Transactions on Combinatorics
ISSN (print): 2251-8657, ISSN (on-line): 2251-8665
Vol. x No. x (201x), pp. xx-xx.

www.combinatorics.ir

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THE LOG-CONVEXITY OF THE FUBINI NUMBERS

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Communicated by Saeid Azam

Abstract. Let f_n denote the nth Fubini number. In this paper, first we give upper and lower bounds for the Fubini numbers f_n. Then the log-convexity of the Fubini numbers has been obtained. Furthermore we also give the monotonicity of the sequence $\{\sqrt[n]{f_n}\}_{n\geq 1}$ by using the aforementioned bounds.

1. Introduction

The nth Fubini number (also called ordered Bell number, or geometric number [1], or surjection number [2]), which is denoted by f_n for $n \geq 0$, counts all the possible ordered partitions of a set with n elements. The Fubini numbers are also the number of different ways to arrange the ordering of sums and integrals in Fubini's theorem [3]. The first several terms of the Fubini numbers are

$$f_0 = 1, \;\; f_1 = 1, \;\; f_2 = 3, \;\; f_3 = 13, \;\; f_4 = 75, \;\; f_5 = 541, \;\; f_6 = 4683, \;\; f_7 = 47293.$$

The Fubini numbers can be given by the following exponential generating function,

$$\sum_{n=0}^{\infty} f_n \frac{x^n}{n!} = \frac{1}{2-e^x}.$$

The nth Fubini number may also be given by a summation formula involving the Stirling numbers of the second kind $S(n,k)$, which count the number of partitions of an n elements set into k nonempty subsets,

$$f_n = \sum_{k=1}^{n} k! S(n,k).$$

MSC(2010): Primary: 05A20; Secondary: 05A10, 11B83.

Keywords: Fubini number, log-convexity, monotonicity.

Received: 19 May 2017, Accepted: 25 August 2017.

One of the most beautiful properties that holds for Fubini numbers is the modular periodicity. For example, for large n,

$$f_{n+4} \equiv f_n \pmod{10},$$

$$f_{n+20} \equiv f_n \pmod{100},$$

$$f_{n+100} \equiv f_n \pmod{1000},$$

$$f_{n+500} \equiv f_n \pmod{10000}.$$

Fubini numbers can also be characterized through the following recurrence relation [4] which is a consequence of the exponential generating function for the Fubini numbers,

$$f_n = \sum_{j=0}^{n-1} \binom{n}{j} f_j. \tag{1.1}$$

Actually, this recursive formula is equivalent to the following recurrence relation obtained in [5, 6],

$$f_n = \sum_{j=1}^{n} \binom{n}{j} f_{n-j}.$$

In [4], Dil and Kurt stated that

$$f_n = -2 \sum_{j=0}^{n-1} \binom{n}{j} (-1)^{n-j} f_j.$$

This formula shows that all Fubini numbers, not including f_0 are even. However, after looking at the first several terms of Fubini numbers, one can easily find that the identity above is not correct. In fact, the correct version should be

$$2 \sum_{j=0}^{n} \binom{n}{j} (-1)^j f_j = (-1)^n f_n + 1,$$

which indicates that all the Fubini numbers must be odd.

In this paper, we will mainly focus on the log-behavior of Fubini numbers.

A sequence $\{a_n\}_{n\geq 0}$ of positive numbers is said to be log-convex (resp. log-concave) if for all $n \geq 1$,

$$a_n^2 \leq a_{n-1}a_{n+1} \quad (\text{resp. } a_n^2 \geq a_{n-1}a_{n+1}),$$

and it is said to be strictly log-convex (resp. strictly log-concave) if the above inequalities is strict. Log-convexity (or log-concavity) is an important property of combinatorial sequences. They are fertile sources of inequalities. In [7], Sun raised many conjectures on the log-behavior and monotonicity of combinatorial sequences of positive integers. After Sun raised those conjectures, the log-behavior of some combinatorial sequences have been studied in some literatures (see for example [8, 9, 10, 11]). It seems that the log-behavior of $\{f_n\}_{n\geq 0}$ has not been investigated. In this paper, we will discuss the log-convexity of $\{f_n\}_{n\geq 0}$.

2. Bounds for Fubini numbers

Before showing the log-convexity of Fubini numbers, we would like to discuss on upper and lower bounds of Fubini numbers first.

First of all, let us give an upper bound for Fubini numbers.

Theorem 2.1. *For $n \geq 1$, $f_n < (n+1)^n$.*

Proof. In [12], Barthelemy proved that

$$f_n = \frac{n!}{2(\log 2)^{n+1}} + o((n-1)!).$$

If we follow the idea of Barthelemy and use the fact that $2(\log 2)^{n+1} < 1$ for $n \geq 1$, we can derive that when $n \geq 1$,

$$f_n < \frac{n!}{2(\log 2)^{n+1}} + n! < \frac{n!}{2(\log 2)^{n+1}} + \frac{n!}{2(\log 2)^{n+1}} = \frac{n!}{(\log 2)^{n+1}}.$$

Next, we use induction to show that for $n \geq 1$,

$$\frac{n!}{(\log 2)^{n+1}} < (n+1)^n. \tag{2.1}$$

The base case $n = 1$ is straightforward. Now, suppose (2.1) holds for $n-1$, i.e.,

$$\frac{(n-1)!}{(\log 2)^n} < n^{n-1}.$$

Then

$$\begin{aligned}\frac{n!}{(\log 2)^{n+1}} &= \frac{n}{\log 2} \cdot \frac{(n-1)!}{(\log 2)^n} \\ &< \frac{n}{\log 2} \cdot n^{n-1} = \frac{1}{\log 2} \cdot n^n \\ &< \frac{3}{2} n^n < (n+1)^n.\end{aligned}$$

The last inequalities is due to the easily checked fact that for $n \geq 1$,

$$\frac{2}{3}(1 + \frac{1}{n})^n > 1.$$

Thus, we get that for $n \geq 1$,

$$f_n < \frac{n!}{(\log 2)^{n+1}} < (n+1)^n.$$

This completes the proof. □

Remark 2.2. *From the process of the proof, one can find that $\frac{n!}{(\log 2)^{n+1}}$ is a better upper bound for f_n. While the upper bound $(n+1)^n$ is more useful in the proof of the monotonicity of the sequence $\{\sqrt[n]{f_n}\}_{n \geq 1}$.*

Remark 2.3. *If we combine this upper bound with Lemma 3.1 below, we can get the by-product that*

$$\sum_{k=0}^{\infty} \frac{k^n}{2^k} < \frac{2n!}{(\log 2)^{n+1}}, \quad \sum_{k=0}^{\infty} \frac{k^n}{2^k} < 2(n+1)^n.$$

Motivated by this upper bound, we are going to find the lower bound. Here is the lower bound we found.

Theorem 2.4. *For $n \geq 3$, $f_n > 2^n$.*

Proof. Let us prove it by induction.

For $n = 3$, $f_n = 13 > 2^3 = 8$. So, the base case holds true. We suppose the conclusion holds for $n-1$. That is

$$f_{n-1} = \sum_{j=0}^{n-2} \binom{n-1}{j} f_j > 2^{n-1}.$$

Next, we show the conclusion also holds for n. Pascal's recurrence [13] reads as

$$\binom{n}{j} = \binom{n-1}{j} + \binom{n-1}{j-1}.$$

Then

$$\begin{aligned} f_n &= \sum_{j=0}^{n-1} \binom{n}{j} f_j = \sum_{j=0}^{n-1} \binom{n-1}{j} f_j + \sum_{j=1}^{n-1} \binom{n-1}{j-1} f_j \\ &= \sum_{j=0}^{n-2} \binom{n-1}{j} f_j + \binom{n-1}{n-1} f_{n-1} + \sum_{j=1}^{n-1} \binom{n-1}{j-1} f_j \\ &> 2^{n-1} + 2^{n-1} + \sum_{j=1}^{n-1} \binom{n-1}{j-1} f_j \\ &> 2^{n-1} + 2^{n-1} = 2^n. \end{aligned}$$

So, the conclusion also holds for n. Hence we get that for $n \geq 3$, $f_n > 2^n$. □

3. The log-convexity of the Fubini numbers

The goal of section is to show the log-convexity of the Fubini numbers and the monotonicity of the sequence $\{\sqrt[n]{f_n}\}_{n\geq 1}$.

Before giving the log-convexity of the Fubini numbers, two lemmas need to be introduced.

Lemma 3.1. *For $n \geq 1$, we have*

$$f_n = \frac{1}{2} \sum_{k=1}^{\infty} \frac{k^n}{2^k}.$$

Proof. This lemma follows from the infinite series representation of the Fubini numbers obtained in [5, 14],

$$f_n = \frac{1}{2} \sum_{k=0}^{\infty} \frac{k^n}{2^k}, \quad n \geq 0.$$

Our lemma exclude the case $n = 0$ since $n = 0$ is a very special case that cannot be formulated by the form given in the lemma. □

Lemma 3.2. *Let α be any nonzero constant. Let $\{a_k\}_{k\geq 1}$ and $\{b_k\}_{k\geq 1}$ be two positive sequences and*

$$a_i \neq a_j \text{ for } i \neq j.$$

Define $\{c_n\}_{n\geq 0}$ as

$$c_n = \alpha \sum_{k=1}^{\infty} \frac{a_k^n}{b_k}.$$

Then $\{c_n\}_{n\geq 0}$ is strictly log-convex.

Proof. By the definition of c_n, we have

$$\begin{aligned}
&c_{n+1}c_{n-1} - c_n^2 \\
=&\alpha \sum_{k=1}^{\infty} \frac{a_k^{n+1}}{b_k} \cdot \alpha \sum_{k=1}^{\infty} \frac{a_k^{n-1}}{b_k} - \alpha \sum_{k=1}^{\infty} \frac{a_k^n}{b_k} \cdot \alpha \sum_{k=1}^{\infty} \frac{a_k^n}{b_k} \\
=&\alpha^2 \sum_{j>i\geq 1} \frac{\frac{1}{b_i}\frac{1}{b_j}\left(\frac{1}{a_i^2} - \frac{1}{a_j^2} - 2\frac{1}{a_i a_j}\right)}{\left(\frac{1}{a_i}\right)^{n+1}\left(\frac{1}{a_j}\right)^{n+1}} \\
=&\alpha^2 \sum_{j>i\geq 1} \frac{a_i^{n+1}a_j^{n+1}\left(\frac{1}{a_i} - \frac{1}{a_j}\right)^2}{b_i b_j} \\
>&0.
\end{aligned}$$

The last inequality holds since $a_i \neq a_j$ when $i \neq j$ and $\{a_k\}_{k\geq 1}$ and $\{b_k\}_{k\geq 1}$ are two positive sequences. Thus we get that $c_{n+1}c_{n-1} > c_n^2$, which implies that $\{c_n\}_{n\geq 0}$ is strictly log-convex. □

With these two lemmas in hand, we can prove the following conclusion.

Theorem 3.3. *The sequence $\{f_n\}_{n\geq 0}$ is strictly log-convex.*

Proof. Let

$$\alpha = \frac{1}{2}, \quad a_k = k, \quad b_k = 2^k$$

in Lemma 3.2, we can get that $\{f_n\}_{n\geq 1}$ is strictly log-convex.

Furthermore, since $f_0 = 1$, then we can check that

$$1 = f_1^2 < f_0 \cdot f_2 = 3.$$

Hence, $\{f_n\}_{n\geq 0}$ is strictly log-convex. □

Along with the usual research routine, after investigating the log-convexity of a combinatorial sequences, the next thing is to research the monotonicity of the sequence $\{\sqrt[n]{f_n}\}_{n\geq 1}$. For this point, we have the following conclusion.

Theorem 3.4. *The sequence $\{\sqrt[n]{f_n}\}_{n\geq 1}$ is strictly increasing.*

Proof. In order to prove that $\{\sqrt[n]{f_n}\}_{n\geq 1}$ is strictly increasing, it suffices to show that

$$\frac{\sqrt[n+1]{f_{n+1}}}{\sqrt[n]{f_n}} > 1,$$

which is equivalent to show

$$\frac{f_{n+1}}{f_n \cdot \sqrt[n]{f_n}} > 1.$$

Since

$$\begin{aligned}\frac{f_{n+1}}{f_n \cdot \sqrt[n]{f_n}} &= \frac{\sum_{j=0}^{n}\binom{n+1}{j} f_j}{f_n \cdot \sqrt[n]{f_n}} = \frac{(n+1)f_n + \sum_{j=0}^{n-1}\binom{n+1}{j} f_j}{f_n \cdot \sqrt[n]{f_n}} \\ &= \frac{n+1}{\sqrt[n]{f_n}} + \frac{\sum_{j=0}^{n-1}\binom{n+1}{j} f_j}{f_n \cdot \sqrt[n]{f_n}} \\ &> \frac{n+1}{\sqrt[n]{f_n}} > 1.\end{aligned}$$

The last inequality is due to Theorem 2.1. So, $\{\sqrt[n]{f_n}\}_{n\geq 1}$ is strictly increasing.

This completes the proof. □

4. **Conclusion**

In this paper, we first obtained bounds for the Fubini numbers. With the bounds, we showed that the sequence $\{\sqrt[n]{f_n}\}_{n\geq 1}$ is strictly increasing. We also proved that a sequence of positive integers of a given form must be log-convex, see Lemma 3.2. By which we also showed that $\{f_n\}_{n\geq 0}$ is strictly log-convex.

Acknowledgments

The author expresses the gratitude to the referees for a careful reading of the manuscript and many valuable suggestions.

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