

THE HARMONIC INDEX OF SUBDIVISION GRAPHS

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ABSTRACT. The harmonic index of a graph G is defined as the sum of the weights $\frac{2}{\deg_G(u) + \deg_G(v)}$ of all edges uv of G , where $\deg_G(u)$ denotes the degree of a vertex u in G . In this paper, we study the harmonic index of subdivision graphs, t -subdivision graphs and also, S -sum and S_t -sum of graphs.

1. Introduction

Throughout this paper, all graphs are finite, simple, undirected and connected. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. As usual, the maximum and minimum degrees of G are denoted by Δ and δ , respectively. We will use P_n , C_n and K_n to denote the path, the cycle and the complete graph of order n , respectively.

For a graph G , the subdivision graph $S(G)$ is a graph obtained from G by replacing each edge of G by a path of length 2. The t -subdivision graph $S_t(G)$ of G is a graph obtained from G by replacing each edge of G by a path of length $t + 1$. Obviously, $S_1(G) = S(G)$.

Let G_1 and G_2 be two graphs. The S -sum $G_1 +_S G_2$ is a graph with vertex set $(V(G_1) \cup E(G_1)) \times V(G_2)$ in which two vertices (u_1, v_1) and (u_2, v_2) of $G_1 +_S G_2$ are adjacent if and only if $[u_1 = u_2 \in V(G_1) \text{ and } v_1 v_2 \in E(G_2)]$ or $[v_1 = v_2 \text{ and } u_1 u_2 \in E(S(G))]$ [4].

$P_3 +_S P_2$ is shown in Fig. 1.

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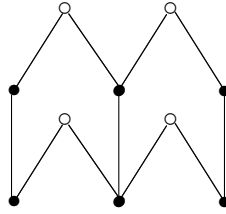


Fig. 1. $P_3 +_S P_2$

For a graph G , the harmonic index $H(G)$ is defined as

$$H(G) = \sum_{uv \in E(G)} \frac{2}{\deg_G(u) + \deg_G(v)}.$$

In recent years, this vertex-degree-based topological index has been extensively studied. For example, Zhong [18, 19, 20] gave the minimum and maximum values of the harmonic index for simple graphs, trees, unicyclic graphs and graphs with girth at least k ($k \geq 3$) and characterized the corresponding extremal graphs, respectively. Deng et al. [3] considered the relation between the harmonic index of a graph and its chromatic number. Lv et al. [9, 10] established the relationship between the harmonic index of a graph and its matching number. Shwetha et al. [15] derived expressions for the harmonic index of the join, corona product, Cartesian product, composition and symmetric difference of graphs. Recently, Onagh investigated the harmonic index of product graphs, vertex-semi-total graphs, edge-semi-total graphs, total graphs and F -sum of graphs, where $F \in \{R, Q, T\}$ [12, 13, 14]. More results on this index can be found in [8, 16, 17, 21, 22].

In this paper, we present some sharp bounds for the harmonic index of subdivision and t -subdivision graphs and characterize graphs for which these bounds are best possible. Also, we obtain some upper bounds for the harmonic index of S -sum and S_t -sum of graphs.

2. Lower bounds for the harmonic index of subdivision graphs

In this section, we give some lower bounds for the harmonic index of subdivision and t -subdivision graphs and obtain some inequalities between the harmonic index of these graphs and some topological indices such as the inverse degree and the first Zagreb index. Hereafter, we deal with nontrivial graphs.

Let G be a graph of size $m \geq 1$. By definition of the harmonic index, we have

$$(2.1) \quad H(S(G)) = \sum_{uv \in E(G)} \left(\frac{2}{\deg_G(u) + 2} + \frac{2}{2 + \deg_G(v)} \right),$$

and

$$\begin{aligned} H(S_t(G)) &= \sum_{uv \in E(G)} \left(\frac{2}{\deg_G(u) + 2} + \overbrace{\frac{2}{2+2} + \dots + \frac{2}{2+2}}^{(t-1)\text{-times}} + \frac{2}{2 + \deg_G(v)} \right) \\ &= \sum_{uv \in E(G)} \left(\frac{2}{\deg_G(u) + 2} + \frac{2}{2 + \deg_G(v)} \right) + \frac{1}{2}(t-1)m \\ &= H(S(G)) + \frac{1}{2}(t-1)m. \end{aligned}$$

It is easy to see that $H(G) < H(S(G))$ and then $H(G) < H(S_t(G))$.

Example 2.1.

- (i) For any $n \geq 2$, $H(S(P_n)) = \frac{4}{3} + n - 2$ and $H(S_t(P_n)) = \frac{4}{3} + n - 2 + \frac{1}{2}(t - 1)(n - 1)$,
- (ii) for any $n \geq 3$, $H(S(C_n)) = n$ and $H(S_t(C_n)) = n + \frac{1}{2}(t - 1)n$,
- (iii) for any $n \geq 2$, $H(S(K_n)) = \frac{2n(n - 1)}{n + 1}$ and $H(S_t(K_n)) = \frac{2n(n - 1)}{n + 1} + \frac{1}{4}(t - 1)n(n - 1)$.

Example 2.2. Let G be a k -regular graph of order n . Then $H(S(G)) = \frac{2kn}{k + 2}$ and $H(S_t(G)) = \frac{2kn}{k + 2} + \frac{1}{4}(t - 1)kn$.

In the following, simpler formulas are given for the harmonic index of subdivision graphs.

Theorem 2.3. Let G be a graph of order n . Then

$$H(S(G)) = 2n - \sum_{u \in V(G)} \frac{4}{\deg_G(u) + 2}.$$

Proof. Let u be a vertex of G . For each neighbor of u , $\frac{2}{\deg_G(u) + 2}$ appears exactly once in

$\sum_{uv \in E(G)} \left(\frac{2}{\deg_G(u) + 2} + \frac{2}{2 + \deg_G(v)} \right)$. Thus,

$$(2.2) \quad H(S(G)) = \sum_{u \in V(G)} \overbrace{\left(\frac{2}{\deg_G(u) + 2} + \dots + \frac{2}{\deg_G(u) + 2} \right)}^{\deg_G(u) \text{-times}} = \sum_{u \in V(G)} \frac{2 \deg_G(u)}{\deg_G(u) + 2}.$$

So, $H(S(G)) = \sum_{u \in V(G)} \left(2 - \frac{4}{\deg_G(u) + 2} \right) = 2n - \sum_{u \in V(G)} \frac{4}{\deg_G(u) + 2}$, as desired. □

Corollary 2.4. Let G be a graph of order n and size m . Then

$$H(S_t(G)) = 2n + \frac{1}{2}(t - 1)m - \sum_{u \in V(G)} \frac{4}{\deg_G(u) + 2}.$$

Remark 2.5. Let G_1 and G_2 be two graphs with the same degree sequence. Then $H(S(G_1)) = H(S(G_2))$ and $H(S_t(G_1)) = H(S_t(G_2))$.

A graph G is called a (Δ, δ) -bidegred if whose vertices have degree either Δ or δ ($\Delta \neq \delta$).

We will use Schweitzer’s inequality [2, 7] in order to present a sharp lower bound for the harmonic index of subdivision graphs.

Schweitzer’s inequality. Let x_1, \dots, x_n be positive real numbers such that for $1 \leq i \leq n$ holds $m \leq x_i \leq M$. Then

$$\left(\sum_{i=1}^n x_i \right) \left(\sum_{i=1}^n \frac{1}{x_i} \right) \leq \frac{n^2(m + M)^2}{4mM}.$$

Equality holds if and only if $x_1 = \dots = x_n = m = M$ or n is even, $x_1 = \dots = x_{\frac{n}{2}} = m$ and $x_{\frac{n}{2}+1} = \dots = x_n = M$, where $m < M$ and $x_1 \leq \dots \leq x_n$.

Theorem 2.6. *Let G be a graph of order n and size m . Then*

$$(2.3) \quad 2n - \frac{n^2(\delta + \Delta + 4)^2}{2(n + m)(\delta + 2)(\Delta + 2)} \leq H(S(G)),$$

with equality if and only if G is a regular graph or a (Δ, δ) -bidegreed graph.

Proof. For every vertex u in G , we have $\delta + 2 \leq \deg_G(u) + 2 \leq \Delta + 2$. Using $\sum_{u \in V(G)} (\deg_G(u) + 2) = 2m + 2n$ and Schweitzer's inequality, we get

$$\sum_{u \in V(G)} \frac{1}{\deg_G(u) + 2} \leq \frac{n^2(\delta + \Delta + 4)^2}{8(n + m)(\delta + 2)(\Delta + 2)}.$$

Now, Theorem 2.3 implies the desired inequality.

Moreover, by Schweitzer's inequality, equality holds in (2.3) if and only if $\delta = \Delta$ or $\frac{n}{2}$ vertices of G have degree δ and the remaining $\frac{n}{2}$ vertices have degree Δ , i.e., G is regular or (Δ, δ) -bidegreed. \square

Corollary 2.7. *Let G be a graph of order n and size m . Then*

$$2n - \frac{n^2(\delta + \Delta + 4)^2}{2(n + m)(\delta + 2)(\Delta + 2)} + \frac{1}{2}(t - 1)m \leq H(S_t(G)),$$

with equality if and only if G is a regular graph or a (Δ, δ) -bidegreed graph.

The first Zagreb index of a graph G is originally defined as $M_1(G) = \sum_{u \in V(G)} \deg_G^2(u)$. This topological index can be also expressed as $M_1(G) = \sum_{uv \in E(G)} (\deg_G(u) + \deg_G(v))$ [1, 6, 11].

Theorem 2.8. *Let G be a graph of size m . Then*

$$2m - \frac{1}{2}M_1(G) < H(S(G)).$$

Proof. Let $f(x) = \frac{1}{x}$. Since f is a convex function on $(0, +\infty)$, by Jensen's inequality, for every edge $uv \in E(G)$, we have

$$(2.4) \quad \frac{2}{\deg_G(u) + 2} + \frac{2}{2 + \deg_G(v)} \geq \frac{8}{4 + \deg_G(u) + \deg_G(v)},$$

with equality if and only if $\deg_G(u) = \deg_G(v)$.

So,

$$\begin{aligned}
 H(S(G)) &= \sum_{uv \in E(G)} \left(\frac{2}{\deg_G(u) + 2} + \frac{2}{2 + \deg_G(v)} \right) \\
 &\geq \sum_{uv \in E(G)} \frac{8}{4 + \deg_G(u) + \deg_G(v)} \\
 &= 2 \sum_{uv \in E(G)} \left(1 + \frac{\deg_G(u) + \deg_G(v)}{4} \right)^{-1} \\
 \text{(by Bernoulli's inequality)} \quad &> 2 \sum_{uv \in E(G)} \left(1 - \frac{\deg_G(u) + \deg_G(v)}{4} \right) \\
 &= 2m - \frac{1}{2} \sum_{uv \in E(G)} (\deg_G(u) + \deg_G(v)) \\
 &= 2m - \frac{1}{2} M_1(G).
 \end{aligned}$$

It completes the proof. □

Corollary 2.9. *Let G be a graph of size m . Then*

$$2m - \frac{1}{2} M_1(G) + \frac{1}{2} (t - 1)m < H(S_t(G)).$$

The inverse degree of a graph G is defined as $r(G) = \sum_{u \in V(G)} \frac{1}{\deg_G(u)}$ [5].

Theorem 2.10. *Let G be a graph of order n . Then*

$$(2.5) \quad \frac{3}{2}n - r(G) \leq H(S(G)),$$

with equality if and only if G is a disjoint union of cycles.

Proof. Similar to (2.4), for every vertex $u \in V(G)$, we have

$$\frac{1}{\deg_G(u) + 2} \leq \frac{1}{4} \left(\frac{1}{\deg_G(u)} + \frac{1}{2} \right),$$

with equality if and only if $\deg_G(u) = 2$.

Thus,

$$\sum_{u \in V(G)} \frac{4}{\deg_G(u) + 2} \leq \sum_{u \in V(G)} \frac{1}{\deg_G(u)} + \frac{1}{2}n = r(G) + \frac{1}{2}n.$$

By Theorem 2.3, we have the required inequality.

Furthermore, equality holds in (2.5) if and only if for every vertex $u \in V(G)$, $\deg_G(u) = 2$, i.e., G is the disjoint union of cycles. □

Corollary 2.11. *Let G be a graph of order n and size m . Then*

$$\frac{3}{2}n + \frac{1}{2}(t - 1)m - r(G) \leq H(S_t(G)),$$

with equality if and only if G is a disjoint union of cycles.

3. Upper bounds for the harmonic index of subdivision graphs

In this section, we present some sharp upper bounds for the harmonic index of subdivision and t -subdivision graphs and characterize the corresponding extremal graphs.

The following corollary is an immediate consequence of Theorem 2.3.

Corollary 3.1. *Let G be a graph of order n . Then*

$$H(S(G)) \leq \frac{2n(n-1)}{n+1} \quad \text{and} \quad H(S_t(G)) \leq \frac{2n(n-1)}{n+1} + \frac{1}{4}(t-1)n(n-1).$$

In both cases, equality holds if and only if $G \cong K_n$.

In the following, some upper bounds for the harmonic index of subdivision graphs are given in terms of order and size.

Theorem 3.2. *Let G be a graph of order n and size m . Then*

$$(3.1) \quad H(S(G)) \leq \frac{n+m}{2},$$

with equality if and only if G is a disjoint union of cycles.

Proof. For every vertex $u \in V(G)$, we have

$$\frac{2 \deg_G(u)}{\deg_G(u) + 2} \leq \frac{2 + \deg_G(u)}{4},$$

with equality if and only if $\deg_G(u) = 2$.

By (2.2), we get

$$H(S(G)) \leq \sum_{u \in V(G)} \frac{2 + \deg_G(u)}{4} = \frac{1}{2}n + \frac{1}{4} \sum_{u \in V(G)} \deg_G(u) = \frac{1}{2}n + \frac{1}{4}(2m) = \frac{n+m}{2},$$

as desired.

Moreover, equality holds in (3.1) if and only if for every vertex $u \in V(G)$, $\deg_G(u) = 2$, i.e., G is the disjoint union of cycles. \square

Corollary 3.3. *Let G be a graph of order n and size m . Then*

$$(3.2) \quad H(S_t(G)) \leq \frac{n+tm}{2},$$

with equality if and only if G is a disjoint union of cycles.

Theorem 3.4. *Let G be a graph of order n and size m . Then*

$$(3.3) \quad H(S(G)) \leq \frac{2nm}{n+m},$$

with equality if and only if G is a regular graph.

Proof. By Cauchy-Schwarz' inequality, we have

$$\left(\sum_{u \in V(G)} (\deg_G(u) + 2) \right) \left(\sum_{u \in V(G)} \frac{1}{\deg_G(u) + 2} \right) \geq \left(\sum_{u \in V(G)} \sqrt{\deg_G(u) + 2} \frac{1}{\sqrt{(\deg_G(u) + 2)}} \right)^2,$$

with equality if and only if all the $\deg_G(u)$'s are equal.

Since $\sum_{u \in V(G)} (\deg_G(u) + 2) = 2(n + m)$, we get $\sum_{u \in V(G)} \frac{1}{\deg_G(u) + 2} \geq \frac{n^2}{2(n + m)}$. By Theorem 2.3, this yields the required inequality.

Furthermore, equality holds in (3.3) if and only if all the $\deg_G(u)$'s are equal, i.e., G is regular. \square

Corollary 3.5. *Let G be a graph of order n and size m . Then*

$$(3.4) \quad H(S_t(G)) \leq \frac{2nm}{n + m} + \frac{1}{2}(t - 1)m,$$

with equality if and only if G is a regular graph.

Remark 3.6. *The inequalities (3.3) and (3.4) are always better than the inequalities (3.1) and (3.2), respectively.*

Let G be a graph of order n and size m . If $m = n - 1, n$ and $n + 1$ then G is called a tree, a unicyclic graph and a bicyclic graph, respectively.

Corollary 3.7. *Let G be a tree of order n . Then*

$$H(S(G)) \leq \frac{2n(n - 1)}{2n - 1} \quad \text{and} \quad H(S_t(G)) \leq \frac{2n(n - 1)}{2n - 1} + \frac{1}{2}(t - 1)(n - 1).$$

In both cases, equality holds if and only if $G \cong P_2$.

Corollary 3.8. *Let G be a unicyclic graph of order n . Then*

$$H(S(G)) \leq n \quad \text{and} \quad H(S_t(G)) \leq \frac{1}{2}(t + 1)n.$$

In both cases, equality holds if and only if $G \cong C_n$.

Corollary 3.9. *Let G be a bicyclic graph of order n . Then*

$$H(S(G)) < \frac{2n(n + 1)}{2n + 1} \quad \text{and} \quad H(S_t(G)) < \frac{2n(n + 1)}{2n + 1} + \frac{1}{2}(t - 1)(n + 1).$$

A pendant vertex in a graph is a vertex of degree 1. Now, an upper bound for the harmonic index of subdivision graphs is given in terms of order, the maximum degree and the number of pendant vertices.

Theorem 3.10. *Let G be a graph of order n with p pendant vertices. Then*

$$(3.5) \quad H(S(G)) \leq 2n - \frac{4}{3}p - \frac{4(n - p)}{\Delta + 2},$$

with equality if and only if G is a regular graph or a $(\Delta, 1)$ -bidegreed graph.

Proof. Note that

$$\begin{aligned}
 \sum_{u \in V(G)} \frac{1}{\deg_G(u) + 2} &= \overbrace{\left(\frac{1}{1+2} + \cdots + \frac{1}{1+2} \right)}^{p\text{-times}} + \sum_{u \in V(G), \deg_G(u) > 1} \frac{1}{\deg_G(u) + 2} \\
 &= \frac{1}{3}p + \sum_{u \in V(G), \deg_G(u) > 1} \frac{1}{\deg_G(u) + 2} \\
 (\Delta \geq \deg_G(u)) \quad &\geq \frac{1}{3}p + \overbrace{\left(\frac{1}{\Delta+2} + \cdots + \frac{1}{\Delta+2} \right)}^{(n-p)\text{-times}} \\
 &= \frac{1}{3}p + \frac{n-p}{\Delta+2}.
 \end{aligned}$$

Now, by Theorem 2.3, we get the desired inequality.

Moreover, equality holds in (3.5) if and only if for every non-pendant vertex $u \in V(G)$, $\deg_G(u) = \Delta$. If $p = 0$ then for every vertex $u \in V(G)$, $\deg_G(u) = \Delta$, i.e., G is Δ -regular, where $2 \leq \Delta \leq n - 1$. Now, suppose that $p > 0$. If there is no non-pendant vertex in G then $G \cong K_2$ and otherwise, G is $(\Delta, 1)$ -bidegreed.

Conversely, one can see easily that the equality holds in (3.5) for regular graph or $(\Delta, 1)$ -bidegreed graph. □

Corollary 3.11. *Let G be a graph of order n and size m with p pendant vertices. Then*

$$H(S_t(G)) \leq 2n - \frac{4}{3}p - \frac{4(n-p)}{\Delta+2} + \frac{1}{2}(t-1)m,$$

with equality if and only if G is a regular graph or a $(\Delta, 1)$ -bidegreed graph.

Corollary 3.12. *Let G be a graph of order n and size m with $\delta > 1$. Then*

$$H(S(G)) \leq 2n - \frac{4n}{\Delta+2} \quad \text{and} \quad H(S_t(G)) \leq 2n - \frac{4n}{\Delta+2} + \frac{1}{2}(t-1)m.$$

In both cases, equality holds if and only if G is a Δ -regular graph, where $2 \leq \Delta \leq n - 1$.

Let G be a graph. For two vertices u and v in G , the distance $d_G(u, v)$ from u to v is the length of a shortest path connecting u and v . For a vertex u in G , the eccentricity $\epsilon(u)$ of u is $\max_{v \in V(G)} d_G(u, v)$. The minimum eccentricity among the vertices of G is the radius of G , denoted by $rad(G)$, and the maximum eccentricity is its diameter $diam(G)$. A vertex u in G is a central vertex if $\epsilon(u) = rad(G)$.

A graph G is self-centered if $\epsilon(u) = rad(G)$ for all vertices $u \in V(G)$.

For any even n , the cocktail party graph CP_n is the unique regular graph with n vertices of degree $n - 2$; it is obtained from K_n by removing $\frac{n}{2}$ disjoint edges.

Theorem 3.13. *Let G be a graph of order n . Then*

$$(3.6) \quad H(S(G)) \leq \frac{2n(n - rad(G))}{n - rad(G) + 2},$$

with equality if and only if $G \cong K_n$ or $G \cong CP_n$.

Proof. Let $n_i(u)$ be the number of vertices at distance i from the vertex u in G . One can see that $\deg_G(u) \leq n - \epsilon(u)$ (*), with equality if and only if $\epsilon(u) = 1$ and $\deg_G(u) = n - 1$ or $\epsilon(u) \geq 2$ and $n_2(u) = n_3(u) = \dots = n_{\epsilon(u)}(u) = 1$. Thus, for every vertex $u \in V(G)$, we have

$$\frac{1}{\deg_G(u) + 2} \geq \frac{1}{n - \epsilon(u) + 2} \geq \frac{1}{n - \text{rad}(G) + 2}.$$

This implies that $\sum_{u \in V(G)} \frac{1}{\deg_G(u) + 2} \geq \frac{n}{n - \text{rad}(G) + 2}$. Now, Theorem 2.3 yields the required inequality.

Suppose that equality holds in (3.6). Then G is self-centered and for every vertex $u \in V(G)$, equality holds in (*). If $\epsilon(u) = 1$ for some vertex $u \in V(G)$ then $\deg_G(u) = n - 1$ and $\epsilon(v) \leq 2$ for all vertices $v \neq u$. Since G is self-centered, $\epsilon(u) = 1$ for all vertices $u \in V(G)$. Thus $G \cong K_n$.

Now, suppose that $\epsilon(u) \geq 2$ for all vertices $u \in V(G)$. If $\epsilon(v) \geq 3$ for some vertex v , then $\text{diam}(G) = 3$ (otherwise, there exist at least two neighbors at distance 2 for the central vertex) and $G \cong P_4$. This contradicts that G is self-centered. So, $\epsilon(u) = 2$ for all vertices $u \in V(G)$ and then $\deg_G(u) = n - 2$ for all vertices $u \in V(G)$. It implies that $G \cong CP_n$.

It can be easily seen that the upper bound is attained for K_n or CP_n . □

Corollary 3.14. *Let G be a graph of order n and size m . Then*

$$(3.7) \quad H(S_t(G)) \leq \frac{2n(n - \text{rad}(G))}{n - \text{rad}(G) + 2} + \frac{1}{2}(t - 1)m,$$

with equality if and only if $G \cong K_n$ or $G \cong CP_n$.

4. The harmonic index for S -sum and S_t -sum of graphs

In the following theorem, we give an upper bound for the harmonic index of $G_1 +_S G_2$ in terms of $H(S(G_1))$, $H(G_2)$, $r(G_1)$ and $r(G_2)$.

Theorem 4.1. *Let G_1 and G_2 be two graphs. Then*

$$H(G_1 +_S G_2) < \frac{1}{4} \left(n_2 H(S(G_1)) + n_1 H(G_2) + m_2 r(G_1) + 4m_1 r(G_2) \right),$$

where $n_i = |V(G_i)|$ and $m_i = |E(G_i)|$, $i = 1, 2$.

Proof. Let $\deg(u, v) = \deg_{G_1 +_S G_2}(u, v)$ be the degree of a vertex (u, v) in $G_1 +_S G_2$. By definition of the harmonic index, we have

$$\begin{aligned} H(G_1 +_S G_2) &= \sum_{u \in V(G_1)} \sum_{v_1 v_2 \in E(G_2)} \frac{2}{\deg(u, v_1) + \deg(u, v_2)} \\ &+ \sum_{v \in V(G_2)} \sum_{u_1 u_2 \in E(S(G_1))} \frac{2}{\deg(u_1, v) + \deg(u_2, v)} \\ &:= \sum 1 + \sum 2. \end{aligned}$$

Then,

$$\begin{aligned} \sum 1 &= \sum_{u \in V(G_1)} \sum_{v_1 v_2 \in E(G_2)} \frac{2}{(\deg_{G_1}(u) + \deg_{G_2}(v_1)) + (\deg_{G_1}(u) + \deg_{G_2}(v_2))} \\ &= \sum_{u \in V(G_1)} \sum_{v_1 v_2 \in E(G_2)} \frac{2}{2 \deg_{G_1}(u) + (\deg_{G_2}(v_1) + \deg_{G_2}(v_2))}. \end{aligned}$$

By Jensen's inequality, for every vertex $u \in V(G_1)$ and every edge $v_1 v_2 \in E(G_2)$, we have

$$(4.1) \quad \frac{2}{2 \deg_{G_1}(u) + (\deg_{G_2}(v_1) + \deg_{G_2}(v_2))} \leq \frac{1}{4} \left(\frac{1}{\deg_{G_1}(u)} + \frac{2}{\deg_{G_2}(v_1) + \deg_{G_2}(v_2)} \right),$$

with equality if and only if $2 \deg_{G_1}(u) = \deg_{G_2}(v_1) + \deg_{G_2}(v_2)$.

So,

$$\begin{aligned} \sum 1 &\leq \frac{1}{4} \sum_{u \in V(G_1)} \sum_{v_1 v_2 \in E(G_2)} \frac{1}{\deg_{G_1}(u)} + \frac{1}{4} \sum_{u \in V(G_1)} \sum_{v_1 v_2 \in E(G_2)} \frac{2}{\deg_{G_2}(v_1) + \deg_{G_2}(v_2)} \\ &= \frac{1}{4} \sum_{u \in V(G_1)} \left(m_2 \times \frac{1}{\deg_{G_1}(u)} \right) + \frac{1}{4} \sum_{u \in V(G_1)} H(G_2) \\ &= \frac{1}{4} \left(m_2 r(G_1) + n_1 H(G_2) \right). \end{aligned}$$

Also,

$$\begin{aligned} \sum 2 &= \sum_{v \in V(G_2)} \sum_{u_1 u_2 \in E(S(G_1))} \frac{2}{\deg_{S(G_1)}(u_1) + \deg_{G_2}(v) + \deg_{S(G_1)}(u_2)} \\ &= \sum_{v \in V(G_2)} \sum_{u_1 u_2 \in E(S(G_1))} \frac{2}{(\deg_{S(G_1)}(u_1) + \deg_{S(G_1)}(u_2)) + \deg_{G_2}(v)}. \end{aligned}$$

Similarly, for every vertex $v \in V(G_2)$ and every edge $u_1 u_2 \in E(S(G_1))$, we have

$$(4.2) \quad \frac{2}{(\deg_{S(G_1)}(u_1) + \deg_{S(G_1)}(u_2)) + \deg_{G_2}(v)} \leq \frac{1}{4} \left(\frac{2}{\deg_{S(G_1)}(u_1) + \deg_{S(G_1)}(u_2)} + \frac{2}{\deg_{G_2}(v)} \right),$$

with equality if and only if $\deg_{S(G_1)}(u_1) + \deg_{S(G_1)}(u_2) = \deg_{G_2}(v)$.

Thus,

$$\begin{aligned} \sum 2 &\leq \frac{1}{4} \sum_{v \in V(G_2)} \sum_{u_1 u_2 \in E(S(G_1))} \frac{2}{\deg_{S(G_1)}(u_1) + \deg_{S(G_1)}(u_2)} + \frac{1}{2} \sum_{v \in V(G_2)} \sum_{u_1 u_2 \in E(S(G_1))} \frac{1}{\deg_{G_2}(v)} \\ &= \frac{1}{4} \sum_{v \in V(G_2)} H(S(G_1)) + \frac{1}{2} \sum_{v \in V(G_2)} \left(2m_1 \times \frac{1}{\deg_{G_2}(v)} \right) \\ &= \frac{1}{4} \left(n_2 H(S(G_1)) + 4m_1 r(G_2) \right). \end{aligned}$$

Therefore,

$$(4.3) \quad H(G_1 +_S G_2) \leq \frac{1}{4} \left(n_2 H(S(G_1)) + n_1 H(G_2) + m_2 r(G_1) + 4m_1 r(G_2) \right).$$

Now, suppose that there exist two graphs G_1 and G_2 such that equality holds in (4.3). Then, the inequalities (4.1) and (4.2) must be equalities. So, G_1 and G_2 are k_1 -regular and k_2 -regular graphs, respectively, such that $2k_1 = k_2 + k_2$ and $k_1 + 2 = k_2$, a contradiction. This completes the proof. \square

Example 4.2. For any $n \geq 2$,

$$H(P_n +_S P_2) = \left(1 + \frac{1}{3}(n - 2)\right) + \left(2 + \frac{8}{5}(n - 2)\right) = 3 + \frac{29}{15}(n - 2).$$

In the following, we extend the definition of S -sum to S_t -sum of graphs.

Definition 4.3. Let G_1 and G_2 be two graphs. The S_t -sum $G_1 +_{S_t} G_2$ is a graph with vertex set $V(S_t(G_1)) \times V(G_2)$ in which two vertices (u_1, v_1) and (u_2, v_2) of $G_1 +_{S_t} G_2$ are adjacent if and only if $[u_1 = u_2 \in V(G_1) \text{ and } v_1 v_2 \in E(G_2)]$ or $[v_1 = v_2 \text{ and } u_1 u_2 \in E(S_t(G_1))]$.

Note that $G_1 +_{S_t} G_2$ has $|V(G_2)|$ copies of the graph $S_t(G_1)$ and we can label these copies by vertices of G_2 . The vertices in each copy have two situations: the vertices in $V(G_1)$ (black vertices) and the vertices in $V(S_t(G_1)) - V(G_1)$ (white vertices). We join only black vertices with the same name in $S_t(G_1)$ in which their corresponding labels are adjacent in G_2 .

$P_3 +_{S_4} P_2$ is shown in Fig. 2.

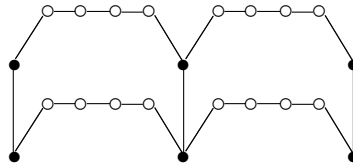


Fig. 2. $P_3 +_{S_4} P_2$

Now, we obtain an upper bound for the harmonic index of $G_1 +_{S_t} G_2$ in terms of $H(S_t(G_1))$, $H(G_2)$, $r(G_1)$ and $r(G_2)$.

Theorem 4.4. Let G_1 and G_2 be two graphs. Then

$$H(G_1 +_{S_t} G_2) < \frac{1}{4} \left(n_2 H(S(G_1)) + n_1 H(G_2) + m_2 r(G_1) + 4m_1 r(G_2) + 2(t - 1)n_2 m_1 \right),$$

where $n_i = |V(G_i)|$ and $m_i = |E(G_i)|$, $i = 1, 2$.

Proof. Let $\deg(u, v) = \deg_{G_1 +_{S_t} G_2}(u, v)$ be the degree of a vertex (u, v) in $G_1 +_{S_t} G_2$. By definition of the harmonic index, we have

$$\begin{aligned} H(G_1 +_{S_t} G_2) &= \sum_{u \in V(G_1)} \sum_{v_1 v_2 \in E(G_2)} \frac{2}{\deg(u, v_1) + \deg(u, v_2)} \\ &+ \sum_{v \in V(G_2)} \sum_{u_1 u_2 \in E(S_t(G_1))} \frac{2}{\deg(u_1, v) + \deg(u_2, v)} \\ &:= \sum 1 + \sum 2. \end{aligned}$$

Similar to the proof of Theorem 4.1, we can get $\sum 1 \leq \frac{1}{4} (m_2 r(G_1) + n_1 H(G_2))$. It suffices to compute the value of $\sum 2$. Note that

$$\begin{aligned} \sum 2 &= \sum_{v \in V(G_2)} \sum_{\substack{u_1 u_2 \in E(S_t(G_1)) \\ u_1 \in V(G_1), u_2 \in V(S_t(G_1)) - V(G_1)}} \frac{2}{\deg(u_1, v) + \deg(u_2, v)} \\ &+ \sum_{v \in V(G_2)} \sum_{\substack{u_1 u_2 \in E(S_t(G_1)) \\ u_1, u_2 \in V(S_t(G_1)) - V(G_1)}} \frac{2}{\deg(u_1, v) + \deg(u_2, v)} \\ &:= \sum 2' + \sum 2''. \end{aligned}$$

By similar argument in the proof of Theorem 4.1, we can show that $\sum 2' \leq \frac{1}{4} (n_2 H(S(G_1)) + 4m_1 r(G_2))$. On the other hand,

$$\begin{aligned} \sum 2'' &= \sum_{v \in V(G_2)} \sum_{\substack{u_1 u_2 \in E(S_t(G_1)) \\ u_1, u_2 \in V(S_t(G_1)) - V(G_1)}} \frac{2}{\deg(u_1, v) + \deg(u_2, v)} \\ &= \sum_{v \in V(G_2)} \sum_{\substack{u_1 u_2 \in E(S_t(G_1)) \\ u_1, u_2 \in V(S_t(G_1)) - V(G_1)}} \frac{2}{2 + 2} \\ &= \sum_{v \in V(G_2)} \left(\frac{1}{2} (t - 1) m_1 \right) = \frac{1}{2} (t - 1) n_2 m_1. \end{aligned}$$

Hence,

$$\sum 2 \leq \frac{1}{4} (n_2 H(S(G_1)) + 4m_1 r(G_2) + 2(t - 1) n_2 m_1).$$

Therefore,

$$H(G_1 +_{S_t} G_2) \leq \frac{1}{4} (n_2 H(S(G_1)) + n_1 H(G_2) + m_2 r(G_1) + 4m_1 r(G_2) + 2(t - 1) n_2 m_1).$$

One can see that equality in above inequality cannot occur. This completes the proof. □

Example 4.5. For any $n \geq 2$,

$$H(P_n +_{S_t} P_2) = \left(1 + \frac{1}{3}(n - 2)\right) + \left(2 + \frac{8}{5}(n - 2)\right) + (t - 1)(n - 1) = 3 + \frac{29}{15}(n - 2) + (t - 1)(n - 1).$$

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