' [21, Theorem 2.3.]. For general undirected graphs they are called *threshold graphs* and its suitable list pair *threshold sequence* [11, Theorem 1]. Strongly connected with the concept of threshold graphs are *Ferrers matrices*. For a given list pair (a, b) the corresponding Ferrers matrix F is the $n \times n'$-matrix where each row i consists of b_i consecutive 1's followed by $(n' - b_i)$ 0's. Ferrers introduced this notion in the context of partitions. For more information see Sylvester [29], or the overview about Norman Macleod Ferrers in [15]. A Ferrers matrix can be seen as the bi-adjacency matrix of a bipartite graph for list pair (a', b) where $a' = (a'_1, \ldots, a'_n)$ is the list of its column sums. Specifically, a Ferrers matrix is a unique realisation of (a', b) because a Ferrers matrix cannot contain a *swap* (or *switch*), i.e. one cannot find 2×2-sub-matrices $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, or $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. This is indeed equivalent to the uniqueness of a suitable realisation which can be proven by a classical but often overlooked result of Ryser [28] from 1957; for a given list pair each realisation can be yielded from another one by a series of swaps (in his paper called *interchanges*). This brings to mind the definition of threshold graphs. Indeed, it turns

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out that Ferrers matrices and threshold graphs are equivalent. Let us consider the $n \times n'$-adjacency matrix A of bipartite threshold graph G. Since the existence of entries $A_{ik} = 0$, $A_{il} = 1$ for $k < l$ would imply the existence of a swap (because a' is non-increasing), the rows must be constructed by consecutive 1's followed by 0's which is a Ferrers matrix. Clearly, each list pair (a, b) (following our definition with $0 \leq a_i \leq n$ and $0 \leq b_i \leq n'$) possesses a Ferrers matrix. In the context of integer partitions, list a' is the *conjugate partition* of b. For simplicity we always use in this paper the term 'Ferrers matrix' and for its list pair (a', b) *threshold list pair*. For more results in the area of threshold graphs we recommend the book of Mahadev and Peled [**?**].

The question for us was if there exists a kind of 'contrary list pair' such that for fixed n, n' and $m := \sum_{i=1}^{n} a_i$ the number of realisations is maximum? The methods in our work are strongly connected to the concept of majorisation. The majorization relation $\prec$ on real n-tuples was introduced by Hardy, Littlewood and Polya [20].

Definition 1.1 (Majorization)**.** *We define the* majorisation *relation* $\prec$ *on* $\mathbb{Z}^+ \times \mathbb{Z}^+$ *by* $a \prec a'$ *if and only if*

$$\begin{aligned} \sum_{i=1}^{k} a_i &\leq \sum_{i=1}^{k} a'_i \text{ for all } k \in \{1, \ldots, n-1\} \text{ and} \\ \sum_{i=1}^{n} a_i &= \sum_{i=1}^{n} a'_i. \end{aligned}$$

n-tuple a is said to be majorized *by a' or a'* majorizes *a, respectively.*

Notice that this classical definition includes the case that list a equals a'. We cite the Gale-Ryser theorem [16, 28] which gives the conditions to decide if a given list pair is bigraphic.

Theorem 1.2 (Gale-Ryser Theorem)**.** *Let* $(a, b) = ((a_1, \ldots, a_n), (b_1, \ldots, b_{n'}))$ *be a list pair*[1]*, and let* (a', b) *be its threshold list pair. Then* (a, b) *is bigraphic if and only if* $a \prec a'$*.*

Indeed, Ryser also gives in his proof a method for constructing a bi-adjacency matrix A with list pair (a, b) starting from the corresponding Ferrers matrix F with threshold list pair (a', b). One basically creates, step by step, a sequence of bi-adjacency matrices $F =: A_1, A_2, \ldots, A_k := A$ with the corresponding sequence of *bigraphic* list pairs $(a', b) =: (a^1, b), (a^2, b), \ldots, (a^k, b) := (a, b)$. The matrices in this sequence are constructed by correcting the number of 1's from the first to the last column (whenever necessary) in shifting 1's from other columns to the current one.

Our Contribution. We generalise the Gale-Ryser Theorem. Let $N(a, b)$ denote the number of realisations[2] for list pair (a, b).

[1] A list pair (a, b) is defined as a pair of non-negative, non-increasing integer lists a and b with $0 \leq a_i \leq n'$ and $0 \leq b_i \leq n$ and $\sum_{i=1}^{n} a_i = \sum_{i=1}^{n'} b_i$.

[2] Recall that we use the term realisation synonymously for a bipartite graph and its bi-adjacency matrix.

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Theorem 3.10. *Let $(a,b) = ((a_1,\dots,a_n),(b_1,\dots,b_{n'}))$ and (a',b) be two list pairs*[1]*. Then $N(a,b) \geq N(a',b)$ if and only if $a \prec a'$. If (a',b) is bigraphic than $N(a,b) > N(a',b)$.*

This generalises the Gale-Ryser Theorem in the following way. If (a',b) is the list pair of the corresponding Ferrers matrix of (a,b) then our result states that (a,b) has more realisations than (a',b). Since (a',b) has exactly one realisation this means (a,b) is bigraphic. More precisely, we show for a sequence of *bigraphic* list pairs $(a',b) =: (a^1,b),(a^2,b),\dots,(a^k,b) := (a,b)$ with $a^k \prec \dots \prec a^2 \prec a^1$ that $N(a^k,b) > \dots > N(a^2,b) > N(a^1,b)$. Another result is a generalisation of the Gale-Ryser Theorem.

Theorem 2.3. *Let $(a,b) = ((a_1,\dots,a_n),(b_1,\dots,b_{n'}))$ and (a',b) be two list pairs*[1]*. Then (a,b) is bigraphic if and only if (a',b) is bigraphic.*

To the best of our knowledge, Theorem 2.3 was never been proven for the bipartite case. A corresponding version for simple graphs was given by Aigner and Triesch [1]. Theorem 3.10 leads to the insight that one can find for a threshold list pair (a',b) a list pair (a,b) with the largest number of realisations under all lists a with $a \prec a'$. This happens when a cannot majorise any other non-increasing list pairs. If n divides m, list a has the form $a = (\frac{m}{n},\dots,\frac{m}{n})$, and is so a constant list. If n does not divide m, list a must be $a = (\lceil\frac{m}{n}\rceil,\cdots,\lceil\frac{m}{n}\rceil,\lfloor\frac{m}{n}\rfloor,\dots,\lfloor\frac{m}{n}\rfloor)$ where $\lceil\frac{m}{n}\rceil$ occurs $(m \mod n)$ times in a. Exchanging the roles of a and b and applying our main result again, leads to the insight that *regular list pairs* $(a,b) = ((\frac{m}{n},\dots,\frac{m}{n}),(\frac{m}{n'},\dots,\frac{m}{n'}))$ have the largest number of realisations for fixed n, n' and m when n and n' divide m. If n and n' do not divide m we get the generalised result that so-called *minconvex list pairs* $(a,b) = ((\lceil\frac{m}{n}\rceil,\dots,\lceil\frac{m}{n}\rceil,\lfloor\frac{m}{n}\rfloor,\dots,\lfloor\frac{m}{n}\rfloor),(\lceil\frac{m}{n'}\rceil,\dots,\lceil\frac{m}{n'}\rceil,\lfloor\frac{m}{n'}\rfloor,\dots,\lfloor\frac{m}{n'}\rfloor))$ possess the largest number of realisations for fixed n, n' and m. Minconvex list pairs are in a certain sense the contrary 'threshold list pairs'.

Overview. In Section 2 we generalise the Gale-Ryser theorem to a version where the majorising list is not necessarily a threshold list. Section 3 states our main result, i.e. a majorised list pair has more realisations than the majorising list. Furthermore, we introduce minconvex list pairs as list pairs with the largest number of realisations.

2. Generalizations of Characterizations of Degree Lists

Similar but not identical to Mahadev and Peled [MP95, Definition 3.1.2], and Marshall and Olkin [23] we define *transfers* on integer lists. Let the ith *single list* be the n-tuple e_i having 1 in coordinate i and 0 elsewhere.

Definition 2.1 (transfer)**.** *For an integer list a' with $a'_i \geq a'_j + 2$ for i,j such that $1 \leq i < j \leq n$, the list obtained from a' by an (i,j)-transfer (written $t_{i,j}(a')$) is the list $a' - e_i + e_j$. Sometimes, we simply use the term* transfer *without specifying the indices.*

We repeat a classical result of Muirhead [24] in a version by Mahadev and Peled [MP95, Theorem 3.1.3] with a small distinction for our specific investigations. This proof gives also an algorithm for obtaining a non-increasing list from a list that majorizes it.

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Theorem 2.2 (Muirhead 1902 [24])**.** *If a and a' are non-negative, non-increasing integer lists and $a \prec a'$, then a can be obtained from a' by κ successive single transfers, where $\kappa = \frac{1}{2}\sum_{j=1}^{n} |a'_j - a_j|$.*

The constructive proof of Muirhead generates explicitly a sequence of integer lists $a := a^1, \ldots, a^\kappa =: a'$ such that list a^{i+1} can be achieved by a_i via one transfer. Furthermore, we need his observation that we find for $a \neq a'$ (as given in his Theorem) indices $k < \ell$ with

$$a'_k \geq a'_\ell + 2. \tag{*}$$

Let us denote by ℓ the first index such that $\sum_{i=1}^{\ell} a_i < \sum_{i=1}^{\ell} a'_i$; by definition, $\ell < n$. Since this is the first position with a difference, $a'_\ell > a_\ell$. Since $\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} a'_i$, there is a smallest index k such that $a'_k < a_k$ and $k > \ell$. Since a is non-increasing, we get $a'_\ell > a_\ell \geq a_k > a'_k$ and thus our claim.

It turns out that Muirhead's Lemma can be used for the construction of a realisation[2] if one starts with the corresponding threshold list (a', b) of a given bigraphic list pair (a, b) in constructing a sequence of bigraphic list pairs $(a', b) = (a^1, b), (a^2, b), \ldots, (a^\kappa, b) =: (a, b)$ and corresponding realisations $A_1, \ldots, A_\kappa$. A crucial observation in constructing realisations is that applying any transfer to a bigraphic list pair (a^i, b) yields a bigraphic list pair (a^{i+1}, b), since $a^i_k \geq a^i_\ell + 2$ (see (*)) implies the existence of $A^i_{j,k} = 1$ and $A^i_{j,\ell} = 0$ for a $j \in \{1, \ldots, n'\}$, and can be replaced by $A^{i+1}_{j,k} = 0$ and $A^{i+1}_{j,\ell} = 1$.

Theorem 2.3. *Let $(a, b) = ((a_1, \ldots, a_n), (b_1, \ldots, b_{n'}))$ and (a', b) be two list pairs*[1]*. Then (a, b) is bigraphic if and only if (a', b) is bigraphic.*

Proof. Integer list a can be obtained from a' by successive single transfers via κ integer lists a^i (see Theorem 2.2). We show by induction on κ that each (a^i, b) is a bigraphic list pair. Let A^i be a realisation of list pair (a^i, b), and let (a^{i+1}, b) obtained by a (k, ℓ)-transfer, i.e. $a^{i+1} = a^i - e_\ell + e_k$. With (*) we have $a^i_k \geq a^i_\ell + 2$. Hence, there exist $A^i_{j,k} = 1$ and $A^i_{j,\ell} = 0$ for a $j \in \{1, \ldots, n'\}$. The matrix A^{i+1} with $A^{i+1}_{s,t} := A^i_{s,t}$ for all pairs $(s, t) \neq (j, k), (j, \ell)$, $A^{i+1}_{j,k} := 0$ and $A^{i+1}_{j,\ell} := 1$ is a realisation of list pair (a^{i+1}, b). □

The variant for simple graphs was first given by Aigner and Triesch [1] and later rediscovered by Mahadev and Peled [MP95, Corollary 3.1.4]. Aigner and Triesch called the order 'dominance' rather than 'majorization', and several papers continue that usage.

We introduce a special notion for sequences $(a^1, b), (a^2, b), \ldots, (a^r, b)$ of bigraphic list pairs as they appear in Theorem 2.3.

Definition 2.4 (transfer path)**.** *Sequences $(a^1, b), (a^2, b), \ldots, (a^\kappa, b)$ of bigraphic list pairs (a^i, b), where a^i yields a^{i+1} by a single transfer, are called* transfer paths*. We denote the value $(\kappa - 1)$ as the* length *of a transfer path.*

Note that the theorem gives a proof for the existence of at least one transfer path between two bigraphic lists pairs. Indeed, there often exist several transfer paths with different lengths. We give an example.

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Example 2.5. *We consider the two bigraphic list pairs* (a, b) *and* (a', b) *with* $a = (2, 2, 2, 0)$*,* $b = (1, 1, 2, 2)$ *and* $a' = (4, 2, 0, 0)$*. Then we find the following transfer paths.*

1: $a' = (4, 2, 0, 0), (3, 3, 0, 0), (3, 2, 1, 0), (2, 2, 2, 0) = a$

2: $a' = (4, 2, 0, 0), (3, 2, 1, 0), (2, 2, 2, 0) = a$

3: $a' = (4, 2, 0, 0), (4, 1, 1, 0), (3, 2, 1, 0), (2, 2, 2, 0) = a$

Transfer path 2 would have been constructed in the proof of Theorem 2.3.

3. The Connection between Majorization and the Number of Realizations

In a first step, we define the set $R(a, b)$ of all realisations[2] for a given bigraphic list pair (a, b) and denote by $N(a, b) := |R(a, b)|$ the number of realisations of (a, b). We need a connection between a single (i, j)-transfer on a bigraphic list pair (a', b) yielding (a, b), and a corresponding operation on a realisation $A' \in R(a', b)$ leading to a realisation $A \in R(a, b)$. Then a'_i, a'_j are the sums of the ith or jth columns in A', respectively. Since a' yields a by one single transfer we have $a'_i \geq a'_j + 2$. Recalling that $|a'_i - a'_j| \geq 2$, it is possible to shift at least two 1's from the ith column to the jth column. So, we can construct $|a'_i - a'_j| \geq 2$ different realisations in $R(a, b)$ from only **o**ne realisation $A' \in R(a', b)$.

Definition 3.1. *We call an exchange of* $A'_{ki} = 1$ *and* $A'_{kj} = 0$ *for* $k \in \{1, \ldots, n\}$ *on the* $(n \times n')$ *bi-adjacency matrix* A' *with* $i < j$ *an* (i, j)-shift on A'*. We denote the subset of realisations from* $R(a, b)$ *which are constructed by* (i, j)*-shifts from a matrix* A' *by* $\mathrm{Shift}_{ij}(A')$*.*

In the beginning of this section we have already seen that a shift which corresponds to a transfer creates at least two matrices from one. What happens when we consider all possible (i, j)-shifts on **two** matrices A'_1 and A'_2 for list pair (a', b). Then it occurs that we produce a matrix A more than once, i.e. $\mathrm{Shift}_{i,j}(A'_1) \cap \mathrm{Shift}_{i,j}(A'_2) \neq \emptyset$. We give an example.

Example 3.2. *We consider the bi-adjacency matrices* A'_1 *and* A'_2 *for bigraphic list pair* $(a', b) := ((4, 1), (1, 1, 1, 1, 1))$*, and apply in* A'_1 *a* $(1, 2)$*-shift in row three, and in* A'_2 *a* $(1, 2)$*-shift in row two. Then we create from*

$$A'_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{pmatrix} \text{ and } A'_2 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{pmatrix} \text{ the common solution } A = \begin{pmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{pmatrix} \text{ for list pair } (a, b) =$$

$((3, 2), (1, 1, 1, 1, 1))$*. More generally, we get* $\mathrm{Shift}_{1,2}(A'_1) \cap \mathrm{Shift}_{1,2}(A'_2) = \{A\}$*. However, it is easy to see that* $N(a', b) = \binom{5}{4}$*, and* $N(a, b) = \binom{5}{3}$*. The idea that more realisations are generated by shifts still seems to work.*

Carefully comparing the bi-adjacency matrices in Example 3.2 we observe that they exactly differ by a 2×2-matrix. This difference corresponds to a swap which was already introduced in the Introduction. It turns out that two matrices A'_1 and A'_2 can only generate the same matrix A by $(i, j)-$shifts if they differ by exactly one swap which also occurs in the ith and jth column.

Proposition 3.3. *Let A'_1 and A'_2 be bi-adjacency matrices of list pair*[1] (a', b), *and let* (a, b) *be a list pair such that* a' *yields* a *by one single* (i, j)*-transfer. Then* $\text{Shift}_{i,j}(A'_1) \cap \text{Shift}_{i,j}(A'_2) \neq \emptyset$ *if and only if* A'_1 *and* A'_2 *differ by a swap.*

Proof. We consider a matrix $A \in \text{Shift}_{i,j}(A'_1) \cap \text{Shift}_{i,j}(A'_2)$. Then there exist two different (i, j)-shifts – one in A'_1, and one in A'_2, say an (i, j)-shift in row k of A'_1, and an $(i, j)-shift$ in row k' of A'_2. Since both shifts lead to the matrix A, we can conclude that entries (k', j) in A'_1, and (k, j) in A'_2 are 1. Furthermore, we find that entries (k', i) in A'_1 and (k, i) in A'_2 are 0. Considering the 2×2 sub-matrices for row indices k, k' and column indices i, j in A'_1 and A'_2 reveals that they differ by exactly one swap. The converse implication holds trivially. For the two matrices A'_1 and A'_2 which differ by exactly one swap in 2×2 sub-matrix for row indices k, k' and column indices i, j, we apply in matrix A'_1 w.l.o.g. an (i, j)-shifts in row k, and in A'_2 one in row k'. Clearly, this results in the same matrix A. □

We call two matrices A'_1, A'_2 which differ by a swap like in Proposition 3.3, (i, j)*-adjacent*. We define the subset of all realisations $R(a', b)$ which possess at least one (i, j)-adjacent realisation by

$$M_{i,j}(a', b) := \{A' \in R(a', b)| \text{ it exists } A'_2 \in R(a', b) \text{ such that } A' \text{ and } A'_2 \text{ are } (i, j)\text{-adjacent.}\}$$

Applying (i, j)-shifts on realisations in $M_{i,j}(a', b)$ can lead to identical realisations of (a', b) which was shown in Proposition 3.3, and makes it more difficult to estimate the number of created realisations. Contrary, the calculation of the realisation number from a set of matrices which are not (i, j)-adjacent is quite easy because Proposition 3.3 states that all (i, j)-shifts on this set generate pairwise disjoint matrices. Hence, the number of realisations which can be constructed by (i, j)-shifts from elements in $R(a', b) \setminus M_{i,j}(a', b)$ is at least $|a'_i - a'_j| \cdot |R(a', b) \setminus M_{i,j}(a', b)| \geq 2 \cdot |R(a', b) \setminus M_{i,j}(a', b)|$. This means we get at least twice the number of realisations by applying all (i, j)-shifts.

Proposition 3.4. *Let* (a, b) *and* (a', b) *be two bigraphic list pairs*[1] *such that* a' *yields* a *by a single* (i, j)*-transfer. Applying all possible* (i, j)*-shifts realisations in* $R(a', b) \setminus M_{i,j}(a', b)$ *generates a subset of* $R(a, b)$ *that has at least twice the cardinality of* $|R(a', b) \setminus M_{i,j}(a', b)|$*. In particular, we get with* $M_1 := \left|\bigcup_{A' \in (R(a',b) \setminus M_{i,j}(a',b))} \text{Shift}_{i,j}(A')\right|$;

$$M_1 \geq |a'_i - a'_j| \cdot |R(a', b) \setminus M_{i,j}(a', b)| \geq 2 \cdot |R(a', b) \setminus M_{i,j}(a', b)|.$$

Let us consider an extremal example in which we apply $(k, k+1)$-shifts for $3 \leq k \leq n-1$ on matrices which are not $(k, k+1)$-adjacent.

Example 3.5. *We consider threshold list pair* $(a', b) := ((n-1, n-2, \ldots, 1, 0), (0, 1, \ldots, n-2, n-1))$*. Its Ferrers matrix has* $0's$ *above, and on its main diagonal. The rest is filled with* $1's$*. The first possible shift from column* 1 *can be done to column* 3 *because the difference of these column sums is two. We find two possible* $(1, 3)$*-shifts, and get two matrices for list pair* (a^2, b)*. Since, the* 4*th columns have not changed yet, these matrices cannot be* $(3, 4)$*-adjacent. On each of them we apply the two possible* $(3, 4)$*-shift, and get* 4 *matrices. One after another we apply now* ***two*** $(4, 5), \ldots, (n-1, n)$*-shifts on the current matrix set, and get all matrices for the list pair* $((a, b) = (n-2, n-2, n-3, n-$

$4, \ldots, 3, 2, 1, 1), (0, 1, \ldots, n-2, n-1))$. *In each step we have two possibilities for a shift. Hence, we create pairwise disjoint matrix sets in each step which leads to* $N(a, b) \geq 2^{n-2}$.

Note that (a', b) in our example can yield (a, b) by a single $(1, n)$-transfer. That is, only one shift would have been sufficient to achieve bigraphic list pair (a, b). In this case we only have $n-1$ possible $(1, n)$-shifts, and so our estimation for a lower bound of $N(a, b)$ was $n-1$. Hence, the kind of transfer paths plays an important role for the estimation of a possible lower bound for the number of realisations. In particular, the existence of two different (i, j)-adjacent realisations of a list pair (a^κ, b) on a transfer path $(a^1, b), (a^2, b), \ldots, (a^\kappa, b)$ ((a^1, b) is threshold list pair) could only appear, if there were (i, c)-shifts and (d, j)-shifts on transfer sub-path $(a^1, b), \ldots, (a^{\kappa-1}, b)$.

Corollary 3.6. *Let* (a, b) *be a bigraphic list pair*[1] *and* (a', b) *its threshold list pair. If there exists a transfer path* $(a', b) := (a^1, b), \ldots, (a^\kappa, b) =: (a, b)$ *of length* $\kappa - 1$ *such that we have for each pair* $(i, j), (c, d)$ *of* (i, j)*- and* (c, d)*-transfers that* $i \neq c$ *and* $j \neq d$*, then* $N(a, b) \geq 2^{r-1}$.

Note that it is possible that we have $j = c$ or $i = d$ as in Example 3.5. For the next steps we state a combinatorial insight for binomial coefficients.

Proposition 3.7. *Let* $d := c - \ell$ *where* $c, d, \ell \in \mathbb{N}$*. If* $\ell \geq 2$ *then* $\binom{2c-\ell}{c-\ell+1} > \binom{2c-\ell}{c}$.

Proof. We consider Pascal's triangle in row $2c - \ell$. For an even $2c - \ell$, we find the maximum binomial coefficient $\binom{2c-\ell}{c-\frac{\ell}{2}}$. In this case ℓ must be even and therefore $\ell \geq 2$. Clearly, the binomial coefficient decreases symmetrically starting on the maximum middle binomial coefficient in the directions of both borders of Pascal's triangle. Since, $|(c) - (c - \frac{\ell}{2})| = \frac{\ell}{2}$ and $|(c - \frac{\ell}{2}) - (c - \ell + 1)| = \frac{\ell}{2} - 1$, binomial coefficient $\binom{2c-\ell}{c-\ell+1}$ is closer to maximum binomial coefficient than $\binom{2c-\ell}{c}$. Hence, $\binom{2c-\ell}{c-\ell+1} > \binom{2c-\ell}{c}$. For an odd $2c-\ell$, ℓ must be odd, and $\ell \geq 3$. We find the two maximum binomial coefficients $\binom{2c-\ell}{c-\frac{1}{2}(\ell+1)}$ and $\binom{2c-\ell}{c-\frac{1}{2}(\ell-1)}$ in row $2c - \ell$ of Pascal's triangle. Again, the binomial coefficients decrease symmetrically starting on the two maximum middle binomial coefficients in the directions of both borders of Pascal's triangle. We get the following 'distances' d_1, d_2 between binomial coefficients and the 'middle' binomial coefficients;

(1) $\binom{2c-\ell}{c}$ is closer to $\binom{2c-\ell}{c-\frac{1}{2}(\ell-1)}$ with distance $d_1 := \min\{|c - (c - \frac{1}{2}(\ell+1))|, |c - (c - \frac{1}{2}(\ell-1))|\} = \frac{1}{2}(\ell - 1)$, and
(2) $\binom{2c-\ell}{c-\ell+1}$ is closer to $\binom{2c-\ell}{c-\frac{1}{2}(\ell+1)}$ with distance $d_2 := \min\{|(c - \ell + 1) - (c - \frac{1}{2}(\ell+1))|, |(c - \ell + 1) - (c - \frac{1}{2}(\ell - 1))|\} = \frac{1}{2}(\ell - 3)$.

Since $d_1 > d_2$ we find that $\binom{2c-\ell}{c-\ell+1}$ is closer to the 'right' maximum binomial coefficient than $\binom{2c-\ell}{c}$ to the 'left' maximum binomial coefficient in Pascal's triangle. Together with the even case it follows $\binom{2c-\ell}{c-\ell+1} > \binom{2c-\ell}{c}$ for $\ell \geq 2$. □

We prove that the set $M_{i,j}(a', b)$ of (i, j)-adjacent matrices in $R(a', b)$ creates always a larger number of matrices after applying all possible single (i, j)-shifts.

Proposition 3.8. *Let* (a, b) *and* (a', b) *be two different bigraphic list pairs*[1] *such that* a' *yields* a *by a single* (i, j)*-transfer and* $M_{i,j}(a', b) \neq \emptyset$*. Then the application of all possible* (i, j)*-shifts on*

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set $M_{i,j}(a',b) \subset R(a',b)$ generates a subset $M_2 := \left|\bigcup_{A' \in M_{i,j}(a',b)} \mathrm{Shift}_{i,j}(A')\right|$ of $R(a,b)$ with $M_2 > |M_{i,j}(a',b)|$.

Proof. Two (i,j)-adjacent bi-adjacency matrices in $M_{i,j}(a',b)$ do only differ in the ith and jth columns, and in the kth and k'th rows. Specifically, all other rows and columns are identical. Hence, the number c of possible (i,j)-shifts in both matrices must be identical, and $2 \le |a'_i - a'_j| \le c \le a'_i$. We conclude for those matrices that there exist $c \le a'_i$ rows with entry 1 in column i whereas the entries in the jth column of these rows are 0. Consider a schematic picture of such a matrix where the permutation of indices has been ignored. That means, the rows have been permuted to form four different blocks. $A' :=$

$$
\begin{array}{c}
\\ 1 \\ \vdots \\ c \\ \vdots \\ c+d \\ \vdots \\ \\ n \\ \\
\end{array}
\begin{array}{ccccccc}
1 & \cdots & i & \cdots & j & \cdots & n \\
\end{array}
\left(\begin{array}{ccccccc}
 & & \boxed{\begin{matrix}1\\1\\\vdots\\1\end{matrix}} & & \boxed{\begin{matrix}0\\0\\\vdots\\0\end{matrix}} & & \\
 & & \boxed{\begin{matrix}0\\\vdots\\0\end{matrix}} & & \boxed{\begin{matrix}1\\\vdots\\1\end{matrix}} & & \\
 & & \boxed{\begin{matrix}1\\\vdots\\1\end{matrix}} & & \boxed{\begin{matrix}1\\\vdots\\1\end{matrix}} & & \\
 & & \boxed{\begin{matrix}0\\\vdots\\0\end{matrix}} & & \boxed{\begin{matrix}0\\\vdots\\0\end{matrix}} & &
\end{array}\right)
\begin{array}{l}
\left.\vphantom{\begin{matrix}1\\1\\\vdots\\1\end{matrix}}\right\} c \\
\left.\vphantom{\begin{matrix}0\\\vdots\\0\end{matrix}}\right\} d
\end{array}
$$

$$a'_i \qquad\qquad a'_j$$

Since $a'_i \ge a'_j + 2$, we find in the ith column $d = c - \ell$ rows with entries 0 ($\ell := |a'_i - a'_j| \ge 2$) whereas the entries of these rows in the jth column are 1. There are different possibilities for c and d, but for a pair of (i,j)-adjacent matrices these values are fixed with Proposition 3.3. Notice, that we only consider matrices $A' \in M_{ij}(a',b)$ in a fixed scenario F, i.e. all entries that do not belong to the ith and jth column of a matrix in scenario F are identical. The set of all possible scenarios we denote by $\mathcal{F}$. Furthermore, $d \ge 1$ and $c \ge 3$. Otherwise there is no possibility for finding an (i,j)-adjacency for two matrices. We partition set $M_{i,j}(a',b)$ in disjoint subsets $m_F(c,d)$ for each fixed pair (c,d) and each scenario F, i.e. we create a set $m_F(c,d)$ for each F, and pair c,d with $3 \le c \le a'_i$ and $d \in \{c - \ell \mid d \ge 1, \ell \ge 2\}$. Notice, that only matrices within a same set $m_F(c,d)$ can be (i,j)-adjacent, and are able to generate identical matrices by single shifts. Each $m_F(c,d)$ is either empty, or $|m_F(c,d)| = \binom{c+d}{c} = \binom{2c-\ell}{c}$ because the entries in the corresponding $(c+d)$ rows in columns i and j can be permuted and maintain the row sums. We apply all possible (i,j)-shifts on $m_F(c,d)$, and get $\binom{c+d}{d+1} = \binom{2c-l}{c-\ell+1}$ different realisations of (a,b). Since, $\binom{2c-\ell}{c-\ell+1} > \binom{2c-\ell}{c}$ for all non-empty $m_F(c,d)$ with $\ell \ge 2$ (Proposition 3.7) it follows $M_2 > M_{i,j}(a',b)$. $\square$

Putting all results together we find that the all possible single shifts on a set of matrices with list pair (a', b) always generate more matrices.

Theorem 3.9. *Let (a, b) and (a', b) be two different list pairs*[1] *such that a' yields a by a single (i, j)-transfer. Then it follows $N(a, b) \geq N(a', b)$. If (a', b) is a bigraphic list pair, then we have $N(a, b) > N(a', b)$.*

Proof. If (a, b) is not a bigraphic list pair then $N(a, b) = 0$, and the Gale-Ryser Theorem 1.2 proves the claim. Let us consider the non-trivial case that (a', b) is a bigraphic list pair. Then (a, b) is also a bigraphic list pair by Theorem 2.3. A realisation $G' \in R(a', b)$ is either in $M_2 := M_{i,j}(a', b)$, or, in $M_1 := R(a', b) \setminus M_{i,j}(a', b)$. We apply on $R(a', b)$ all possible (i, j)-shifts, and generate set $M := \bigcup_{A' \in R(a',b)} \text{Shift}_{i,j}(A')$ with $M \subset R(a, b)$. (Notice that $M \neq R(a, b)$ is possible.) We apply Propositions 3.4 and 3.8, and get $N(a, b) \geq M > |M_1| + |M_2| = N(a', b)$. □

If we now consider a transfer path between two bigraphic list pairs (a, b) and (a', b) where a' majorizes a, we can easily conclude the main result of our paper.

Theorem 3.10. *Let $(a, b) = ((a_1, \ldots, a_n), (b_1, \ldots, b_{n'}))$ and (a', b) be two list pairs*[1]*. Then $N(a, b) \geq N(a', b)$ if and only if $a \prec a'$. If (a', b) is bigraphic than $N(a, b) > N(a', b)$.*

Proof. By Theorem 2.3 there exists at least one transfer path $(a^1, b), \ldots, (a^\kappa, b)$ with $(a^1, b) := (a', b)$, $(a^\kappa, b) := (a, b)$ and $a^{i+1} \prec a^i$. We show with induction on κ the correctness of our claim. For $\kappa = 2$ we apply Theorem 3.9, and get $N(a', b) < N(a, b)$. We consider the transfer path $(a^2, b), \ldots, (a^\kappa, b)$. With our induction hypothesis we can conclude $N(a^2, b) < N(a^\kappa, b)$. For (a^1, b) and (a^2, b) we apply again Theorem 3.9. This yields $N(a^1, b) < N(a^2, b) < N(a^\kappa, b))$. □

We now define list pairs which show regarding the number of realisations a 'contrary' behaviour in comparison to threshold list pairs, i.e. for fixed n, n' and m they possess the largest number of realisations. For given n, n' and m we define the *minconvex list pair* $(\alpha^n, \alpha^{n'}) = ((\alpha_1^n, \ldots, \alpha_n^n), (\alpha_1^{n'}, \ldots, \alpha_{n'}^{n'}))$ with

$$\alpha_i^x := \begin{cases} \lceil \frac{m}{x} \rceil & \text{for } i \in \{1, \ldots, m \bmod (x)\} \\ \lfloor \frac{m}{x} \rfloor & \text{for } i \in \{m \bmod (x) + 1, \ldots, x\} \end{cases}$$

Clearly, $\sum_{i=1}^{x} \alpha_i = m$. If x divides m, then we have $\alpha_i^x = \frac{m}{x}$ for $1 \leq i \leq x$. Hence, list α^x is constant. In our next theorem we show that each integer list with fixed n, m majorizes α^n.

Theorem 3.11. *For each non-increasing integer list a with $\sum_{i=1}^{n} a_i = m$ we have $\alpha^n \prec a$.*

Proof. Assume there exists $k < n$ with $\sum_{i=1}^{k} a_i < \sum_{i=1}^{k} \alpha_i^n$. Let k_0 be the smallest of such ks. Since $\sum_{i=1}^{k_0-1} a_i \geq \sum_{i=1}^{k_0-1} \alpha_i^n$ for $k_0 > 1$, it follows $\alpha_{k_0}^n > a_{k_0} \geq a_{k_0+1} \geq \cdots \geq a_n$. Since $\alpha_{k_0+1}^n, \ldots, \alpha_n^n$ are at least $\alpha_{k_0}^n - 1$ we get $\sum_{i=k_0+1}^{n} a_i \leq \sum_{i=k_0+1}^{n} \alpha_i^n$. Hence, $\sum_{i=1}^{n} a_i < \sum_{i=1}^{n} \alpha_i^n = m$ in contradiction to our assumption. □

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A classical result of Polya, Hardy and Littlewood [19] shows that $\sum_{i=1}^{n} g(a_i) \leq \sum_{i}^{n} g(a'_i)$ for $a \prec a'$ when $g : \mathbb{Z} \mapsto \mathbb{Z}$ denotes an arbitrary convex function. This is the intuition behind notion 'minconvex'. Now we are able to prove that minconvex list pairs have the largest number of realisations.

Corollary 3.12. *If (a, b) is a bigraphic list pair*[1], *$(\alpha^n, \alpha^{n'})$ the corresponding minconvex list pair, and $(a, b) \neq (\alpha^n, \alpha^{n'})$ then $N(\alpha^n, \alpha^{n'}) > N(a, b)$.*

Proof. Due to Theorem 3.11 we can apply Theorem 3.10 on list pairs (a, b) and (α^n, b) and get $N(a, b) \leq N(\alpha^n, b)$. Again, due to Theorem 3.11 we can apply Theorem 3.10 on list pairs (α^n, b) and $(\alpha^n, \alpha^{n'})$ and get $N(\alpha^n, b) \leq N(\alpha^n, \alpha^{n'})$. (The order of lists a and b in the list pair can exchanged without lost of generality.) Since $a \neq \alpha_n$ or $b \neq \alpha^{n'}$ at least one of the inequalities is strict. This proves the claim. □

4. Summary

We established a new connection between the number of realisations for list pairs (a, b) and (a', b) and the classical relation *majorisation*, i.e. whenever list a' majorises list a the number of realisations for list pair (a, b) is larger. This result generalises the classical Gale-Ryser theorem. Moreover, we introduced minconvex list pairs, which turn out to be contrary to threshold list pairs in the sense that they have under all list pairs the largest number of realisations.

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