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## COMBINATORIAL PARAMETERS ON BARGRAPHS OF PERMUTATIONS

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**ABSTRACT.** In this paper, we consider statistics on permutations of length  $n$  represented geometrically as bargraphs having the same number of horizontal steps. More precisely, we find the joint distribution of the descent and up step statistics on the bargraph representations, thereby obtaining a new refined count of permutations of a given length. To do so, we consider the distribution of the parameters on permutations of a more general multiset of which  $\mathcal{S}_n$  is a subset. In addition to finding an explicit formula for the joint distribution on this multiset, we provide counts for the total number of descents and up steps of all its members, supplying both algebraic and combinatorial proofs. Finally, we derive explicit expressions for the sign balance of these statistics, from which the comparable results on permutations follow as special cases.

### 1. Introduction

Bargraphs, also referred to as wall polyominoes [5] or skylines [6], have been an object of recent study and have found several refined enumerations. Recall that a *bargraph* is a self-avoiding random walk in the first quadrant starting from the origin and terminating on the  $x$ -axis and lying strictly above the  $x$ -axis except for the endpoints, where the permissible steps are the up step  $u = (0, 1)$ , down step  $d = (0, -1)$ , and horizontal step  $h = (1, 0)$ . Note that no  $u$  can follow a  $d$ , and vice-versa, and each  $h$  must lie (strictly) above the  $x$ -axis. Earlier authors, such as Prellberg and Brak [11] and Feretić [5], have found generating function formulas involving two variables  $x$  and  $y$  that keep track of the number of horizontal and up steps, respectively. Refined counts of bargraphs were later given by Blecher et al. according to such statistics as levels [1], peaks [2], area [3], and descents [3]. By viewing bargraphs as Motzkin paths without peaks or valleys, Deutsch and Elizalde [4] explored

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further lattice-path type statistics and considered restricted classes of bargraphs such as those that are alternating or symmetric. Bargraphs have connections to statistical physics [10], where they are used to model polymers, and arise in probability theory where they represent frequency diagrams.

Here, we consider a further restricted class of bargraphs, i.e., those all of whose heights are distinct. That is, we view the set of permutations as a subset of the bargraphs and find enumerations related to certain statistics on the latter. More precisely, we count this class of bargraphs according to the descent and up step statistics (marked by  $p$  and  $q$  in what follows) and find a formula for the joint distribution. This extends to permutations recent work on unrestricted bargraphs and integer compositions [3] where some comparable parameters were considered. As a consequence, one gets new refined counts of the set of permutations of a given length. To obtain the featured results on bargraphs of permutations, we consider first a more general class of words which enables one to start the problem by writing a recurrence. We are then able to find the generating function for the distribution of descents and up steps on this more general class which allows one to obtain the comparable results for permutations as special cases. We refer the reader to [7] and references contained therein for previously considered statistics on words and compositions.

This paper is organized as follows. In the next section, we introduce and recall some terminology and identify a set, denoted by  $\mathcal{A}_{n,i}$ , of permutations of a certain multiset in terms of bargraphs (with  $\mathcal{S}_n$  corresponding to the case  $i = 1$ ). In the third section, we find the joint distribution of the descent and up step statistics on  $\mathcal{A}_{n,i}$ . To do so, it is more convenient to consider first the distribution of the number of descents with the statistic recording the difference between area and number of up steps. We also find explicit formulas for the sum of the values of a parameter taken over all members of  $\mathcal{A}_{n,i}$  in the case of descents, number of up steps and the difference between area and up steps, providing both algebraic and combinatorial proofs. In the final section, we derive formulas for the extent to which the members of  $\mathcal{A}_{n,i}$  assuming an even value for one of these parameters outnumber (or are outnumbered by) those members having an odd value.

## 2. Preliminaries

We will study the distributions of certain parameters on various classes of words represented as bargraphs. Let  $\mathcal{P}_n$  denote the set of all words of length  $n$  in the alphabet of positive integers and  $\mathcal{B}_n$  the set of bargraphs having  $n$  horizontal steps (i.e., that terminate at the point  $(n, 0)$ ). We will represent  $w = w_1 w_2 \cdots w_n \in \mathcal{P}_n$  as the member of  $\mathcal{B}_n$  whose  $i$ -th horizontal step has height  $w_i$  for all  $i \in [n] = \{1, 2, \dots, n\}$ , which we denote by  $\text{bar}_w$ .

We focus on the case of permutations of a given length, represented using the one-line notation. For the statistics considered here, finding a recurrence for the distribution on  $\mathcal{S}_n$  directly in terms of distributions on  $\mathcal{S}_m$  for smaller  $m$  with no additional parameters seems difficult. In order to facilitate our study, we consider a more general set of words than the permutations which allows one to view

these statistics in terms of recursive structures. Given  $n \geq 1$  and  $1 \leq i \leq \lfloor (n+1)/2 \rfloor$ , let  $\mathcal{A}_{n,i}$  denote the set of permutations of the multiset  $12 \cdots (n-i)(n-i+1)^i$  in which no two of the letters  $n-i+1$  are adjacent. For example, we have  $\mathcal{A}_{4,2} = \{1323, 2313, 3123, 3132, 3213, 3231\}$ . Note that  $\mathcal{A}_{n,i}$  coincides with the usual set of permutations of  $[n]$  when  $i = 1$ .

We wish to consider various statistics on  $\mathcal{A}_{n,i}$ . A *descent* in a word  $w = w_1w_2 \cdots w_n \in \mathcal{P}_n$  is an index  $i \in [n-1]$  such that  $w_i > w_{i+1}$ . Let  $des(w)$  denote respectively the number of descents in  $w$ , which is equivalently the number of runs of down steps in  $\text{bar}_w$ . Let  $up(w)$  and  $area(w)$  denote the number of up steps in and the first quadrant area subtended by  $\text{bar}_w$ . Note that  $area(w)$  is simply the sum of the entries of  $w$ . For example, if  $\pi = 4785138286 \in \mathcal{A}_{10,3}$  (see Figure 1), then  $des(\pi) = 4$ ,  $up(\pi) = 21$  and  $area(\pi) = 52$ . By a slight abuse of notation, we will at times regard members  $\pi \in \mathcal{A}_{n,i}$  as belonging to the subset of  $\mathcal{B}_n$  comprising  $\text{bar}_\pi$  for all  $\pi$ . In the subsequent section, we consider the joint distribution of the descent and up step statistics on  $\mathcal{A}_{n,i}$  represented in terms of bargraphs.

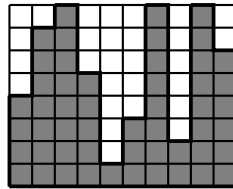


FIGURE 1. Bargraph representation of  $\pi = 4785138286 \in \mathcal{A}_{10,3}$ .

That is, we find the generating function

$$d(n, i; p, q) = \sum_{\pi \in \mathcal{A}_{n,i}} p^{des(\pi)} q^{up(\pi)}, \quad 1 \leq i \leq \lfloor (n+1)/2 \rfloor.$$

In order to do so, it is more convenient to study the joint distribution of descents with the statistic recording area minus number of up steps. Given  $w \in \mathcal{P}_n$ , let  $dif(w) = area(w) - up(w)$ , which is always seen to be non-negative. Define the distribution polynomial  $a(n, i; p, q)$  by

$$a(n, i; p, q) = \sum_{\pi \in \mathcal{A}_{n,i}} p^{des(\pi)} q^{dif(\pi)}, \quad 1 \leq i \leq \lfloor (n+1)/2 \rfloor.$$

For example, if  $n = 4$  and  $i = 2$ , then  $a(4, 2; p, q) = pq^4(1+p)(2+q)$ . Since  $area(\pi) = \binom{n-i+1}{2} + i(n-i+1)$  for all  $\pi \in \mathcal{A}_{n,i}$ , one can recover  $d(n, i; p, q)$  from  $a(n, i; p, q)$  via the relation  $d(n, i; p, q) = q^{\binom{n-i+1}{2} + i(n-i+1)} a(n, i; p, q^{-1})$ . Taking  $i = 1$  in either gives the corresponding distribution on  $\mathcal{S}_n$ .

We remark that the  $dif$ -statistic may be of some interest in its own right. Since the number of up steps always equals the number of down steps within  $\text{bar}_w$  for a word  $w$ , the quantity  $dif(w)$  is seen geometrically to count the number of squares in  $\text{bar}_w$  that are bordered on the right by another square. Thus  $dif(w)$  corresponds to the sum of the lengths of the internal vertical dividing lines within the bargraph representation of  $w$ . For example,  $dif(\pi) = 31$  where  $\pi$  is shown below. Furthermore,

$dif(w)$  may be given directly in terms of  $w = w_1 w_2 \cdots w_n$  as  $dif(w) = \sum_{i=1}^{n-1} \min\{w_i, w_{i+1}\}$ , which is related to certain kinds of variation statistics for words [9].

Using the definitions, one can obtain the boundary conditions  $a(2n-1, n; p, q) = p^{n-1} q^{n(n-1)} (n-1)!$  and  $a(2n, n; p, q) = \frac{p^{n-1} q^{n^2} (1+p)(n-1)!}{(1-q)^2} (n(1-q) - q(1-q^n))$  for all  $n \geq 1$ . Note that the forbidden adjacency requirement of the  $n-i+1$  letters implies  $\mathcal{A}_{n,i}$  is empty if  $i > \lfloor (n+1)/2 \rfloor$ , in which case  $a(n, i; p, q)$  will be taken to be zero.

### 3. Bargraph statistics on permutations

We first determine a recurrence satisfied by the array  $a(n, i; p, q)$ .

**Lemma 3.1.** *If  $n \geq 2$  and  $1 \leq i \leq \lfloor (n+1)/2 \rfloor$ , then*

$$(3.1) \quad a(n, i; p, q) = (i+1)a(n, i+1; p, q) + (1+p)q^{n-i}ia(n-1, i; p, q) + pq^{2(n-i)}(i-1)a(n-2, i-1; p, q),$$

with  $a(1, 1; p, q) = 1$ .

*Proof.* We first count members of  $\mathcal{A}_{n,i}$  in which  $n-i$  is not adjacent to any of the  $n-i+1$ . Let  $\mathcal{A}_{m,j}^*$  denote the set of “marked” members of  $\mathcal{A}_{m,j}$  wherein one the letters  $m-j+1$  is marked. Given  $\lambda \in \mathcal{A}_{n,i+1}^*$ , let  $\lambda'$  be obtained by replacing each  $n-i$  within  $\lambda$  that is not marked with  $n-i+1$ . Then the mapping  $\lambda \mapsto \lambda'$  is a weight-preserving bijection between  $\mathcal{A}_{n,i+1}^*$  and members of  $\mathcal{A}_{n,i}$  in which  $n-i$  is not adjacent to any of the  $n-i+1$ , whence their weight is  $(i+1)a(n, i+1; p, q)$ .

Next, observe that members of the subset of  $\mathcal{A}_{n,i}$  in which  $n-i$  is adjacent to exactly one of the  $n-i+1$  may be obtained by inserting an extra copy of the letter  $n-i$  either directly before or after one of the  $n-i$  within a member of  $\mathcal{A}_{n-1,i}$ , and then replacing all of the letters  $n-i$  with  $n-i+1$  except for the one that was inserted. Note that this operation increases the value of  $dif$  by  $n-i$  due to the contribution from the inserted letter  $n-i$ . Furthermore, there are  $2i$  possible choices for the insertion point of  $n-i$ , each resulting in a distinct member of  $\mathcal{A}_{n,i}$ , where a descent is introduced with exactly half of these choices. Thus, the weight of the aforementioned subset of  $\mathcal{A}_{n,i}$  is given by  $(1+p)q^{n-i}ia(n-1, i; p, q)$ .

Finally, members of  $\mathcal{A}_{n,i}$  in which the string  $(n-i+1)(n-i)(n-i+1)$  occurs may be obtained by inserting the letters  $n-i, n-i+1$  directly after the marked letter  $n-i$  within a member of  $\mathcal{A}_{n-2,i-1}^*$  and then in the resulting word replacing all letters  $n-i$  by  $n-i+1$  except for the one that was added (and also leaving the  $n-i+1$  that was inserted unchanged). This operation is seen to define a bijection between  $\mathcal{A}_{n-2,i-1}^*$  and the subset of  $\mathcal{A}_{n,i}$  in which  $n-i$  comes directly between two of the  $n-i+1$ , where the weight changes by a factor of  $pq^{2(n-i)}$ . Thus, the weight of the aforementioned subset of  $\mathcal{A}_{n,i}$  is given by  $pq^{2(n-i)}(i-1)a(n-2, i-1; p, q)$ . Combining all of the previous cases implies (3.1).  $\square$

Rather than working directly with the usual generating function  $\sum_{i=1}^{\lfloor (n+1)/2 \rfloor} a(n, i; p, q)w^i$ , it is more convenient to consider  $b_n(w) = b_n(w; p, q) = \sum_{i=1}^{n+1} a(n + i, i; p, q)w^i$ . We have the following explicit formula for  $b_n(w)$ .

**Theorem 3.2.** Define  $\alpha_n(w) = \alpha_n(w; p, q) = 1 + (1 + p)q^n w + pq^{2n}w^2$  and  $\rho_n(w) = \rho_n(w; p, q)$  by

$$\rho_n(w) = \alpha_n(w) \frac{d}{dw} \left( \alpha_{n-1}(w) \frac{d}{dw} \left( \cdots \alpha_2(w) \frac{d}{dw} (\alpha_1(w)) \right) \cdots \right), \quad n \geq 1,$$

with  $\rho_0(w) = w$ . For all  $n \geq 1$ ,

$$(3.2) \quad b_n(w) = \rho_n(w) - [w^1]\rho_{n-1}(w),$$

where  $[w^1]f(w)$  denotes the coefficient of  $w$  in the polynomial  $f(w)$ .

*Proof.* Define  $b(n, i) = b(n, i; p, q) = a(n + i, i; p, q)$  for all  $1 \leq i \leq n + 1$  and  $n \geq 1$ , with  $b(0, 1) = 1$  and  $b(0, 0) = 0$ . Then Lemma 3.1 implies

$$b(n, i) = (i + 1)b(n - 1, i + 1) + (1 + p)q^n i b(n - 1, i) + pq^{2n}(i - 1)b(n - 1, i - 1), \quad 1 \leq i \leq n + 1.$$

The last recurrence relation can be written as

$$b_n(w) = \alpha_n(w) \frac{d}{dw} b_{n-1}(w) - b(n - 1, 1), \quad n \geq 1,$$

with  $b_0(w) = w$ . By induction on  $n$ , we see that  $b_n(w) = \rho_n(w) - b(n - 1, 1)$ . Since the coefficient of  $w^1$  in  $b_n(w)$  equals  $[w^1]\rho_n(w)$ , we complete the proof.  $\square$

**Corollary 3.3.** We have

$$(3.3) \quad a(n, i; 1, 1) = (n - i)! \binom{n - i + 1}{i}, \quad 1 \leq i \leq \lfloor (n + 1)/2 \rfloor.$$

*Proof.* By the definitions, we have  $\rho_n(w; 1, 1) = (1 + w)^2 \frac{d}{dw} \rho_{n-1}(w; 1, 1)$  for  $n \geq 1$ , with  $\rho_0(w; 1, 1) = w$ . By induction on  $n$ , we get  $\rho_n(w; 1, 1) = n!(1 + w)^{n+1}$  for  $n \geq 1$ . Hence, by (3.2), we have

$$b_n(w; 1, 1) = n!(1 + w)^{n+1} - (n - 1)! [w^1](1 + w)^n = n!(1 + w)^{n+1} - n!,$$

which implies that  $b(n, i; 1, 1) = n! \binom{n+1}{i}$  for  $i = 1, 2, \dots, n + 1$ . Thus,  $a(n, i; 1, 1) = b(n - i, i; 1, 1) = (n - i)! \binom{n-i+1}{i}$  for  $1 \leq i \leq \lfloor (n + 1)/2 \rfloor$ , as required.

One can also provide a combinatorial proof of (3.3) as follows. First note that members  $\pi \in \mathcal{A}_{n,i}$  can be expressed as

$$\pi = \alpha_1(n - i + 1) \cdots \alpha_i(n - i + 1) \alpha_{i+1},$$

where the  $\alpha_j$  are words in  $[n - i]$  that are mutually disjoint, comprise all of  $[n - i]$  when taken together, and are non-empty for  $2 \leq j \leq i$ . If  $a_j = |\alpha_j|$  for  $1 \leq j \leq i + 1$ , then  $a_1 + a_2 + \cdots + a_{i+1} = n - i$  with  $a_2, \dots, a_i > 0$  and  $a_1, a_{i+1} \geq 0$ , whence there are  $\binom{n-2i+1+i}{i} = \binom{n-i+1}{i}$  choices for the cardinalities of the  $\alpha_j$ . Once the  $a_j$  have been determined, there are  $(n - i)!$  ways to arrange the elements of

$[n - i]$  as they can occur in any order going from left to right, which implies  $a(n, i; 1, 1) = |\mathcal{A}_{n,i}| = (n - i)! \binom{n-i+1}{i}$ .  $\square$

**Theorem 3.4.** *If  $n \geq 1$  and  $1 \leq i \leq \lfloor (n + 1)/2 \rfloor$ , then the sum of the values of  $\text{dif}(\pi)$  taken over all  $\pi \in \mathcal{A}_{n,i}$  is given by*

$$\frac{1}{3}(n - i)! \binom{n - i + 1}{i} (n^2 - i^2 + i - 1).$$

*Proof.* By the definitions, we have

$$\frac{d}{dq} \rho_n(w; 1, q) \Big|_{q=1} = 2nw(1 + w) \frac{d}{dw} \rho_{n-1}(w; 1, 1) + (1 + w)^2 \frac{d}{dw} \frac{d}{dq} \rho_{n-1}(w; 1, q) \Big|_{q=1},$$

with  $\frac{d}{dq} \rho_0(w; 1, q) \Big|_{q=1} = 0$ . Define  $\rho'_n(w) = \frac{d}{dq} \rho_n(w; 1, q) \Big|_{q=1}$ . By the proof of Corollary 3.3, we get

$$\rho'_n(w) = 2n \cdot n!w(1 + w)^n + (1 + w)^2 \frac{d}{dw} \rho'_{n-1}(w), \quad n \geq 1,$$

with  $\rho'_0(w) = 0$ . Define  $R(x, w) = \sum_{n \geq 0} \rho'_n(w) \frac{x^n}{n!}$ . By multiplying the last recurrence by  $\frac{x^{n-1}}{(n-1)!}$  and summing over  $n \geq 1$ , we obtain the first order linear pde

$$(3.4) \quad \frac{d}{dx} R(x, w) = \frac{2w(1 + w)(1 + x + xw)}{(1 - x - xw)^3} + (1 + w)^2 \frac{d}{dw} R(x, w),$$

with  $R(0, w) = 0$ .

To solve (3.4), we first let  $R(x, w) = G(u, v)$ , where  $u = x(1 + w)$  and  $v = w$ . Then (3.4) can be rewritten in terms of  $G$  as

$$(3.5) \quad (1 - u) \frac{d}{du} G(u, v) = \frac{2v(1 + u)}{(1 - u)^3} + (1 + v) \frac{d}{dv} G(u, v).$$

Letting  $H(u, v) = (1 - u)^3 G(u, v)$  transforms (3.5) to

$$(1 - u) \frac{d}{du} H(u, v) - (1 + v) \frac{d}{dv} H(u, v) + 3H(u, v) = 2v(1 + u),$$

where  $H(0, v) = 0$ . Letting  $H(u, v) = 2uv + f(u)$  implies  $f$  satisfies the first-order linear ode  $(1 - u)f'(u) + 3f(u) = 2u$  with  $f(0) = 0$ . Hence,  $f(u) = u^2 - \frac{u^3}{3}$  and  $H(u, v) = 2uv + u^2 - \frac{u^3}{3}$ . Then  $R(x, w) = G(u, v) = \frac{H(u,v)}{(1-u)^3}$  implies

$$(3.6) \quad R(x, w) = \frac{x(1 + w)(6w + 3x(1 + w) - x^2(1 + w)^2)}{3(1 - x - xw)^3}.$$

Extracting the coefficient of  $x^n/n!$  in (3.6) gives  $\rho'_n(w) = \frac{1}{3}(n + 1)!(1 + w)^n(3nw + n - 1)$ ,  $n \geq 1$ . By (3.2), we have

$$\begin{aligned} \frac{d}{dq} b_n(w; 1, q) \Big|_{q=1} &= \frac{1}{3}(n + 1)!(1 + w)^n(3nw + n - 1) - \frac{1}{3}n![w^1](1 + w)^{n-1}(3(n - 1)w + n - 2) \\ &= \frac{1}{3}(n + 1)!(1 + w)^n(3nw + n - 1) - \frac{(n - 1)(n + 1)!}{3}, \end{aligned}$$

so for  $i = 1, 2, \dots, n + 1$ ,

$$\frac{d}{dq} b(n, i; 1, q) \Big|_{q=1} = n(n + 1)! \binom{n}{i - 1} + \frac{n - 1}{3}(n + 1)! \binom{n}{i} = \frac{1}{3}n! \binom{n + 1}{i} (2ni + n^2 + i - 1).$$

Hence, we have  $\frac{d}{dq}a(n, i; 1, q) |_{q=1} = \frac{d}{dq}b(n - i, i; 1, q) |_{q=1} = \frac{1}{3}(n - i)! \binom{n-i+1}{i} (2(n - i)i + (n - i)^2 + i - 1)$ , for  $1 \leq i \leq \lfloor (n + 1)/2 \rfloor$ , which yields the result.  $\square$

Subtracting the previous result from the total area of all members of  $\mathcal{A}_{n,i}$ , that is,

$$(n - i)! \binom{n - i + 1}{i} \left[ \binom{n - i + 1}{2} + i(n - i + 1) \right],$$

gives the following explicit formula for the total number of up steps.

**Corollary 3.5.** *The number of up steps within the bargraph representations of all members of  $\mathcal{A}_{n,i}$  is given by*

$$\frac{1}{3}(n - i)! \binom{n - i + 1}{i} \left[ \binom{n + 2}{2} - \binom{i}{2} \right], \quad 1 \leq i \leq \lfloor (n + 1)/2 \rfloor.$$

*In particular, the number of up steps within all bargraphs of permutations of length  $n$  is given by  $(n - 1)! \binom{n+2}{3}$  for  $n \geq 1$ .*

*Remark:* A proof similar to that given for Theorem 3.4 above shows

$$\frac{d}{dp}a(n, i; p, 1) |_{p=1} = (n - i)! \binom{n - 1}{2} \binom{n - i + 1}{2}.$$

This formula may also be obtained by noting that exactly half of the adjacencies within all members of  $\mathcal{A}_{n,i}$  correspond to descents, which follows from applying the reversal operation to  $\mathcal{A}_{n,i}$ .

It is possible to provide a combinatorial explanation of the prior theorem.

**Combinatorial proof of Theorem 3.4.**

We first consider the contribution towards the total *dif*-value coming from adjacent letters  $ab$  where  $a, b \in [n - i]$ . Note that given any choice of the remaining letters, the string  $ab$  contributes

$$\begin{aligned} \sum_{a=1}^{n-i} \left( \sum_{b=1}^{a-1} b + \sum_{b=a+1}^{n-i} a \right) &= \sum_{a=1}^{n-i} \left( \binom{a}{2} + (n - i - a)a \right) \\ &= (n - i) \binom{n - i + 1}{2} - \binom{n - i + 2}{3} = 2 \binom{n - i + 1}{3} \end{aligned}$$

towards the total value. Given  $i \geq 1$  and  $n \geq 2i$ , let  $\mathcal{U}_{n,i}$  denote the set of binary words  $w$  of length  $n$  having  $i$  1's in which no two 1's are adjacent and in which some occurrence of the string 00 (i.e., two consecutive 0's) within  $w$  is circled and let  $u(n, i) = |\mathcal{U}_{n,i}|$ . Given  $a$  and  $b$ , there are  $u(n, i)$  choices concerning the arrangement of the  $n - i + 1$  letters relative to those in  $[n - i]$  together with the choice for the position of the string  $ab$  (it is understood that  $(n - i + 1)$ 's are to occupy the positions of the 1's within a member of  $\mathcal{U}_{n,i}$ , with  $ab$  corresponding to the circled 00). There are then  $(n - i - 2)!$  ways to arrange the remaining letters of  $[n - i]$  as they can occur in any order. Thus, there is a contribution of  $2(n - i - 2)! \binom{n-i+1}{3} u(n, i) = \frac{(n-i+1)!}{3} u(n, i)$  towards the total value of *dif* in this case.

We now determine  $u(n, i)$ . To do so, we divide  $\mathcal{U}_{n,i}$  into classes according to those (i) not ending in 1, with a 1 not directly preceding the circled 00 string, (ii) not ending in 1, with a 1 preceding the 00,

(iii) ending in 1, with a 1 not preceding the 00, and (iv) ending in 1, with a 1 preceding the 00. Let  $\mathcal{R}_{n,i}$  denote the set of “marked” square-and-domino tilings of length  $n$  having  $i$  dominoes in which one of the dominoes is marked. One may regard words  $w$  in class (i) above as tilings in  $\mathcal{R}_{n,i+1}$ , with the circled 00 string and occurrences of 10 in  $w$  corresponding to the marked domino and to the remaining unmarked dominoes, respectively. Thus, there are  $(i + 1)\binom{n-i-1}{i+1}$  members of  $\mathcal{U}_{n,i}$  in class (i). Next observe that deleting the second 0 in the circled 00 string and identifying the position of the 1 that directly precedes this string as the left half of the marked domino and then identifying the remaining occurrences of 10 as (unmarked) dominoes defines a bijection between class (ii) and  $\mathcal{R}_{n-1,i}$ , whence there are  $i\binom{n-i-1}{i}$  members in this class. By similar reasoning, there are  $i\binom{n-i-1}{i}$  and  $(i - 1)\binom{n-i-1}{i-1}$  words in classes (iii) and (iv), respectively. Thus, altogether we have

$$\begin{aligned} u(n,i) &= (i + 1)\binom{n - i - 1}{i + 1} + 2i\binom{n - i - 1}{i} + (i - 1)\binom{n - i - 1}{i - 1} \\ &= (i - 1)\left[\binom{n - i - 1}{i + 1} + 2\binom{n - i - 1}{i} + \binom{n - i - 1}{i - 1}\right] + 2\binom{n - i - 1}{i + 1} + 2\binom{n - i - 1}{i} \\ &= (i - 1)\binom{n - i + 1}{i + 1} + 2\binom{n - i}{i + 1}. \end{aligned}$$

Next, we determine the contribution from the string  $ab$  where  $a = n - i + 1$  and  $b \in [n - i]$ . Note that, by symmetry, this equals the contribution from when  $a \in [n - i]$  and  $b = n - i + 1$ . For any choice of position for the remaining letters, the string  $ab$  gives  $\sum_{b=1}^{n-i} b = \binom{n-i+1}{2}$ . If  $i \geq 1$  and  $n \geq 2i - 1$ , then let  $\mathcal{V}_{n,i}$  denote the set of binary words  $w$  of length  $n$  containing  $i$  1’s in which no 1’s are adjacent wherein a 1 is marked such that the marked 1 is not the final letter of  $w$ , with  $v(n,i) = |\mathcal{V}_{n,i}|$ . Given each choice of  $a$  and  $b$ , there are  $v(n,i)$  ways to arrange the  $n - i + 1$  letters and the letter  $b$  relative to the others (note that  $b$  corresponds to the 0 directly following the marked 1) and  $(n - i - 1)!$  ways to arrange the remaining members of  $[n - i]$  (in slots containing 0’s). Thus, the contribution from  $ab$  when one of  $a, b$  is  $n - i + 1$  equals  $2(n - i - 1)!\binom{n-i+1}{2}v(n,i) = (n - i + 1)v(n,i)$ . Considering whether or not a member of  $\mathcal{V}_{n,i}$  ends in 1 implies

$$v(n,i) = (i - 1)\binom{n - i}{i - 1} + i\binom{n - i}{i} = (i - 1)\binom{n - i + 1}{i} + \binom{n - i}{i}.$$

Thus, the sum of the *dif*-values of all members of  $\mathcal{A}_{n,i}$  is given by

$$\begin{aligned} &\frac{(n - i + 1)!}{3}u(n,i) + (n - i + 1)v(n,i) \\ &= \frac{(n - i + 1)!}{3}\left[(i - 1)\binom{n - i + 2}{i + 1} + 2\binom{n - i + 1}{i + 1} + 2(i - 1)\binom{n - i + 1}{i} + \binom{n - i}{i}\right] \\ &= \frac{(n - i + 1)!}{3}\binom{n - i + 1}{i}\left[\frac{(i - 1)(n - i + 2)}{i + 1} + \frac{2(n - 2i + 1)}{i + 1} + 2(i - 1) + \frac{n - 2i + 1}{n - i + 1}\right] \\ &= \frac{(n - i)!}{3}\binom{n - i + 1}{i}(n^2 - i^2 + i - 1), \end{aligned}$$

which completes the proof. □



In the next two results, we find the upper and lower bounds for the up step statistic on  $\mathcal{A}_{n,i}$ .

**Proposition 3.6.** *Let  $n \geq 1$  and  $1 \leq i \leq \lfloor (n+1)/2 \rfloor$ . Then the maximum number of up steps in the bargraph representation of a member of  $\mathcal{A}_{n,i}$  is given by  $m(m+1) - \binom{i}{2}$  if  $n = 2m$  and by  $(m+1)^2 - \binom{i}{2}$  if  $n = 2m+1$ . Furthermore, these maximum values are achieved by  $\frac{2(m!)^2}{i!}$  members of  $\mathcal{A}_{n,i}$  if  $n = 2m$  and by  $\frac{m!(m+1)!}{i!}$  members if  $n = 2m+1$ .*

*Proof.* Equivalently, we minimize the statistic value  $dif(\pi) = \binom{n-i+1}{2} + i(n-i+1) - up(\pi)$  for  $\pi \in \mathcal{A}_{n,i}$ . If  $\pi = \pi_1\pi_2 \cdots \pi_n$ , then  $dif(\pi) = \sum_{i=1}^{n-1} \min\{\pi_i, \pi_{i+1}\}$  since the sum is seen to equal area minus the number of up steps of  $\text{bar}_\pi$ . If  $n = 2m$ , then this sum is minimized when each of the letters in  $[m-1]$  is counted twice and  $m$  is counted once (note that a letter  $\pi_j$  is counted twice in  $dif(\pi)$  if  $\pi_{j-1} > \pi_j < \pi_{j+1}$ , and once if exactly one of these inequalities is reversed). Note that this minimum is in fact achieved by

$$\pi = (n-i+1)1(n-i+1)2 \cdots (n-i+1)i(n-i)(i+1) \cdots (n-m+1)m.$$

Thus, the minimum value of  $dif$  is  $2\binom{m}{2} + m = m^2$ , which implies the maximum number of up steps is given by  $\binom{2m-i+1}{2} + i(2m-i+1) - m^2 = m(m+1) - \binom{i}{2}$ . Similarly, if  $n = 2m+1$ , then  $dif(\pi)$  is minimized by

$$\pi = (n-i+1)1(n-i+1)2 \cdots (n-i+1)i(n-i)(i+1) \cdots (n-m+1)m(n-m),$$

implying that the maximum number of up steps is given by  $\binom{2m-i+2}{2} + i(2m-i+2) - m(m+1) = (m+1)^2 - \binom{i}{2}$ .

We now count, equivalently, the number of members of  $\mathcal{A}_{n,i}$  for which the minimum value of  $dif$  is achieved (we will refer to members of  $\mathcal{A}_{n,i}$  where this occurs as *minimal*). If  $n = 2m$ , then the preceding shows that the only element of  $[m]$  that can possibly start or end a minimal member  $\pi$  of  $\mathcal{A}_{n,i}$  is  $m$ , in which case each element of  $[m-1]$  separates two letters of  $I = [m+1, 2m-i+1]$ . This yields  $\frac{2m!(m-1)!}{i!}$  possible members of  $\mathcal{A}_{n,i}$  since the elements of  $[m]$  can be arranged in  $2(m-1)!$  ways and the letters in  $I$  can be arranged in  $\frac{m!}{i!}$  ways (as  $2m-i+1$  occurs with multiplicity  $i$ ). Otherwise,  $\pi$  both starts and ends with letters in  $I$ , no two of which are directly adjacent, with  $m$  next to exactly one element of  $[m-1]$ . This yields  $\frac{2(m-1)m!(m-1)!}{i!}$  possibilities since there are  $2(m-1)(m-1)!$  choices concerning positions of elements of  $[m]$  (as there are  $2(m-1)$  ways to arrange  $m$ ) and  $\frac{m!}{i!}$  choices concerning the letters in  $I$ . Combining the two previous cases gives  $\frac{2(m!)^2}{i!}$  minimal members of  $\mathcal{A}_{n,i}$ . If  $n = 2m+1$ , then the preceding shows that for a minimal  $\pi$ , there are  $m!$  ways to arrange the elements of  $[m]$  and  $\frac{(m+1)!}{i!}$  ways to arrange those in  $[m+1, 2m-i+2]$ , yielding  $\frac{m!(m+1)!}{i!}$  possibilities, which completes the proof.  $\square$

*Remark:* Taking  $i = 1$  in the preceding proposition implies that the maximum number of up steps within a bargraph of a permutation of length  $n$  equals  $m(m+1)$  if  $n = 2m$  and  $(m+1)^2$  if  $n = 2m+1$ .

**Proposition 3.7.** *Let  $n \geq 1$  and  $1 \leq i \leq \lfloor (n + 1)/2 \rfloor$ . Then the minimum number of up steps in the bargraph representation of a member of  $\mathcal{A}_{n,i}$  is given by  $n + \binom{i-1}{2}$ , which is achieved by  $2^{n-2i+1}(i-1)!$  members of  $\mathcal{A}_{n,i}$ .*

*Proof.* We characterize those  $\pi = \pi_1\pi_2 \cdots \pi_n \in \mathcal{A}_{n,i}$  such that  $up(\pi)$  is minimal. Let  $a = n - i + 1$ . Note first that the contribution towards  $up(\pi)$  derived from the columns that correspond to a string of letters  $\pi_{r+1}\pi_{r+2} \cdots \pi_s$  where  $\pi_r = \pi_s = a$  and  $\pi_{r+1}, \dots, \pi_{s-1} \in [n - i]$  is at least  $a - m$ , where  $m = \min\{\pi_{r+1}, \dots, \pi_{s-1}\}$ . To see this, note that once  $bar_\pi$  drops to the level  $m$ , then at least  $a - m$  up steps are required to attain the level  $a$  once again. Thus, if  $\ell_1$  and  $\ell_2$  denote the smallest and the largest indices  $j$  such that  $\pi_j = a$ , then the minimum possible contribution towards  $up(\pi)$  coming from the string  $L = \pi_{\ell_1+1} \cdots \pi_{\ell_2}$  is  $1 + 2 + \cdots + i - 1 = \binom{i}{2}$  since at least one element of  $[n - i]$  must separate any two letters  $a$ . This occurs when the elements of  $[n - i]$  lying within  $L$  are precisely those in  $[n - 2i + 2, n - i]$ . The minimum possible contribution towards  $up(\pi)$  from the letters in  $\pi - L$  equals  $a$  since  $\pi_{\ell_1} = a$ , which occurs if and only if the strings  $\pi_1\pi_2 \cdots \pi_{\ell_1}$  and  $\pi_{\ell_2+1} \cdots \pi_n$  (if non-empty) are increasing and decreasing, respectively. Combining the previous observations, there are  $(i - 1)!$  possible orderings for the letters in  $L$  and  $2^{n-2i+1}$  possibilities for the letters in  $\pi - L$  (as the string  $\pi_1 \cdots \pi_{\ell_1-1}$  can comprise any subset of  $[n - 2i + 1]$ , possibly empty, which then determines the rest of the letters in  $\pi - L$ ). Thus, the minimum number of up steps is  $\binom{i}{2} + a = \binom{i-1}{2} + n$ , which is achieved by  $2^{n-2i+1}(i-1)!$  members of  $\mathcal{A}_{n,i}$ . □

#### 4. Sign balance of bargraph parameters

In this section, we determine by how much the subset of  $\mathcal{A}_{n,i}$  whose members have an even value of a parameter differs in cardinality with the subset of  $\mathcal{A}_{n,i}$  where the value is odd in the case of the parameters  $up$  and  $des$ . Given a structure  $\mathcal{S}$  and a parameter  $\alpha$ , let  $\mathcal{E}$  and  $\mathcal{O}$  denote the subsets of  $\mathcal{S}$  having an even or an odd  $\alpha$ -value, respectively. Then the sign balance  $|\mathcal{E}| - |\mathcal{O}|$  with respect to  $\alpha$  can be obtained by substituting the value  $-1$  into the generating function for  $\alpha$  over  $\mathcal{S}$ .

We apply this strategy and first find the sign balance of the  $up$  statistic. To do so, we consider *dif* once more. Theorem 3.2 with  $p = -q = 1$  gives

$$\rho_n(w; 1, -1) = (1 + (-1)^n w)^2 \frac{d}{dw} \rho_{n-1}(w; 1, -1), \quad \text{with } \rho_0(w; 1, -1) = w.$$

Clearly,  $\rho_2(w; 1, -1) = -2A^2B$  and  $\rho_3(w; 1, -1) = -4AB^3 + 2A^2B^2$ , where  $A = 1 + w$  and  $B = 1 - w$ . By induction on  $n$ , we obtain the following recurrences for all  $n \geq 2$ :

$$(4.1) \quad \rho_{2n}(w; 1, -1) = \sum_{j=2}^{2n} (-1)^{j-1} \alpha_{2n,j} A^j B^{2n+1-j},$$

$$(4.2) \quad \rho_{2n+1}(w; 1, -1) = \sum_{j=1}^{2n} (-1)^j \alpha_{2n+1,j} A^j B^{2n+2-j},$$

where

$$(4.3) \quad \alpha_{2n,j} = (j - 1)\alpha_{2n-1,j-1} + (2n + 2 - j)\alpha_{2n-1,j-2}, \quad j = 2, 3, \dots, 2n,$$

$$(4.4) \quad \alpha_{2n+1,j} = (j + 1)\alpha_{2n,j+1} + (2n + 1 - j)\alpha_{2n,j}, \quad j = 1, 2, \dots, 2n,$$

with  $\alpha_{2,2} = \alpha_{3,2} = 2$  and  $\alpha_{3,1} = 4$ . For example,

$\alpha_{n,j}$	$j = 1$	$j = 2$	$j = 3$	$j = 4$
$n = 2$		2		
$n = 3$	4	2		
$n = 4$		4	16	4
$n = 5$	8	60	48	4.

**Theorem 4.1.** *If  $n \geq 1$ , then*

$$(4.5) \quad \alpha_{2n,2n-j} = \sum_{i=1}^{j+1} (-1)^{j+i-1} \binom{2n+1}{j+1-i} i^n (i+1)^n, \quad 0 \leq j \leq 2n - 2,$$

$$(4.6) \quad \alpha_{2n+1,2n-j} = \sum_{i=1}^{j+1} (-1)^{j+i-1} \binom{2n+2}{j+1-i} i^{n+1} (i+1)^n, \quad 0 \leq j \leq 2n - 1.$$

*Proof.* We first show that formula (4.5) also holds for  $j = 2n - 1$  and that (4.6) holds for  $j = 2n$  by virtue of  $\alpha_{2n,1} = \alpha_{2n+1,0} = 0$ . In the first case, this amounts to verifying the identity

$$\sum_{i=1}^{2n} (-1)^i \binom{2n+1}{i+1} i^n (i+1)^n = 0, \quad n \geq 1.$$

Since  $\binom{2n+1}{i+1} i^n (i+1)^n = (2n+1) \binom{2n}{i} i^n (i+1)^{n-1}$ , the preceding identity follows from expanding  $(i+1)^{n-1}$  and using the fact that  $\sum_{i=1}^{2n} (-1)^i \binom{2n}{i} i^\ell = 0$  if  $1 \leq \ell < 2n$ . A similar proof applies to (4.6) when  $j = 2n$ .

To complete the proof, we proceed by induction on  $n$  and prove (4.5) and (4.6) for all  $j$  given  $n$ , the  $n = 1$  case holding since  $\alpha_{2,2} = 2$ ,  $\alpha_{3,2} = 2$  and  $\alpha_{3,1} = 4$ . Formulas (4.3) and (4.4) give for  $j = 0, 1, \dots, 2n - 1$  the recurrences

$$\begin{aligned} \alpha_{2n,2n-j} &= (2n - 1 - j)\alpha_{2n-1,2n-1-j} + (j + 2)\alpha_{2n-1,2n-2-j}, \\ \alpha_{2n+1,2n-j} &= (2n + 1 - j)\alpha_{2n,2n+1-j} + (j + 1)\alpha_{2n,2n-j}. \end{aligned}$$

If  $n \geq 2$  and  $0 \leq j \leq 2n - 2$ , then we have by the induction hypothesis

$$\begin{aligned}
\alpha_{2n,2n-j} &= (2n - 1 - j) \sum_{i=1}^j (-1)^{j+i-2} \binom{2n}{j-i} i^n (i+1)^{n-1} \\
&\quad + (j+2) \sum_{i=1}^{j+1} (-1)^{j+i-1} \binom{2n}{j+1-i} i^n (i+1)^{n-1} \\
&= \sum_{i=1}^{j+1} (-1)^{j+i-1} \left( -(2n-1-j) \binom{2n}{j-i} + (j+2) \binom{2n}{j+1-i} \right) i^n (i+1)^{n-1} \\
&= \sum_{i=1}^{j+1} (-1)^{j+i-1} \binom{2n+1}{j+1-i} i^n (i+1).
\end{aligned}$$

Also, for  $0 \leq j \leq 2n - 1$ , we have

$$\begin{aligned}
\alpha_{2n+1,2n-j} &= (2n+1-j) \sum_{i=1}^j (-1)^{j+i-2} \binom{2n+1}{j-i} i^n (i+1)^n \\
&\quad + (j+1) \sum_{i=1}^{j+1} (-1)^{j+i-1} \binom{2n+1}{j+1-i} i^n (i+1)^n \\
&= \sum_{i=1}^{j+1} (-1)^{j+i-1} \left( -(2n+1-j) \binom{2n+1}{j-i} + (j+1) \binom{2n+1}{j+1-i} \right) i^n (i+1)^n \\
&= \sum_{i=1}^{j+1} (-1)^{j+i-1} \binom{2n+2}{j+1-i} i^{n+1} (i+1)^n,
\end{aligned}$$

which completes the induction. □

The above theorem, together with (4.1)-(4.2), yields the following for  $n \geq 1$ :

$$\begin{aligned}
\rho_{2n}(w; 1, -1) &= \sum_{j=2}^{2n} \left( \sum_{i=1}^{2n+1-j} (-1)^i \binom{2n+1}{j+i} i^n (i+1)^n \right) (1+w)^j (1-w)^{2n+1-j}, \\
\rho_{2n+1}(w; 1, -1) &= \sum_{j=1}^{2n} \left( \sum_{i=1}^{2n+1-j} (-1)^{i-1} \binom{2n+2}{j+i+1} i^{n+1} (i+1)^n \right) (1+w)^j (1-w)^{2n+2-j}.
\end{aligned}$$

Thus, by (3.2), we have

$$\begin{aligned}
 b_{2n}(w; 1, -1) &= \sum_{j=2}^{2n} \left( \sum_{i=1}^{2n+1-j} (-1)^i \binom{2n+1}{j+i} i^n (i+1)^n \right) (1+w)^j (1-w)^{2n+1-j} \\
 &\quad - [w^1] \sum_{j=1}^{2n-2} \left( \sum_{i=1}^{2n-1-j} (-1)^{i-1} \binom{2n}{j+i+1} i^n (i+1)^{n-1} \right) (1+w)^j (1-w)^{2n-j}, \\
 b_{2n+1}(w; 1, -1) &= \sum_{j=1}^{2n} \left( \sum_{i=1}^{2n+1-j} (-1)^{i-1} \binom{2n+2}{j+i+1} i^{n+1} (i+1)^n \right) (1+w)^j (1-w)^{2n+2-j} \\
 &\quad - [w^1] \sum_{j=2}^{2n} \left( \sum_{i=1}^{2n+1-j} (-1)^i \binom{2n+1}{j+i} i^n (i+1)^n \right) (1+w)^j (1-w)^{2n+1-j},
 \end{aligned}$$

which gives for all  $1 \leq k \leq 2n + 1$ ,

$$\begin{aligned}
 b(2n, k; 1, -1) &= \sum_{j=2}^{2n} \left( \sum_{i=1}^{2n+1-j} (-1)^i \binom{2n+1}{j+i} i^n (i+1)^n \right) \sum_{\ell=0}^k (-1)^\ell \binom{j}{k-\ell} \binom{2n+1-j}{\ell}, \\
 b(2n+1, k; 1, -1) &= \sum_{j=1}^{2n} \left( \sum_{i=1}^{2n+1-j} (-1)^{i-1} \binom{2n+2}{j+i+1} i^{n+1} (i+1)^n \right) \sum_{\ell=0}^k (-1)^\ell \binom{j}{k-\ell} \binom{2n+2-j}{\ell}.
 \end{aligned}$$

Since  $\text{dif}(\pi) = \binom{n+1}{2} + k(n+1) - \text{up}(\pi)$  for  $\pi \in \mathcal{A}_{n+k,k}$ , the sign balance with respect to the up step statistic on  $\mathcal{A}_{n+k,k}$  is given by

$$(-1)^{\binom{n+1}{2} + k(n+1)} a(n+k, k; 1, -1).$$

Recalling  $b(n, k; p, q) = a(n+k, k; p, q)$ , we get the following result.

**Theorem 4.2.** *Let  $n \geq 2$  and  $1 \leq k \leq n + 1$ . Then the sign balance of the up step statistic on  $\mathcal{A}_{n+k,k}$  is given by*

$$(-1)^{m+k} \sum_{j=2}^n \left( \sum_{i=1}^{n+1-j} (-1)^i \binom{n+1}{j+i} i^m (i+1)^m \right) \sum_{\ell=0}^k (-1)^\ell \binom{j}{k-\ell} \binom{n+1-j}{\ell},$$

if  $n = 2m$ , and by

$$(-1)^m \sum_{j=1}^{n-1} \left( \sum_{i=1}^{n-j} (-1)^i \binom{n+1}{j+i+1} i^{m+1} (i+1)^m \right) \sum_{\ell=0}^k (-1)^\ell \binom{j}{k-\ell} \binom{n+1-j}{\ell},$$

if  $n = 2m + 1$ .

Taking  $k = 1$  in the previous result gives the following formula.

**Corollary 4.3.** *If  $n \geq 3$ , then the sign balance of the up step statistic on  $\mathcal{S}_n$  is given by*

$$(-1)^{n/2} \sum_{j=2}^{n-1} (n-2j+2) \sum_{i=1}^{n-j} (-1)^i \binom{n}{j+i} i^{n/2} (i+1)^{(n-2)/2},$$

*if  $n$  is even, and by*

$$(-1)^{(n-1)/2} \sum_{j=2}^{n-1} (n-2j) \sum_{i=1}^{n-j} (-1)^i \binom{n}{j+i} i^{(n-1)/2} (i+1)^{(n-1)/2},$$

*if  $n$  is odd.*

We now focus on the case  $p = -q = -1$ . Theorem 3.2 with  $p = -1$  gives

$$b_n(w; -1, q) = \rho_n(w; -1, q) - [w^1] \rho_{n-1}(w; -1, q),$$

where

$$\rho_n(w; -1, q) = (1 - q^{2n} w^2) \frac{d}{dw} \rho_{n-1}(w; -1, q), \quad n \geq 1,$$

with  $\rho_0(w; -1, q) = w$ . Define  $T(x, w; q) = \sum_{n \geq 0} \rho_n(w; -1, q) \frac{x^n}{n!}$ . Multiplying the last recurrence by  $\frac{x^{n-1}}{(n-1)!}$ , and summing over  $n \geq 1$ , we obtain

$$\frac{d}{dx} T(x, w; q) = \frac{d}{dw} T(x, w; q) - q^2 w^2 \frac{d}{dw} T(q^2 x, w; q),$$

with  $T(0, w; q) = w$ . In particular, we have for  $q = 1$ ,

$$\frac{d}{dx} T(x, w; 1) = (1 - w^2) \frac{d}{dw} T(x, w; 1),$$

with  $T(0, w; 1) = w$ . By the method of characteristics, the solution of this pde is found to be

$$T(x, w; 1) = \tanh(x + \operatorname{arctanh} w) = \frac{e^{2x} - 1}{e^{2x} + 1} + \frac{4we^{2x}}{(1 - w + (1 + w)e^{2x})(e^{2x} + 1)}.$$

The Euler numbers of order  $\alpha \geq 1$ , denoted by  $E_n^{(\alpha)}$ , are defined by (see [13, p. 66, Eqn. 65])

$$\left( \frac{2e^x}{e^{2x} + 1} \right)^\alpha = \sum_{n \geq 0} E_n^{(\alpha)} \frac{x^n}{n!}.$$

In [8], it was shown that the  $E_n^{(\alpha)}$  have explicit formula

$$E_n^{(\alpha)} = \sum_{\ell=0}^n \frac{(-1)^\ell}{2^\ell} \binom{\alpha+n}{n-\ell} \binom{\alpha+\ell-1}{\ell} \sum_{j=0}^{\ell} \binom{\ell}{j} (\ell-2j)^n.$$

Thus, the coefficient of  $w^k$ ,  $k \geq 1$ , in  $T(x, w; 1)$  is given by

$$\begin{aligned} [w^k]T(x, w; 1) &= \frac{4e^{2x}(1 - e^{2x})^{k-1}}{(e^{2x} + 1)^{k+1}} \\ &= \frac{1}{2^{k-1}} \sum_{j=0}^{k-1} (-1)^j \binom{k-1}{j} e^{(2j-k+1)x} \left( \sum_{r \geq 0} E_r^{(k+1)} \frac{x^r}{r!} \right) \\ &= \frac{1}{2^{k-1}} \left( \sum_{j=0}^{k-1} (-1)^j \binom{k-1}{j} \sum_{m \geq 0} \frac{(2j - k + 1)^m x^m}{m!} \right) \left( \sum_{r \geq 0} E_r^{(k+1)} \frac{x^r}{r!} \right). \end{aligned}$$

Then since

$$b(n, k; -1, 1) = [x^n/n!] \frac{4e^{2x}(1 - e^{2x})^{k-1}}{(e^{2x} + 1)^{k+1}}, \quad 1 \leq k \leq n + 1,$$

we get the following result.

**Theorem 4.4.** *If  $n \geq 0$  and  $1 \leq k \leq n + 1$ , then the sign-balance of the descent statistic on  $\mathcal{A}_{n+k,k}$  is given by*

$$(4.7) \quad a(n + k, k; -1, 1) = \frac{1}{2^{k-1}} \sum_{j=0}^{k-1} \left( (-1)^j \binom{k-1}{j} \sum_{m=0}^n \binom{n}{m} (2j - k + 1)^m E_{n-m}^{(k+1)} \right).$$

*Remark:* Taking  $p = -q = -1$  in the formulas found above for  $a(2n, n; p, q)$  and  $a(2n + 1, n + 1; p, q)$  yields 0 and  $(-1)^n n!$ , respectively, for  $n \geq 1$ . Comparing with the  $k = n$  and  $k = n + 1$  cases of (4.7) yields a pair of binomial identities that involve Euler numbers which we were unable to find in the literature.

We can find a second formula for the sign balance of *des* by starting with the Maclaurin series for  $\tan^\ell(x)$  given by

$$\tan^\ell(x) = \sum_{n \geq \ell} \left( (-1)^{(n-\ell)/2} \sum_{j=\ell}^n (-2)^{n-j} j! S_2(n, j) \binom{j-1}{\ell-1} \right) \frac{x^n}{n!}, \quad \ell \geq 1,$$

where  $S_2(n, j)$  is the Stirling number of the second kind (see [12, A059419]). Thus,

$$\tanh^\ell(x) = (-1)^\ell \sum_{n \geq \ell} \left( \sum_{j=\ell}^n (-1)^j 2^{n-j} j! S_2(n, j) \binom{j-1}{\ell-1} \right) \frac{x^n}{n!}.$$

We also have

$$T(x, w; 1) = \tanh(x + \operatorname{arctanh} w) = \frac{w + \tanh(x)}{1 + w \tanh(x)} = \tanh(x) + \sum_{j \geq 1} (\tanh^2(x) - 1) \tanh^{j-1}(x) (-w)^j,$$

which implies

$$[w^k]T(x, w; 1) = (-1)^k (\tanh^{k+1}(x) - \tanh^{k-1}(x)), \quad k \geq 1.$$

Hence, by (3.2), we have for all  $n \geq 0$ ,

$$b(n, k; -1, 1) = (-1)^k [x^n/n!] (\tanh^{k+1}(x) - \tanh^{k-1}(x)), \quad 1 \leq k \leq n + 1.$$

This leads to the following result, where it is assumed that  $\binom{m}{\ell} = 0$  if  $m$  or  $\ell$  is negative.

**Theorem 4.5.** *If  $n \geq 1$  and  $1 \leq k \leq n + 1$ , then the sign-balance of the descent statistic on  $\mathcal{A}_{n+k,k}$  is given by*

$$(4.8) \quad a(n+k, k; -1, 1) = \sum_{j=k-1}^n (-1)^{j-1} 2^{n-j} j! S_2(n, j) \left( \binom{j-1}{k} - \binom{j-1}{k-2} \right).$$

Taking  $k = 1$  in Theorems 4.4 and 4.5 yields the following identity.

**Corollary 4.6.** *The sign balance of the descent statistic on  $\mathcal{S}_{n+1}$  is given by*

$$E_n^{(2)} = \sum_{j=1}^n (-1)^{j-1} 2^{n-j} (j-1) j! S_2(n, j), \quad n \geq 1.$$

## REFERENCES

- [1] A. Blecher, C. Brennan and A. Knopfmacher, Levels in bargraphs, *Ars Math. Contemp.*, **9** (2015) 297–310.
- [2] A. Blecher, C. Brennan and A. Knopfmacher, Peaks in bargraphs, *Trans. Royal Soc. S. Afr.*, **71** (2016) 97–103.
- [3] A. Blecher, C. Brennan and A. Knopfmacher, Combinatorial parameters in bargraphs, *Quaest. Math.*, **39** (2016) 619–635.
- [4] E. Deutsch and S. Elizalde, Statistics on bargraphs viewed as cornerless Motzkin paths, *Discrete Appl. Math.*, **221** (2017) 54–66.
- [5] S. Feretić, A perimeter enumeration of column-convex polyominoes, *Discrete Math. Theor. Comput. Sci.*, **9** (2007) 57–83.
- [6] A. Geraschenko, An investigation of skyline polynomials, <http://people.brandeis.edu/gessel/47a/geraschenko.pdf>.
- [7] S. Heubach and T. Mansour, Combinatorics of Compositions and Words, *Discrete Math. Appl.*, Chapman & Hall/CRC, Boca Raton, (2009).
- [8] Q.-M. Luo, An explicit formula for the Euler numbers of higher order, *Tamkang J. Math.*, **36** (2005) 315–317.
- [9] T. Mansour, Enumeration of words by sum of differences between adjacent letters, *Discrete Math. Theor. Comput. Sci.*, **11** (2009) 173–185.
- [10] J. Osborn and T. Prellberg, Forcing adsorption of a tethered polymer by pulling, *J. Stat. Mech.*, **2010** (2010) 1–18.
- [11] T. Prellberg and R. Brak, Critical exponents from nonlinear functional equations for partially directed cluster models, *J. Stat. Phys.*, **78** (1995) 701–730.
- [12] N. J. Sloane, *The On-Line Encyclopedia of Integer Sequences*, <http://oeis.org>, 2010.
- [13] H. M. Srivastava and J. Choi, *Series Associated with the Zeta and Related Functions*, Kluwer Academic Publishers, Dordrecht, 2001.

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