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PRODUCTS OF GRAPHS AND NORDHAUS-GADDUM TYPE INEQUALITIES

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ABSTRACT. In this paper, we obtain α as coefficient for the $G = K_{\alpha n} \cup \overline{K_{(1-\alpha)n}}$ and by which we discuss Nikiforov's conjecture for λ_1 and Aouchiche and Hansen's conjecture for q_1 in Nordhaus-Gaddum type inequalities. Furthermore, by the properties of the products of graphs we put forward a new approach to find some bounds of Nordhaus-Gaddum type inequalities.

1. Introduction

Let G be a simple connected graph with n vertices and m edges. Suppose that A is the adjacency matrix of G and d_1, d_2, \dots, d_n the vertex degrees of G such that $\Delta = d_1 \geq d_2 \geq \dots \geq d_n = \delta$. We denote the complement graph of G by \overline{G} . The matrices $L(G) = D(G) - A(G)$ and $Q(G) = D(G) + A(G)$ are called the Laplacian and the signless Laplacian of G where $D(G)$ is the diagonal matrix whose diagonal entries are the vertex degrees of G . The eigenvalues of the matrices $A(G)$, $L(G)$ and $Q(G)$ are denoted by $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$, $\mu_1 \geq \mu_2 \geq \dots \geq \mu_n$, and $q_1 \geq q_2 \geq \dots \geq q_n$, respectively. Suppose that \mathbf{v} and \mathbf{w} are eigenvectors corresponding to λ_1 and q_1 and by Perron-Frobenius theorem, assume that all entries of \mathbf{v} and \mathbf{w} are positive. We call these vectors Perron-eigenvectors. Also we consider the maximum entries of \mathbf{v} and \mathbf{w} are 1. In case we have several maximum entries for Perron-eigenvector, we just choose and fix one of them. For any vertex $u \in V(G)$, we denote the entry of the eigenvector \mathbf{v} on u by $\mathbf{v}(u)$. The set of the neighbors of vertex v_i are denoted by $N(v_i)$ and $\overline{N(v_i)}$ is the set $V - N(v_i)$. Suppose G and H are two disjoint graphs, denote by $G \cup H$ and $G \nabla H$ disjoint union and join of G and H , respectively. The Cartesian product of two simple graphs G and H denoted by $G \square H$.

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A classical paper of Nordhaus and Gaddum [9] established the following inequalities for the chromatic numbers $\chi(G)$ and $\chi(\overline{G})$

$$2\sqrt{n} \leq \chi(G) + \chi(\overline{G}) \leq n + 1,$$

$$n \leq \chi(G)\chi(\overline{G}) \leq \frac{(n+1)^2}{4}.$$

Initially, Aouchiche and Hansen in the paper [1] said that, this type of relation did not attract much attention. Actually, the first detailed study of these relations came almost a decade after the publication of the paper of Nordhaus and Gaddum [9]. But later, there was broader attention to the problem of Nordhaus-Gaddum in obtaining the upper and lower bounds for sum and product parameters of a graph and its complement. We recall some of these results for parameters λ_1 and q_1 . For λ_1 and $\overline{\lambda}_1$ of graphs G and \overline{G} , Nikiforov [8] conjectured that

$$(1.1) \quad \lambda_1 + \overline{\lambda}_1 \leq \frac{4}{3}n + O(1).$$

This conjecture was proved by Terpai [12]. Some other results on $\lambda_1 + \overline{\lambda}_1$ are

$$(1.2) \quad \lambda_1 + \overline{\lambda}_1 \leq \sqrt{2n(n-1) - 4\delta(n-1-\Delta) + 1} - 1, \quad [6]$$

$$(1.3) \quad \lambda_1 + \overline{\lambda}_1 \leq \sqrt{2((n-1)^2 - 2\delta n + 2\Delta\delta - \Delta + 3\delta)}, \quad [10]$$

$$(1.4) \quad \lambda_1 + \overline{\lambda}_1 \leq \frac{n - \Delta + \delta - 3 + \sqrt{2((n-\Delta)^2 + 4n(\Delta-\delta)^2(\delta+1))}}{2}. \quad [10]$$

There was another conjecture for q_1 and \overline{q}_1 ,

$$(1.5) \quad q_1 + \overline{q}_1 \leq 3n - 4. \quad [1]$$

This conjecture was proved by Ashraf and Tayfeh-Rezaie [2]. Other conjecture by Aouchiche and Hansen was

$$(1.6) \quad q_1\overline{q}_1 \leq 2n(n-2), \quad [1]$$

that disproved by Ashraf and Tayfeh-Rezaie [2].

In this paper, we obtain α as coefficient for graph $G = K_{\alpha n} \cup \overline{K_{(1-\alpha)n}}$ and discuss obtained bound by Nikiforov [8] for $\lambda_1 + \overline{\lambda}_1$, also conjectured bound by Aouchiche and Hansen in [1] for $q_1 + \overline{q}_1$. In section 2, we use the structure of graph products to provide a theory to obtain upper bounds for Nordhaus-Gaddum type inequalities for sum or product of the greatest eigenvalues of the adjacency matrix and Laplacian and singless Laplacian matrices of the graphs G and \overline{G} .

2. Extremal Graphs Of Nordhaus- Gaddum Bounds

In this section we provide a value of α as the coefficient of the number of vertices of $G = K_{\alpha n} \cup \overline{K_{(1-\alpha)n}}$ and then occurrence of equality of the two bounds that can be seen in the following.

Conjecture 2.1. [8] *For any graph G on n vertices,*

$$(2.1) \quad \lambda_1 + \overline{\lambda}_1 \leq \frac{4}{3}n + O(1).$$

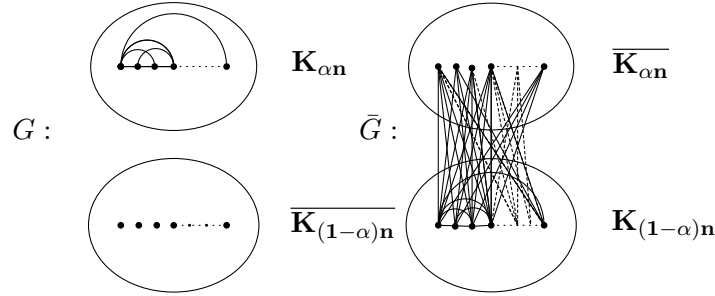


FIGURE 1. $G = K_{\alpha n} \cup \overline{K_{(1-\alpha)n}}$ and $\bar{G} = \overline{K_{\alpha n}} \nabla K_{(1-\alpha)n}$

Conjecture 2.2. [1] *Let G be a simple graph on $n \geq 2$ vertices. Then,*

$$(2.2) \quad q_1 + \bar{q}_1 \leq 3n - 4.$$

For $0 \leq \alpha \leq 1$, suppose that $G = K_{\alpha n} \cup \overline{K_{(1-\alpha)n}}$ and $\bar{G} = \overline{K_{\alpha n}} \nabla K_{(1-\alpha)n}$ that are displayed in Figure 1. To obtain eigenvalues $\bar{\lambda}_1$ and \bar{q}_1 of \bar{G} we use the following theorems.

Theorem 2.3. [11] *Let G_i be an r_i -regular graph on n_i vertices for $i = 1, 2$. If $P(A(G_i), \lambda)$ is the characteristic polynomial of G_i and $i = 1, 2$, then*

$$(2.3) \quad P(A(G_1 \nabla G_2), \lambda) = \frac{P(A(G_1), \lambda)P(A(G_2), \lambda)}{(\lambda - r_1)(\lambda - r_2)} [(\lambda - r_1)(\lambda - r_2) - n_1 n_2].$$

Theorem 2.4. [5] *Let G_i be an r_i -regular graph on n_i vertices for $i = 1, 2$. If $\phi_{G_i}(q)$ is the signless Laplacian characteristic polynomial of G_i and $i = 1, 2$, then*

$$(2.4) \quad \phi(Q(G_1 \nabla G_2), q) = \frac{\phi(Q(G_1), q - n_2)\phi(Q(G_2), q - n_1)}{(q - 2r_1 - n_2)(q - 2r_2 - n_1)} f(q)$$

where $f(q) = q^2 - (2(r_1 + r_2) + (n_1 + n_2))q + 2(2r_1 r_2 + r_1 n_1 + r_2 n_2)$.

First, we consider $\lambda_1 + \bar{\lambda}_1$ for the graphs $G = K_{\alpha n} \cup \overline{K_{(1-\alpha)n}}$ and $\bar{G} = \overline{K_{\alpha n}} \nabla K_{(1-\alpha)n}$,

$$(2.5) \quad (\lambda_1 + \bar{\lambda}_1)(\alpha) = \alpha n - 1 + \frac{1}{2}(n - \alpha n - 1 + \sqrt{n^2 - 3\alpha^2 n^2 + 2\alpha n^2 + 2\alpha n - 2n}).$$

For the maximum value of $(\lambda + \bar{\lambda}_1)(\alpha)$ by using derivative we have,

$$(2.6) \quad \alpha_{max} = \frac{n + 1 + \sqrt{n^2 - n + 1}}{3n}.$$

By substituting α_{max} in relation (2.5), we have

$$(2.7) \quad (\lambda_1 + \bar{\lambda}_1)(\alpha) \leq \frac{4n - 5 + 4\sqrt{n^2 - n + 1}}{6} < \frac{4}{3}n - 1.$$

The obtained α in relation (2.6) tends to $\frac{2}{3}$, when n tends to infinity and relation (2.7) tends to $\frac{4}{3}n$. Now, we conjecture the following:

Conjecture 2.5. *Let G be a simple graph on n vertices. Then*

$$(2.8) \quad \lambda_1 + \bar{\lambda}_1 \leq \frac{4n - 5 + 4\sqrt{n^2 - n + 1}}{6}.$$

Now, we consider $q_1 + \bar{q}_1$ for the graphs $G = K_{\alpha n} \cup \overline{K_{(1-\alpha)n}}$ and $\bar{G} = \overline{K_{\alpha n}} \nabla K_{(1-\alpha)n}$.

$$(2.9) \quad (q_1 + \bar{q}_1)(\alpha) = 2(\alpha n - 1) + \frac{1}{2}(3n - 2\alpha n - 2 + \sqrt{n^2 - 4\alpha^2 n^2 + 4\alpha n^2 - 4n + 4}).$$

As is mentioned above, the maximum value of $(q_1 + \bar{q}_1)(\alpha)$ by using its derivative, is at

$$(2.10) \quad \alpha_{max} = \frac{1 + \sqrt{1 - \frac{2(n-1)}{n^2}}}{2}.$$

By substituting α_{max} in relation (2.9), we have

$$(2.11) \quad (q_1 + \bar{q}_1)(\alpha_{max}) = \sqrt{n^2 - 2n + 2} + 2n - 3.$$

Because $n - 1 < n\alpha_{max} < n$ and $1 \leq n\alpha \leq n - 1$, the maximum value of $q_1 + \bar{q}_1$ for simple graphs is for $n\alpha = n - 1$ and we have

$$(2.12) \quad q_1 + \bar{q}_1 \leq (q_1 + \bar{q}_1)_{(\alpha=\frac{n-1}{n})} = 3n - 4 < \sqrt{n^2 - 2n + 2} + 2n - 3.$$

3. Upper Bound For Problems Of Nordhaus-Gaddum Type By Using Product Of Graph

In the following, some definitions of the products of graphs are recalled :

Definition 3.1. [3] *Direct product of two simple graphs G and H denoted by $G \times H$ has the vertex-set $V(G) \times V(H)$. For any $u, v \in V(G)$ and $x, y \in V(H)$, (u, x) is adjacent to (v, y) if $uv \in E(G)$ and $xy \in E(H)$. The adjacency matrix of $G \times H$ is the direct product of the adjacency matrices of G and H .*

If α and β are eigenvectors for G and H with eigenvalues $\lambda_i (1 \leq i \leq m)$ and $\lambda'_j (1 \leq j \leq n)$, respectively, then the vector $w = \alpha \otimes \beta$ is an eigenvector of $G \times H$ with eigenvalues $\lambda_i \lambda'_j$.

Remark 3.2. *The direct product is also called, the tensor product categorical product, cardinal product, relational product, Kronecker product, weak direct product, or conjunction. The notation $G \times H$ is also sometimes used to represent another construction known as the Cartesian product of graphs, but more commonly refers to the direct product.*

The tensor product of matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ of orders $m \times p$ and $n \times q$, respectively, is the matrix $A \otimes B$ of order $mn \times pq$ defined by

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1p}B \\ \vdots & \vdots & \vdots \\ a_{m1}B & \cdots & a_{mp}B \end{bmatrix}$$

Definition 3.3. [3] *The Cartesian product of two simple graphs G and H denoted by $G \square H$ has the vertex-set $V(G) \times V(H)$. For any $u, v \in V(G)$ and $x, y \in V(H)$, (u, x) is adjacent to (v, y) if $u = v$ and $xy \in E(H)$ or $uv \in E(G)$ and $x = y$. Let A and B be adjacency matrices of graphs G and H of orders m and n , respectively. Its adjacency is $A \otimes I_n + I_m \otimes B$.*

If v and u are eigenvectors for G and H with eigenvalues $\lambda_i (1 \leq i \leq m)$ and $\lambda'_j (1 \leq j \leq n)$, respectively, then the vector $w = \alpha \otimes \beta$ is an eigenvector of $G \square H$ with eigenvalue $\lambda_i + \lambda'_j$.

Definition 3.4. [3] *The strong product of two simple graphs G and H denoted by $G \boxtimes H$ has the vertex-set $V(G) \times V(H)$. For any $u, v \in V(G)$ and $x, y \in V(H)$, (u, x) is adjacent to (v, y) if $u = v$ and $xy \in E(H)$ or $uv \in E(G)$ and $x = y$ or. If A and B are the adjacency matrices of G and H of orders m and n then, $A \otimes B + A \otimes I_n + I_m \otimes B$ is the adjacency matrix of $G \boxtimes H$. It follows that the eigenvalues of $G \boxtimes H$ are the numbers $(\lambda_i + 1)(\lambda'_j + 1) - 1$, where λ_i and λ'_j the eigenvalues of G and H , respectively.*

Theorem 3.5. [5] *Let the Laplacian eigenvalues of graphs G and H are μ_i and μ_j . The Laplacian eigenvalues of the Cartesian product $G \square H$ of graphs G and H are equal to all the possible sums of eigenvalues of the two factors:*

$$\mu_i + \mu_j \quad 1 \leq i \leq |V_G| \quad \text{and} \quad 1 \leq j \leq |V(H)|$$

Remark 3.6. *Suppose that G and H are two arbitrary graphs. We have for the matrices $A(G)$ and $A(H)$*

$$\begin{aligned} Q(G \square H) &= D(G \square H) + A(G \square H) = (D(G) \otimes I + I \otimes D(H)) + (A(G) \otimes I + I \otimes A(H)) \\ &= (D(G) + A(G)) \otimes I + I \otimes (D(H) + A(H)) = Q(G) \otimes I + I \otimes Q(H). \end{aligned}$$

So we can conclude similar to Theorem 3.5 for singless Laplacian eigenvalues.

Using the properties of Cartesian product of adjacency, Laplacian, and singless Laplacian matrices in [3], [5], and Remark 3.6, we have

$$\lambda_1 + \bar{\lambda}_1 = \lambda_1(G \square \bar{G}).$$

$$\mu + \bar{\mu}_1 = \mu_1(G \square \bar{G}).$$

$$q_1 + \bar{q}_1 = q_1(G \square \bar{G}).$$

In conclusion, by using the properties and structure of Cartesian product, we can obtain upper bounds for Nordhaus-Gaddum type inequalities of sum of eigenvalues. In fact, it is hoped that, by using the properties of product of graphs, we can obtain better bounds of Nordhaus-Gaddum type inequalities by easier and better methods.

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REFERENCES

- [1] M. Aouchiche and P. Hansen, A survey of Nordhaus-Gaddum type relations, *Discrete. Appl. Math.*, **161** (2013) 466–546.
- [2] F. Ashraf and B. Tayfeh-Rezaie, Nordhaus-Gaddum type inequalities for Laplacian and signless Laplacian eigenvalues, *Electron. J. Combin.*, **21** (2014).
- [3] E. A. Brouwer and W. H. Haemers, *Spectra of graphs*, Springer Science & Business Media, 2011.

- [4] D. M. Cvetković, M. Doob and H. Sachs, *Spectra of graphs theory and application*, **87**, New York: Academic Press, 1980.
- [5] M. Fiedler, Algebraic connectivity of graphs, *Czechoslovak mathematical journal*, **23** (1973) 298–305.
- [6] X. Li, The relations between the spectral radius of the graphs and their complement, *J. North China Technol. Inst.*, **17** (1996) 297–299.
- [7] X. Liu and P. Lu, Signless Laplacian spectral characterization of some joins, *Elect. J. Linear Algebra*, **30** (2015) 443–454.
- [8] V. Nikiforov, Eigenvalue problems of Nordhaus-Gaddum type, *Discrete Math.*, **307** (2007) 774–780.
- [9] E. A. Nordhaus and J. Gaddum, On complementary graphs, *Amer. Math. Monthly*, **63** (1956) 175–177.
- [10] L. Shi, Bounds on the (Laplacian) spectral radius of graphs, *Linear Algebra Appl.*, **422** (2007) 755–770.
- [11] H. Sayamai, Estimation of Laplacian spectra of direct and strong product graphs, *Discrete Appl. Math.*, **205** (2016) 160–170.
- [12] T. Terpai, Proof of a conjecture of V. Nikiforov, *Combinatorica*, **31** (2011) 739–754.

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