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SOLUTION TO THE MINIMUM HARMONIC INDEX OF GRAPHS WITH GIVEN MINIMUM DEGREE

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Abstract. The harmonic index of a graph G is defined as $H(G) = \sum_{uv\in E(G)} \frac{2}{d(u)+d(v)}$, where $d(u)$ denotes the degree of a vertex u in G. Let $\mathcal{G}(n,k)$ be the set of simple n-vertex graphs with minimum degree at least k. In this work we consider the problem of determining the minimum value of the harmonic index and the corresponding extremal graphs among $\mathcal{G}(n,k)$. We solve the problem for each integer $k(1 \le k \le n/2)$ and show the corresponding extremal graph is the complete split graph $K^*_{k,n-k}$. This result together with our previous result which solve the problem for each integer $k(n/2 \le k \le n-1)$ give a complete solution of the problem.

1. Introduction

All graphs considered in the following will be simple. Let G be a connected graph with vertex set $V(G)$ and edge set $E(G)$. The harmonic index $H(G)$ a graph G is defined as

$$H(G) = \sum_{uv\in E(G)} \frac{2}{d(u)+d(v)}.$$

It has been found that the harmonic index correlate well with the Randić index [4, 5] and the π-electronic energy of benzenoid hydrocarbons [6]. To our best knowledge, this index first appear in [1], in which Favaron et al. considered the relation between harmonic index and the eigenvalues of graphs.

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Let $\mathcal{G}(n,k)$ be the set of n-vertex graphs with minimum degree at least k. In this work we consider the problem of determining the minimum value of the harmonic index and the corresponding extremal graphs among $\mathcal{G}(n,k)$. Zhong [8, 9] solved the problem for the case of $k=1$ and gave some extremal results for trees and unicyclic graphs. Wu et al. [7] solved the result for the case of $k=2$. Recently Liu [2], one of the authors, solved this problem for the cases of each integer $k(n/2 \le k \le n-1)$. Thus, it is still an open problem for the cases of $k(3 \le k < n/2)$. In this work, we solve this problem for each integer $k(1 \le k < n/2)$ using a nonlinear programming model. Thus the problem of determining the minimum value of the harmonic index among $\mathcal{G}(n,k)$ is solved completely.

2. A nonlinear programming model for the harmonic index

We give a nonlinear programming model for the harmonic index in this section. We will see this model play a significant role in the proof of our main result.

Let G be a graphs of $\mathcal{G}(n,k)$. Denote by $x_{i,j}$ ($x_{i,j} \ge 0$), the number of edges joining the vertices of degrees i and j. Denote by n_i the number of vertices of degree of i. Then

$$H(G) = \sum_{k \le i \le j \le n-1} \frac{2}{i+j} x_{i,j} \tag{2.1}$$

$$n_k + n_{k+1} + \cdots + n_{n-1} = n. \tag{2.2}$$

By counting the edges that incident to a vertex of degree i, $i = k, \dots, n-1$, one obtains

$$\sum_{\substack{j=k \\ j \ne i}}^{n-1} x_{i,j} + 2x_{i,i} = in_i, \;\; i.e.\; n_i = \frac{1}{i}\left(\sum_{\substack{j=k \\ j \ne i}}^{n-1} x_{i,j} + 2x_{i,i}\right). \tag{2.3}$$

Substituting Eq.(2.3) back into Eq.(2.2) and performing appropriate rearrangements, we get

$$\sum_{k \le i < j \le n-1} \left(\frac{1}{i} + \frac{1}{j}\right) x_{i,j} + 2\sum_{i=k}^{n-1} \frac{x_{i,i}}{i} = n. \tag{2.4}$$

Now, rewriting Eq.(2.1) as

$$H(G) = \sum_{k \le i < j \le n-1} \frac{2}{i+j} x_{i,j} + \sum_{i=k}^{n-1} \frac{x_{i,i}}{i} \tag{2.5}$$

and combining Eq.(2.4) and Eq.(2.5) so as to eliminate the term $\sum (x_{ii}/i)$, we arrive at

$$n - 2H(G) = \sum_{k \le i < j \le n-1} \left(\frac{1}{i} + \frac{1}{j} - \frac{4}{i+j}\right) x_{i,j} = \sum_{k \le i < j \le n-1} \frac{(i-j)^2}{ij(i+j)} x_{i,j}$$

i.e.,

$$H(G) = \frac{n}{2} - \frac{1}{2}\sum_{k\le i<j\le n-1}\left(\frac{1}{i}+\frac{1}{j}-\frac{4}{i+j}\right)x_{i,j} \tag{2.6}$$

$$= \frac{n}{2} - \frac{1}{2}\sum_{k\le i<j\le n-1}\frac{(i-j)^2}{ij(i+j)}x_{i,j} \tag{2.7}$$

Remark 2.1. *From (2.7), we see that $H(G) \le \frac{n}{2}$ for any n-vertex graph G and the equality holds if and only if G is regular. It is the reason why the problem of determining the maximum value of the harmonic index among $\mathcal{G}(n,k)$ is trivial.*

We will give some linear equalities and nonlinear inequalities which must be satisfied in any above mentioned graph. By Eq. (2.6), the mathematical description of the problem (P) is

$$\min\ H = \frac{n}{2} - \frac{1}{2}\sum_{k\le i<j\le n-1}\left(\frac{1}{i}+\frac{1}{j}-\frac{4}{i+j}\right)x_{i,j}$$

subject to

$$\sum_{\substack{j=k\\ j\ne i}}^{n-1} x_{i,j} + 2x_{i,i} = in_i \text{ for } k\le i\le n-1; \tag{2.8}$$

$$n_k + n_{k+1} + \cdots + n_{n-1} = n; \tag{2.9}$$

$$x_{ij} \le n_i n_j \text{ for } k \le i < j \le n-1; \tag{2.10}$$

$$x_{i,i} \le \binom{n_i}{2} \text{ for } k\le i\le n-1; \tag{2.11}$$

$$x_{i,j},\ n_i \in \mathbb{N},\ k\le i\le j\le n-1. \tag{2.12}$$

We can see the problem (P) together with (2.8)–(2.12) define a nonlinearly constrained optimization problem.

Thereafter the problem of minimization of H becomes the problem of maximization of

$$\gamma = \sum_{k\le i<j\le n-1}\left(\frac{1}{i}+\frac{1}{j}-\frac{4}{i+j}\right)x_{ij},$$

subject to (2.8)–(2.12).

Denote by $w(i,j) := \frac{1}{i}+\frac{1}{j}-\frac{4}{i+j}$ for simplicity. Note that $w(i,i)=0$ for every positive integer i. Now we consider i and j as continuous variables and therefore we can take the partial derivative with respect to i and j, respectively. Then $\frac{\partial w(i,j)}{\partial i} = -\frac{1}{i^2}+\frac{4}{(i+j)^2} = \frac{(3i+j)(i-j)}{i^2(i+j)^2} < 0$ and $\frac{\partial w(i,j)}{\partial j} = -\frac{1}{j^2}+\frac{4}{(i+j)^2} = \frac{(i+3j)(j-i)}{j^2(i+j)^2} > 0$ for $i<j$. Thus, $w(i,j)$ is decreasing in i and increasing in j for $k\le j<j\le n-1$. We will use this property of $w(i,j)$ through the paper.

3. Main results

In this section, we will prove our main result using the nonlinear programming model mentioned above. Let $K^*_{k,n-k}$ be a complete split graph which is arised from a complete bipartite graph $K_{k,n-k}$ by joining each pair of vertices in the partite of k vertices.

Theorem 3.1. *Let G be a graph of $\mathcal{G}(n,k)$ for $1 \leq k < n/2$. Then*

$$H(G) \geq \frac{n}{2} - \frac{k(n-k)}{2}\left(\frac{1}{k} + \frac{1}{n-1} - \frac{4}{k+n-1}\right),$$

where equality holds if and only if $G \cong K^*_{k,n-k}$.

Proof. Let m be the index such that $n_m + n_{m+1} + \cdots + n_{n-1} \geq k$ and $n_{m+1} + n_{m+2} + \cdots + n_{n-1} < k$. We distinguish two subcases: (a) for such m holds $n_m + n_{m+1} + \cdots + n_{n-1} = k$, and (b) $n_m + n_{m+1} + \cdots + n_{n-1} > k$.

Subcase (a). $n_m + n_{m+1} + \cdots + n_{n-1} = k$. We have:

$$\begin{aligned} \max\ \gamma \quad &= \sum_{k \leq i < j \leq n-1} w(i,j)x_{i,j} \\ &= \sum_{i=k}^{m-1}\sum_{j=i+1}^{n-1} w(i,j)x_{i,j} + \sum_{m \leq i < j \leq n-1} w(i,j)x_{i,j}. \end{aligned}$$

We give the maximum possible values of $\sum\limits_{j=i+1}^{n-1} w(i,j)x_{i,j}$ for $k \leq i \leq m-1$. Since $n_m + n_{m+1} + \cdots + n_{n-1} = k$ and $\sum_{j=i+1}^{n-1} x_{i,j} \leq i n_i$, first we join a vertex of degree i to all k vertices of degrees $n-1, \ldots, m$ and last $i-k$ edges joining to $i-k$ vertices of degree $m-1$. Thus, holds:

$$\sum_{j=i+1}^{n-1} w(i,j)x_{i,j} \leq n_i\left(\sum_{j=m}^{n-1} w(i,j)n_j + w(i,m-1)(i-k)\right).$$

Therefore,

$$\max\ \gamma \quad \leq \sum_{i=k}^{m-1} n_i\left(\sum_{j=m}^{n-1} w(i,j)n_j + w(i,m-1)(i-k)\right) + \sum_{m \leq i < j \leq n-1} w(i,j)x_{i,j}.$$

Denote $g(i) = \sum\limits_{j=m}^{n-1} w(i,j)n_j + w(i,m-1)(i-k)$ and $f(x) = x(\frac{1}{x} + \frac{1}{y} - \frac{4}{x+y})$ for $0 < x < y$, then $f'(x) = \frac{1}{y} - \frac{4y}{(x+y)^2} < 0$. We have

$$k \cdot w(k,j) \geq i \cdot w(i,j), \text{ for } k \leq i \leq m-1, m \leq j \leq n-1.$$

Therefore,

$$\begin{aligned} g(i) &\le \frac{k}{i}\left(\sum_{j=m}^{n-1} w(k,j)n_j + w(k,m-1)(i-k)\right) \\ &= \sum_{j=m}^{n-1} w(k,j)n_j + \frac{i-k}{i}\left(w(k,m-1)k - \sum_{j=m}^{n-1} w(k,j)n_j\right) \\ &\le \sum_{j=m}^{n-1} w(k,j)n_j, \end{aligned}$$

because $\sum\limits_{j=m}^{n-k} n_j = k$ and $w(k,m-1) < w(k,j)$ for $m \le j \le n-1$. Since $\sum\limits_{j=k}^{m-1} n_j = n-k$, we have

$$\begin{aligned} &\sum_{i=k}^{m-1} g(i)n_i \le \sum_{i=k}^{m-1} n_i\left(\sum_{j=m}^{n-1} w(k,j)n_j\right) \\ &= (n-k)\sum_{j=m}^{n-1} w(k,j)n_j \\ &= w(k,n-1)(n-k)n_{n-1} + \sum_{j=m}^{n-2} w(k,j)(n-k)n_j \\ &= w(k,n-1)(n-k)(k - \sum_{j=m}^{n-2} n_j) + \sum_{j=m}^{n-2} w(k,j)(n-k)n_j \\ &= w(k,n-1)(n-k)k + \sum_{j=m}^{n-2}\Big(w(k,j) - w(k,n-1)\Big)(n-k)n_j. \end{aligned}$$

Since $x_{i,j} \le n_i n_j$ for $m \le i < j \le n-1$ and $n_{n-1} = k - \sum\limits_{j=m}^{n-2} n_j$, we have:

$$\begin{aligned} &\sum_{m\le i<j\le n-1} w(i,j)x_{i,j} \le \sum_{m\le i<j\le n-1} w(i,j)n_i n_j = \sum_{i=m}^{n-2} w(i,n-1)n_i n_{n-1} + \sum_{m\le i<j\le n-2} w(i,j)n_i n_j \\ &= \sum_{i=m}^{n-2} w(i,n-1)n_i(k - \sum_{j=m}^{n-2} n_j) + \sum_{m\le i<j\le n-2} w(i,j)n_i n_j \\ &= k\sum_{i=m}^{n-2} w(i,n-1)n_i - \sum_{i=m}^{n-2}\sum_{j=m}^{n-2} w(i,n-1)n_i n_j + \sum_{m\le i<j\le n-2} w(i,j)n_i n_j \\ &= k\sum_{i=m}^{n-2} w(i,n-1)n_i - \sum_{i=m}^{n-2} w(i,n-1)n_i^2 - \sum_{m\le i<j\le n-2}\Big(w(i,n-1) + w(j,n-1)\Big)n_i n_j + \sum_{m\le i<j\le n-2} w(i,j)n_i n_j \\ &= k\sum_{i=m}^{n-2} w(i,n-1)n_i - \sum_{i=m}^{n-2} w(i,n-1)n_i^2 - \sum_{m\le i<j\le n-2}\Big(w(i,n-1) + w(j,n-1) - w(i,j)\Big)n_i n_j \end{aligned}$$

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Thus

$$
\begin{aligned}
\gamma \quad & \leq w(k,n-1)(n-k)k + \sum_{j=m}^{n-2}\Big(\big(w(k,j)-w(k,n-1)\big)(n-k)+w(j,n-1)k\Big)n_j - \sum_{j=m}^{n-2} w(j,n-1)n_j^2 \\
& - \sum_{m\leq i<j\leq n-2}\Big(w(i,n-1)+w(j,n-1)-w(i,j)\Big)n_i n_j \\
& \leq w(k,n-1)(n-k)k + \sum_{j=m}^{n-2}\Big(w(k,j)-w(k,n-1)+w(j,n-1)\Big)(n-k)n_j \\
& - \sum_{m\leq i<j\leq n-2}\Big(w(i,n-1)+w(j,n-1)-w(i,j)\Big)n_i n_j \\
& \leq w(k,n-1)(n-k)k,
\end{aligned}
$$

where the second inequality holds because $k \leq \frac{n}{2} \leq n-k$ and $w(j,n-1) > 0$ for $m \leq j \leq n-2$. The last inequality holds because $h_1(j) = w(k,j) - w(k,n-1) + w(j,n-1) = \frac{2}{j} - \frac{4}{k+j} + \frac{4}{k+n-1} - \frac{4}{j+n-1} = \frac{2(k-j)}{j(k+j)} + \frac{4(j-k)}{(j+n-1)(k+n-1)} = 2(j-k)(\frac{2}{(j+n-1)(k+n-1)} - \frac{1}{j(k+j)}) = \frac{2(j-k)}{j(k+j)(j+n-1)(k+n-1)}\Big(k(j-n+1)+(j^2 - j(n-1)) + (j^2-(n-1)^2)\Big) < 0$.

Denote $h_2(i,j) = w(i,n-1)+w(j,n-1)-w(i,j) = \frac{2}{n-1} + \frac{4}{i+j} - \frac{4}{i+n-1} - \frac{4}{j+n-1}$. We consider i and j as continuous variables, then $\frac{\partial h_2(i,j)}{\partial i} = -\frac{4}{(i+j)^2} + \frac{4}{(i+n-1)^2} < 0$ and $\frac{\partial h_2(i,j)}{\partial j} = -\frac{4}{(i+j)^2} + \frac{4}{(j+n-1)^2} < 0$, thus $h_2(i,j) > h_2(n-1,n-1) = 0$. Equality holds when $n_j = 0$ for $k+1 \leq j \leq n-2, n_k = n-k, n_{n-1} = k, x_{k,n-1} = (n-k)k, x_{n-1,n-1} = \binom{k}{2}$, and all other $x_{i,j}$ are equal to zero. Thus, the graph for which the harmonic index attains its minimum value is $K^*_{k,n-k}$.

Subcase (b). We put $n_m = n_{m'} + n_{m''}$, such that $n_{m''} + n_{m+1} + \cdots + n_{n-1} = k$. Then $n_k + \cdots + n_{m-1} + n_{m'} = n-k$. We will color the vertices of degree m with red and white, such that the number of red vertices is $n_{m''}$. Denote by $x_{i,m'}(x_{i,m''})$ for $i \neq m$, the number of edges between vertices of degree i and the white (red) vertices of degree m, by $x_{m',m'}(x_{m'',m''})$ the number of edges between white (red) vertices of degree m, and by $x_{m',m''}$ the number of edges between white and red vertices of degree m. Then $x_{i,m} = x_{i,m'} + x_{i,m''}$ for $i \neq m$, and $x_{m,m} = x_{m',m'} + x_{m',m''} + x_{m'',m''}$. We will replace system (2.8) by:

$$
\begin{aligned}
\sum_{k\leq j\leq n-1, j\neq i} x_{i,j} + 2x_{i,i} = in_i, \text{ for } k \leq i \leq n-1, i \neq m, \\
\sum_{k\leq j\leq n-1, i\neq m} x_{i,m'} + x_{m',m''} + 2x_{m',m'} = mn_{m'}, \\
\sum_{k\leq j\leq n-1, i\neq m} x_{i,m''} + x_{m',m''} + 2x_{m'',m''} = mn_{m''}.
\end{aligned}
$$

We will proceed similarly as in the Subcase (a) and we omit the rest of the proof. □

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An edge x_1x_2 is called *local maximum* if its weight $\frac{2}{d(x_1)+d(x_2)}$ is maximum in its neighborhood, i.e., $\frac{2}{d(x_1)+d(x_2)} \geq \frac{2}{d(x_i)+d(u)}$ for any edge x_iu for $i = 1, 2$.

Lemma 3.2. *Let x_1x_2 be a local maximum edge in graph G, then*

$$H(G) - H(G - x_1x_2) > 0.$$

Proof. We have

$$\begin{aligned}
H(G) - H(G - x_1x_2) &= \frac{2}{d(x_1)+d(x_2)} + \sum_{u \in N(x_1)\setminus\{x_2\}} \left(\frac{2}{d(x_1)+d(u)} - \frac{2}{d(x_1)+d(u)-1} \right) \\
&+ \sum_{v \in N(x_2)\setminus\{x_1\}} \left(\frac{2}{d(x_2)+d(v)} - \frac{2}{d(x_2)+d(v)-1} \right) \\
&\geq \frac{2}{d(x_1)+d(x_2)} + \big(d(x_1)-1\big) \left(\frac{2}{d(x_1)+d(x_2)} - \frac{2}{d(x_1)+d(x_2)-1} \right) \\
&+ \big(d(x_1)-1\big) \left(\frac{2}{d(x_1)+d(x_2)} - \frac{2}{d(x_1)+d(x_2)-1} \right) \\
&= \frac{2}{\big(d(x_1)+d(x_2)-1\big)\big(d(x_1)+d(x_2)\big)} > 0,
\end{aligned}$$

where $N(x_i)$ denotes the vertex set adjoining to x_i for $i = 1, 2$. $\square$

For an edge x_1x_2 associates with a leaf, i.e. a vertex with degree one, we can see that it is a local maximum edge. Thus by Lemma 3.2 we have

Corollary 3.3. *If x_1x_2 is an edge in graph G such that x_1 is a leaf, then*

$$H(G) - H(G - x_1x_2) > 0.$$

For a graph $G \in \mathcal{G}(n, k)$. If $\delta(G) > k$, we always can find a local maximum edge uv and delete it. Then $(G) \geq (G - uv)$ and $H(G) > H(G - uv)$. This is the reason why the minimum degree of the extremal graphs which attaining the minimum value of the harmonic index among $\mathcal{G}(n, k)$ is k. Thus the problem of determining the minimum value of the harmonic index among $\mathcal{G}(n, k)$ is equal with the corresponding problem among n-vertex graphs with the minimum degree k.

4. **Conclusion**

We would like to conclude the problem of determining the minimum value of the harmonic index among $\mathcal{G}(n, k)$ with a complete answer. To do this, we need some notations. For two vertex disjoint graphs, G and F, let $G + F$ denote their join, i.e., the graph obtained by joining edges from every vertex of G to all vertices of F. One of the authors, Liu [2] proved that every graph with the minimum value of the harmonic index in $G(n, k)$ for $k \geq n/2$ is a join between a regular graph and a complete graph. Let $\mathbf{F}(n, r)$ be the set of r-regular n-vertex graphs. We have $\mathbf{F}(n, r) \neq \emptyset$ if and only if $r \leq n-1$ and nr is even (see for example [3]).

Theorem 4.1. [2] *If G is an extremal graph attaining the minimum value of the harmonic index among $\mathcal{G}(n,k)$ for $k \geq n/2$, then G is a join between a regular graph and a complete graph, namely:*
(1) If n is even, then: $H(G) = \frac{n}{2} - \frac{n^2}{8}w(k,n-1)$ and $G \cong F + K_{n/2}$, where $F \in \mathbf{F}(\frac{n}{2}, k - \frac{n}{2})$ for k even and $n = 0$ (mod 4), k odd and $n = 2$ (mod 4); $H(G) = \frac{n}{2} - \frac{n^2-4}{8}w(k,n-1)$ and $G \cong F + K_{\frac{n+2}{2}}$, where $F \in \mathbf{F}(\frac{n-2}{2}, k - \frac{n+2}{2})$ if $k \geq \frac{n+2}{2}$ for k even and $n = 2$ (mod 4), or $H(G) = \frac{n}{2} - \frac{n^2-4}{8}w(k,n-1)$ and $G \cong F + K_{\frac{n-2}{2}}$, where $F \in \mathbf{F}(\frac{n+2}{2}, k - \frac{n-2}{2})$ for k even and $n = 2$ (mod 4).
(2) If n is odd, then: $H(G) = \frac{n}{2} - \frac{n^2-1}{8}w(k,n-1)$ and $G \cong F + K_{\frac{n+1}{2}}$, where $F \in \mathbf{F}(\frac{n-1}{2}, k - \frac{n+1}{2})$ if $k \geq \frac{n+1}{2}$ for k even, k odd and $n = 1$ (mod 4), or $H(G) = \frac{n}{2} - \frac{n^2-1}{8}w(k,n-1)$ and $G \cong F + K_{\frac{n-1}{2}}$, where $F \in \mathbf{F}(\frac{n+1}{2}, k - \frac{n-1}{2})$ for k even, k odd and $n = 3$ (mod 4).

In short, the above theorem states that the extremal graph attaining the minimum value of the harmonic index among $\mathcal{G}(n,k)(k \geq n/2)$ has only two degrees, i.e., degree k and degree $n-1$, and n_k is as close to $n/2$ as possible. Therefore we conclude the paper with the following simple one. Reminding that n_k is the number of vertices of degree k.

Theorem 4.2. *The extremal graph attaining the minimum value of the harmonic index among $\mathcal{G}(n,k)$ has only two degrees, i.e., degree k and degree $n-1$, and $n_k = n-k$ for $1 \leq k < n/2$ and n_k it is as close to $n/2$ as possible for $n/2 \leq k \leq n-1$.*

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References

[1] O. Favaron, M. Mahó and J.F. Saclé, Some eigenvalue properties in graphs (conjectures of Graffiti-II), *Discrete Math.*, **111** (1993) 197–220.

[2] J. Liu, Harmonic index of dense graphs, *Ars Combinatoria*, **120** (2015) 293–304.

[3] L. Lovász, *Combinatorial problems and exercises*, Akadémiai Kiadó, Budapest, 1979.

[4] X. Li and I. Gutman, Mathematical Aspects of Randić-Type Molecular Structure Descriptors, *Mathematical Chemistry Monographs*, Kragujevac, (2006) pp. VI+330.

[5] X. Li and Y. T. Shi, A survey on the Randić index, *MATCH Commun. Math. Comput. Chem.*, **59** (2008) 127–156.

[6] B. Lučić, N. Trinajstić and B. Zhou, Comparison between the sum-connectivity index and product-connectivity index for benzenoid hydrocarbons, *Chem. Phys. Lett.*, **475** (2009) 146–148.

[7] R. Wu, Z. Tang and H. Deng, A lower bound for the harmonic index of a graph with minimum degree at least two, *Filomat*, **27** (2013) 51–55.

[8] L. Zhong, The harmonic index for graphs, *Appl. Math. Lett.*, **25** (2012) 561–566.

[9] L. Zhong, The harmonic index on unicyclic graphs, *Ars Combin.*, **104** (2012) 261–269.

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