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ON MATRIX AND LATTICE IDEALS OF DIGRAPHS

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Abstract. Let G be a simple, oriented connected graph with n vertices and m edges. Let $I(\mathbf{B})$ be the binomial ideal associated to the incidence matrix $\mathbf{B}$ of the graph G. Assume that I_L is the lattice ideal associated to the rows of the matrix $\mathbf{B}$. Also let $\mathbf{B}_i$ be a submatrix of $\mathbf{B}$ after removing the i-th row. We introduce a graph theoretical criterion for G which is a sufficient and necessary condition for $I(\mathbf{B}) = I(\mathbf{B}_i)$ and $I(\mathbf{B}_i) = I_L$. After that we introduce another graph theoretical criterion for G which is a sufficient and necessary condition for $I(\mathbf{B}) = I_L$. It is shown that the heights of $I(\mathbf{B})$ and $I(\mathbf{B}_i)$ are equal to $n-1$ and the dimensions of $I(\mathbf{B})$ and $I(\mathbf{B}_i)$ are equal to $m-n+1$; then $I(\mathbf{B}_i)$ is a complete intersection ideal.

1. Introduction

It is so difficult to get information about the structure of an ideal or scheme by directly examining its defining polynomials. But A good description of generators of an ideal of a ring, provides effective tools to study the structure of the ideal. Monomial ideals in polynomial rings with coefficients in a field are studied from this point of view and by using Gröbner bases, these ideals have been used to study general ideals. The other ideals which are important are binomial ideals. Let $\mathbb{K}$ be a field. A binomial is an element of a polynomial ring $\mathbb{K}[X] = \mathbb{K}[x_1, \ldots, x_m]$ with at most two terms; a binomial ideal is one whose generators can be chosen binomial. This class of ideals introduced by

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Eisenbud and Sturmfels in 1995, and since then, these type of ideals, has been become the subject of intensive research in Algebraic Geometry and Commutative Algebra. Since this class of ideals is the meeting point of several branches of mathematics, i.e., linear algebra, combinatorics, commutative algebra and algebraic geometry, provide the possibility to transfer a knowledge about one concept in a filed to other field and formulate and prove a good theorem. The zero set of binomial ideals are unions of toric varieties, which makes binomial ideals important in algebraic geometry, and is one reason that combinatorial methods are very effective for studying them. These type of ideals has been used to model some phenomena in biology (phylogenetic models), physics (string theory), statistics (independence variables in joint distributions) and many new emerging applications in other areas of mathematics. The importance of the binomial ideals encourages mathematicians to study these ideals deeper and in more details [see [6]].

Beyond their intrinsic interest, binomial ideals arise naturally in various contexts, such as combinatorial game theory, algebraic statistics and dynamics of mass action kinetics. These ideals are also very important in the study of hypergeometric differential equations, as is shown in [1],[3].

In this paper, special binomial ideals are considered which are obtained from the incidence matrix of a connected oriented graph.

Throughout this paper, we assume G is a simple, connected oriented graph with n vertices and m edges. I.e., each edge e of G is an ordered pair (u, v), where u is called the tail of the edge and v is called the head of the edge. Also we let every vertex of this graph has both input and output edges. Moreover, let $\mathbf{B} = (h_{ij})$ be the incidence matrix of G, whose entries is defined as

$$h_{ij} = \begin{cases} -1 & \text{if } e_j \text{ exits from } v_i, \\ 1 & \text{if } e_j \text{ enters to } v_i, \\ 0 & \text{otherwise.} \end{cases}$$

Also let $\mathbf{B}_i$ be a submatrix of $\mathbf{B}$ after removing the i-th row. In addition, assume L is the lattice generated by the rows of $\mathbf{B}$.

In this paper a graph theoretical criterion is defined for the graph G which is a sufficient and necessary condition that guarantees equality of lattice basis ideals of matrices $\mathbf{B}$ and $\mathbf{B}_i$. Also other graph theoretical criterion for the graph G is introduced that again is a sufficient and necessary condition for the lattice basis ideal of the matrix $\mathbf{B}$ to be prime. Finally it is shown that the hieght and dimension of the lattice basis ideal of the matrices $\mathbf{B}$ and $\mathbf{B}_i$ can be computed just by the numbers of the vertices and edges of the graph.

2. Preliminaries

A lattice $L \subseteq \mathbb{Z}^m$ is called saturated, if for any $b \in \mathbb{Z}, \mathbf{Y} \in \mathbb{Z}^m$; $b\mathbf{Y} \in L$ implies that $\mathbf{Y} \in L$. Suppose $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ is a basis for the saturated lattice L, and $\mathbf{B}$ is a matrix with rows $\mathbf{u}_1, \ldots, \mathbf{u}_n$.

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Two binomial ideals are associated to the lattice and the matrix:

$$I_L = \langle X^{\mathbf{u}} - X^{\mathbf{v}} | \mathbf{u}, \mathbf{v} \in \mathbb{N}^m, \mathbf{u} - \mathbf{v} \in L \rangle$$

and

$$I(\mathbf{B}) = \langle X^{\mathbf{u}_i+} - X^{\mathbf{u}_i-} | \mathbf{u}_i = \mathbf{u}_i + -\mathbf{u}_i -, 1 \leq i \leq n \rangle,$$

where $\mathbf{u}+$ and $\mathbf{u}-$ are, respectively, the positive and negative parts of the vector $\mathbf{u}$. That is, if $(\mathbf{u})_i$ is the i-th coordinate of $\mathbf{u}$, then we set the i-th coordinate $(\mathbf{u}+)_i$, of $\mathbf{u}+$ equal to $\max\{0, (\mathbf{u})_i\}$. Similarly, $(\mathbf{u}-)_i = \max\{0, -(\mathbf{u})_i\}$. Also for any $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{Z}^m_{\geq 0}$, we denote the monomial $x_1^{\alpha_1} \ldots x_m^{\alpha_m}$ by X^α.

From definitions of these two ideals, it follows that $I(\mathbf{B}) \subset I_L$ and an important question in this regard (see [8, Remark 2.3]), is to find necessary and sufficient conditions for which this inclusion being equality. In this paper, we are going to answer this question for a special class of incidence matrices coming from oriented graphs.

For our main results, we require that the matrix $\mathbf{B}$ be in a special form, which were defined in [7], For ease of reference, we quote them here:

Definition 2.1. *A matrix $\boldsymbol{B}$ is called **mixed** if every row of $\boldsymbol{B}$ contains both a positive and negative entry.*

In other words, if we consider each row of the matrix $\mathbf{B}$ as a vector $\mathbf{u}$, then this definition means that the conditions $\mathbf{u}+ \neq 0$ and $\mathbf{u}- \neq 0$ should be hold.

Definition 2.2. *A matrix $\boldsymbol{B}$ is called **dominating** if it does not contain a square mixed submatrix.*

By using this notion, a simple characterization for the equality $I(\mathbf{B}) = I_L$ is given, i.e., $I(\mathbf{B}) = I_L$ if and only if $\mathbf{B}$ is a dominating matrix [7, Theorem 2.9].

In [8] it is proved that I_L is a minimal prime of $I(\mathbf{B})$. Moreover, in [8], by introducing the following notion, the authors determined other minimal prime ideals of $I(\mathbf{B})$.

Definition 2.3. *A matrix $\boldsymbol{B}$ is called **irreducible** if:*

(1) *$\boldsymbol{B}$ is a mixed $s \times t$ matrix where $t \leq s$, and*
(2) *one cannot bring $\boldsymbol{B}$ into the following form after permutation its rows and columns,*

$$\boldsymbol{B} = \begin{pmatrix} \boldsymbol{N'} & \boldsymbol{M'} \\ \boldsymbol{0} & \boldsymbol{D'} \end{pmatrix},$$

where $\boldsymbol{N'}$ is a $s' \times t'$ mixed matrix with $t' \leq s'$ and $\boldsymbol{D'}$ is a $(s - s') \times (t - t')$ matrix with $t - t' \geq s - s'$.

In [8, Theorem 2.1], it was proved that P is a minimal prime of $I(\mathbf{B})$ if and only if P corresponds to the following decomposition in which $\mathbf{N}$ is irreducible;

$$\mathbf{B} = \begin{pmatrix} \mathbf{N} & \mathbf{M} \\ \mathbf{0} & \mathbf{D} \end{pmatrix}.$$

3. Main results

Since the sum of the columns of the incidence matrix $\mathbf{B}$ is zero, the rows of this matrix are not independent, and previous definitions are applied to full rank matrices, so we can remove arbitrarily any row of $\mathbf{B}$, say i-th row and denote the new matrix by $\mathbf{B}_i$. Notice that for all $i \in \{1, \cdots, n\}$,

$$-\mathbf{u}_i = \mathbf{u}_1 + \cdots + \mathbf{u}_{i-1} + \mathbf{u}_{i+1} + \cdots + \mathbf{u}_n.$$

Lemma 3.1. *The lattice L is saturated.*

Proof. Since every column of the matrix $\mathbf{B}$ has one $+1$ and one -1, the matrix $\mathbf{B}$ is a totally unimodular matrix, i.e., all subdeterminants equal to $1, -1$ or 0 [10, Theorem 19.3] . Now by [10, Corollary 4.1c], the claim is proved. $\square$

Let $A = \{1, \ldots, n\}$, and $\mathrm{P}(A)$ be the set of all nonempty subsets of A and $\mathbf{u}_1, \ldots, \mathbf{u}_n \in \mathbb{Z}^m$. For any $\mathbf{u} \in \mathbf{Z}^m$ We denote $X^{\mathbf{u}+} - X^{\mathbf{u}-}$ by $F(\mathbf{u})$ and for any $W \in \mathrm{P}(A)$, $\mathbf{u}_W = \sum_{w \in W} \mathbf{u}_w$ and $I(\mathrm{P}(A)) = \langle F(\mathbf{u}_W) | W \in \mathrm{P}(A) \rangle$. Now suppose $\mathbf{u}_1, \ldots, \mathbf{u}_n \in \mathbb{Z}^m$ are the rows of $\mathbf{B}$. By the following proposition we can provide a specific finite set of generators for the ideal I_L .

Proposition 3.2. *With the above mentioned notations $I(P(A)) = I_L$.*

Proof. It is clear that $I(P(A)) \subset I_L$. Hence we should show the reverse inclusion. Let $\mathbf{w} \in L$, then we can write $\mathbf{w} = \sum_{i=1}^{n} \lambda_i \mathbf{u}_i$. Three cases may arise.

Case 1: For every $1 \leq i \leq n$, let $\lambda_i \geq 0$. By induction on $\lambda = \max_i \lambda_i$, we will prove $F(\mathbf{w}) \in I(\mathrm{P}(A))$. If $\lambda = 1$ the assertion is clear. Suppose $\lambda > 1$. Put $\mathbf{w}_1 = \sum_{\lambda_i > 0} (\lambda_i - 1)\mathbf{u}_i$ and $\mathbf{w}_2 = -\sum_{\lambda_i = 0} \mathbf{u}_i$, then $\mathbf{w} = \mathbf{w}_1 + \mathbf{w}_2$. By induction hypothesis, $F(\mathbf{w}_1)$ and $F(\mathbf{w}_2)$ are contained in $I(\mathrm{P}(A))$. If for some $1 \leq j \leq m$, the j-th entry of $\mathbf{w}_2$ is less than 0, then the j-th entry of $\mathbf{w}_1$ is less than or equal to 0, hence by [4, Lemma 1.4], $F(\mathbf{w}) \in I(\mathrm{P}(A))$.

Case 2: Let for all i, $\lambda_i \leq 0$. This case will follow by a similar argument as the case 1.

Case 3: Let for all i, $1 \leq i \leq r$, $\lambda_i \geq 0$ and for all i, $r+1 \leq i \leq n$, $\lambda_i \leq 0$. Put $\mathbf{a} = \sum_{i=1}^{r} \lambda_i \mathbf{u}_i$ and $\mathbf{b} = \sum_{i=r+1}^{n} \lambda_i \mathbf{u}_i$, then $\mathbf{w} = \mathbf{a} + \mathbf{b}$. By case 1, $F(\mathbf{a}) \in I(\mathrm{P}(A))$ and by case 2, $F(\mathbf{b}) \in I(\mathrm{P}(A))$. Let the k-th entry of $\mathbf{a}$ be positive, then the k-th entry of $\mathbf{b}$ is positive or 0, hence by [4, Lemma 1.4], the claim will be followed. $\square$

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Let $L = \langle \mathbf{u}_1, \ldots, \mathbf{u}_n \rangle$ where $\mathbf{u}_i$'s are the rows of the incidence matrix $\mathbf{B}$ of the directed graph G. Also let $Q_1, \ldots, Q_s$ be all directed cycles of G and vertices of every Q_i be $v_{i_1}, \ldots, v_{i_{r_i}}$, and,

$$f_{Q_i} = f_{\mathbf{u}_{i_1}+\cdots+\mathbf{u}_{i_{r_i}}} = F(\mathbf{u}_{i_1} + \cdots + \mathbf{u}_{i_{r_i}}).$$

Since G is a simple directed graph, by [4, Lemma 1.4], it is obvious that,

$$f_{\mathbf{u}_i+\mathbf{u}_j} \in \langle f_{\mathbf{u}_i}, f_{\mathbf{u}_j} \rangle, \quad \text{for all} \quad i, j; \quad 1 \leq i, j \leq n, \quad i \neq j.$$

So we can delete all $f_{\mathbf{u}_i+\mathbf{u}_j}$'s from the generating set of $I(\mathrm{P}(A))$. Now consider $\mathbf{u}_i, \mathbf{u}_j, \mathbf{u}_k$. The only case in which the hypotheses of [4, Lemma 1.4], are not hold, is that the associated vertices v_i, v_j, v_k form a directed cycle in G. Otherwise

$$f_{\mathbf{u}_i+\mathbf{u}_j+\mathbf{u}_k} \in \langle f_{\mathbf{u}_i}, f_{\mathbf{u}_j}, f_{\mathbf{u}_k} \rangle.$$

By arguing in this way, we obtain:

$$I_L = \langle f_{\mathbf{u}_1}, \ldots, f_{\mathbf{u}_n}, f_{Q_1}, \ldots, f_{Q_s} \rangle.$$

By our assumption regarding the graph G, it is clear that $L = \langle \mathbf{u}_1, \ldots, \mathbf{u}_{i-1}, \mathbf{u}_{i+1}, \ldots, \mathbf{u}_n \rangle$ for every $1 \leq i \leq n$. Moreover, let v_i be any vertex of the graph and $Q_{c_1}, \ldots, Q_{c_t}$ be the cycles, if any, of $G \backslash \{v_i\}$. Then we can again reduce the generators of the I_L. This argument can be considered as a proof for the following proposition.

Theorem 3.3. *With the above assumptions, we have:*

$$I_L =< f_{\boldsymbol{u}_1}, \ldots, f_{\boldsymbol{u}_{i-1}}, f_{\boldsymbol{u}_{i+1}}, \ldots, f_{\boldsymbol{u}_n}, f_{Q_{c_1}}, \ldots, f_{Q_{c_t}} > .$$

Remark 3.4. *Note that the generators of I_L, for which proposed in Theorem 3.3, is not minimal. For example, let v_i be the vertex of G which occurs in the most cycles of G, then the graph $G \backslash \{v_i\}$ has the least number of cycles, and hence the number of the generators in the right hand side of the above equality can be lowered.*

Definition 3.5. *If G has a vertex v_i such that by removing it, the graph $G \backslash \{v_i\}$ does not have any directed cycle, then G is called **i-cycleless**.*

Remark 3.6. *Theorem 3.3 shows that if the graph G is an i-cyclelss graph, then we have:*

$$I_L = I(\boldsymbol{B}_i).$$

The converse of this fact will be proved in Proposition 3.15.

Definition 3.7. *An edge of a oriented graph G is called a **hanging edge** of a cycle C in G if it shares only one of its vertices with C.*

*We break down the set of hanging edges of an oriented graph G with respect to a cycle C in G, into the **inward hanging edges** and **outward hanging edges**.*

Theorem 3.8. *Let G be a graph with n vertices and m edges. Then G is an i-cycleless graph if and only if $I(\boldsymbol{B}) = I(\boldsymbol{B}_i)$.*

Proof. Let G be an i-cycleless graph. By Theorem 3.3, $I(\mathbf{B}_i) = I_L$, since $I(\mathbf{B}_i) \subseteq I(\mathbf{B}) \subseteq I_L$ so $I(\mathbf{B}_i) = I(\mathbf{B})$.

Now suppose that for some i, $I(\mathbf{B}_i) = I(\mathbf{B})$. For simplification of the argument let $i = n$. Assume $G \backslash \{v_n\}$ has the following directed cycle, $Q = \{v_{i_1}, \dots, v_{i_r}\}$. Again, for simplification of the argument let $Q = \{v_1, \dots, v_r\}$. Assume that the inward hanging edges of Q enter into the vertices, $\{v_{i_1}, \dots, v_{i_s}\}$, and outward hanging edges of Q exit from the vertices, $\{v_{j_1}, \dots, v_{j_t}\}$. Then we have:

$$f_Q = x_{i_1} \dots x_{i_s} - x_{j_{s+1}} \dots x_{j_t}.$$

Since for all i, $1 \leq i \leq r$, we have:

$$X^{\mathbf{u}_i+} \not| x_{i_1} \dots x_{i_s} \quad , \quad X^{\mathbf{u}_i-} \not| x_{i_1} \dots x_{i_s}$$

and

$$X^{\mathbf{u}_i+} \not| x_{j_{s+1}} \dots x_{j_t} \quad , \quad X^{\mathbf{u}_i-} \not| x_{j_{s+1}} \dots x_{j_t}.$$

Hence f_Q can not be in $\langle f_{\mathbf{u}_1}, \dots, f_{\mathbf{u}_r} \rangle$.

Now let v_{r+1} be one of the vertices (outside of the cycle Q) into which some outward hanging edges of the cycle enters. Suppose that;

$$f_{Q+\mathbf{u}_{r+1}} = X^{\mathbf{a}+} - X^{\mathbf{a}-}.$$

Since Q and v_{r+1} are adjacent with another vertices of the graph, we have:

$$X^{\mathbf{u}_i+} \not| X^{a+} \quad , \quad X^{\mathbf{u}_i-} \not| X^{a+} \quad \text{for all i with} \quad 1 \leq i \leq r+1,$$

or

$$X^{\mathbf{u}_i+} \not| X^{a-} \quad , \quad X^{\mathbf{u}_i-} \not| X^{a-} \quad \text{for all i with} \quad 1 \leq i \leq r+1.$$

Thus

$$f_{Q+\mathbf{u}_{r+1}} \notin \langle f_{\mathbf{u}_1}, \dots, f_{\mathbf{u}_{r+1}} \rangle.$$

If we continue arguing in this way, at first for vertices which some outward hanging edges of the cycle Q enters to those, and second for vertices which are not adjacent to the cycle Q, and lastly for other vertices, it follows that:

$$f_{Q+\mathbf{u}_{r+1}+\dots+\mathbf{u}_{n-1}} \notin \langle f_{\mathbf{u}_1}, \dots, f_{\mathbf{u}_{n-1}} \rangle.$$

As we know $-\mathbf{u}_n = \mathbf{u}_1 + \dots + \mathbf{u}_{n-1}$, so

$$f_{\mathbf{u}_n} \notin \langle f_{\mathbf{u}_1}, \dots, f_{\mathbf{u}_{n-1}} \rangle,$$

which is a contradiction, hence the claim is proved. Finally it is easy to see that if the both sets of inward and outward hanging edges are empty simultaneously, or one of them is empty, the above argument still works. □

Notation 3.9. *We denote the set of edges associated to* $\boldsymbol{u}_k+, \boldsymbol{u}_k-$ *by* $\{\boldsymbol{u}_k+\}$, $\{\boldsymbol{u}_k-\}$ *respectively, that is* $e_i \in \{\boldsymbol{u}_k+\}$ *and* $e_l \in \{\boldsymbol{u}_k-\}$ *if* $(\boldsymbol{u}_k+)_i$ *and* $(\boldsymbol{u}_k-)_l$ *be one respectively. In fact, in the incidence matrix, each column is associated to an edge. Hence* $\{\boldsymbol{u}_k+\}$ *denotes the set of edges of the graph which enter into* v_k*, and* $\{\boldsymbol{u}_k-\}$ *denotes the set of edges of the graph which exit from the vertex* v_k*.*

Definition 3.10. *A directed cycle* Q *in a graph* G *is called* ***terminal*** *if there exist some vertices such that all outward edges of these vertices enter into the cycle* Q*, and there exist some vertices such that all inward edges of these vertices exit from the cycle* Q*.*

A vertex for which all outward edges from it enter into the cycle, is called a ***source vertex****. Moreover, a vertex for which all inward edges into it come from the cycle, is called a* ***sink vertex****.*

Let v_r be one of the sink vertices of the cycle and v_{r+t} be one of the source vertices of the cycle. Also let for all j, $0 \leq j \leq t-1$,

$$\psi_{r+j} \bigcup \{\mathbf{u}_{r+j}-\} = \psi_{r+j+1} \bigcup \{\mathbf{u}_{r+j+1}+\}$$

where ψ_r is the set of outward hanging edges of the cycle which don't enter into v_r, and for all j, $1 \leq j \leq t-1$,

$$\psi_{r+j} = \lambda_Q^{r+j} \cup \bigcup_{k=r}^{r+j+1} \lambda_k^{r+j}.$$

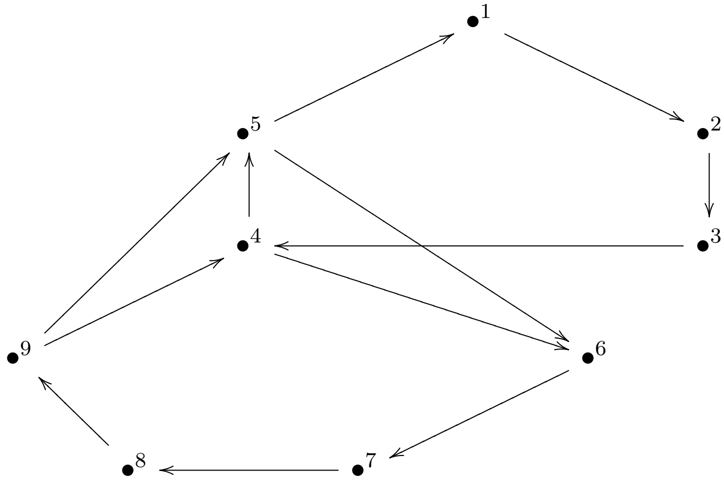
The set λ_Q^{r+j} is the set of outward hanging edges of the cycle which don't enter into v_l's, $r \leq l \leq r+j$, and λ_k^{r+j} is the set of outward edges of v_k which don't enter into v_l's, $k+1 \leq l \leq r+j$.

In other words, there is a labeling for the vertices which don't contain in the cycle, such as $k+1 \leq l \leq r+j$, such that v_r is one of the sink vertices of the cycle, v_{r+t} is one of the source vertices of the cycle, and all inward edges of v_{r+j}, $0 \leq j \leq t$, exit from $Q, v_r, \ldots, v_{r+j-1}$.

Definition 3.11. *The directed cycle* C *which is satisfied the above properties, is called* ***neat terminal****.*

Definition 3.12. *We call a graph* ***neat terminal*** *if all directed cycles of the graph is neat terminal. Also, we call the graph* i***-neat terminal*** *if the graph has a vertex* v_i *such that all directed cycles which don't contain* v_i*, be neat terminal.*

Example 1. *Let G be the following graph:*



This graph is a neat terminal and 7-neat terminal graph.

In general the ideal $I(\mathbf{B})$ is not prime. But the following theorem gives a criterion in terms of properties of the graph G which guarantees this ideal would be prime.

Theorem 3.13. *The ideal $I(\boldsymbol{B})$ is a prime ideal if and only if the graph G is a neat terminal graph.*

Proof. Let G be a neat terminal graph. By Theorem 3.3,

$$I_L = \langle f_{\mathbf{u}_1}, \dots, f_{\mathbf{u}_{n-1}}, f_{\mathbf{u}_n}, f_{Q_1}, \dots, f_{Q_t} \rangle,$$

where $Q_1, \dots, Q_t$ are all the cycles of G. Since G is a neat terminal graph, we have:

$$-f_{Q_i} = X^{\psi_r} f_{\mathbf{u}_r} + X^{\psi_{r+1}} f_{\mathbf{u}_{r+1}} + \cdots + X^{\psi_{r+j}} f_{\mathbf{u}_{r+j}} + \cdots + X^{\psi_{r+t}} f_{\mathbf{u}_{r+t}},$$

where v_r is the sink vertex of the cycle, and v_{r+t} is the source vertex of the cycle, and the other v_{u_j}'s are vertices that the cycle doesn't contain. Also $X^{\psi_{r+j}}$ is the monomial associated to ψ_{r+j}, that is $X^{\psi_{r+j}} = x_1^{a_1} \dots x_m^{a_m}$, where a_i is one, if $e_i \in \psi_{r+j}$, otherwise a_i is zero.

Hence for all i, $1 \leq i \leq t$,

$$f_{Q_i} \in \langle f_{\mathbf{u}_1}, \dots, f_{\mathbf{u}_n} \rangle.$$

That is $I(\mathbf{B}) = I_L$.

Conversely, let $I(\mathbf{B}) = I_L$. By Theorem 3.3,

$$I_L = \langle f_{\mathbf{u}_1}, \dots, f_{\mathbf{u}_n}, f_{Q_1}, \dots, f_{Q_t} \rangle.$$

where $Q_1, \dots, Q_t$ are all the cycles of G.

Since $f_{Q_i} \in \langle f_{\mathbf{u}_1}, \dots, f_{\mathbf{u}_n} \rangle$, f_{Q_i} must be in the following form;

$$-f_{Q_i} = X^a - X^b,$$

where for some k and $k+s$,

$$X^{u_k+}|X^a \quad and \quad X^{u_{k+s}-}|X^b.$$

First $f_{\mathbf{u}_k}$ must multiplied by X^{ϕ_k}, where ϕ_k is a set such that,

$$X^a = X^{\phi_k} X^{u_k+}.$$

Now there must be a vertex, v_{k+1}, and a set ϕ_{k+1} such that

$$X^{\phi_k} X^{\mathbf{u}_k-} = X^{\phi_{k+1}} X^{\mathbf{u}_{k+1}+}.$$

By continuing this way, there must be the sets ϕ_{k+j}'s, such that for all j, $0 \leq j \leq s-1$, we have:

$$X^{\phi_{k+j}} X^{\mathbf{u}_{k+j}-} = X^{\phi_{k+j+1}} X^{\mathbf{u}_{k+j+1}+}.$$

Now assume that v_k and v_{k+s} are the sink and source vertices of the cycle respectively. By setting $\phi_{k+j} = \psi_{k+j}$, and by introduced labeling in the proof, the cycle is terminal. □

Corollary 3.14. *The ideal $I(\boldsymbol{B})$ is a prime ideal if and only if there is some positive integer i such that G is an i-neat terminal graph.*

Proof. Let $I(\mathbf{B})$ is a prime ideal, so $I(\mathbf{B}) = I_L$. By Theorem 3.13, G is a neat terminal graph, hence for all i, $1 \leq i \leq n$, G is an i-neat terminal.

Conversely, suppose that for some i, G is an i-neat terminal graph. By Theorem 3.3, we have:

$$I_L = \langle f_{\mathbf{u}_1}, \ldots, f_{\mathbf{u}_{i-1}}, f_{\mathbf{u}_{i+1}}, \ldots, f_{\mathbf{u}_n}, f_{Q_1}, \ldots, f_{Q_t} \rangle.$$

Therefore, $I(\mathbf{B}) = I_L$. □

Proposition 3.15. *The ideal $I(\boldsymbol{B}_i)$ is prime if and only if G is an i-cycleless graph.*

Proof. First we prove that $\mathbf{B}_i$ is a dominating matrix if and only if G is an i-cycleless graph. Let $\mathbf{B}_i$ be a dominating matrix. Since every mixed square submatrix of $\mathbf{B}_i$ is corresponded to a directed cycle in $G\backslash\{v_i\}$, so G is an i-cycleless graph.

Conversely, let G be an i-cycleless graph. Since every directed cycle of $G\backslash\{v_i\}$ is corresponded to a mixed square submatrix of $\mathbf{B}_i$, so $\mathbf{B}_i$ is a dominating matrix. Now by [7, Theorem 2.9], the claim is followed. □

Note that to determine the minimal prime ideals of $I(\mathbf{B}_i)$, we must study the decompositions of $\mathbf{B}_i$. Due to the properties of $\mathbf{B}_i$, the appropriate decomposition for $\mathbf{B}_i$ such that the block $\mathbf{N}$ being irreducible, occurs only if $\mathbf{N}$ corresponds to a q-vertices directed cycle. Suppose $\{v_{j_1}, \ldots, v_{j_q}\}$ are the q vertices of the directed cycle, then:

$$\mathbf{B}_i = \begin{pmatrix} \mathbf{N} & \mathbf{M} \\ \mathbf{0} & \mathbf{D} \end{pmatrix}, \qquad \mathrm{P} = \langle x_{j_1}, \ldots, x_{j_q} \rangle + I_{\mathbf{D}}$$

where P is a minimal prime of $I(\mathbf{B}_i)$ and $I_{\mathbf{D}}$ is the lattice ideal associated to the matrix $\mathbf{D}$.

Lemma 3.16. *With the above notations, the rank of the matrix $\boldsymbol{D}$ is equal to $n-q-1$.*

Proof. Let C_q be the q-vertices directed cycle and u be a vertex adjacent to C_q via an edge e. Since every entry of the column corresponded to e, except in the u-th corresponding row, is zero, rank of $\mathbf{D}$ will be increased by 1. Now let u be adjacent to v by e' but v not adjacent to C_q. By adding the rows corresponding to v and u, we have a column with entries 0 except one, then again the rank of $\mathbf{D}$ is increased by 1. The same argument about other vertices proves the claim. □

Lemma 3.17. *Let G be an oriented graph with n vertices and incidence matrix $\boldsymbol{B}$. Let $\boldsymbol{B}_i$ be the matrix which is obtained by removing its i-th row.*

(1) *If P is a minimal prime ideal of $I(\boldsymbol{B}_i)$ and If $P \neq I_L$ then* $\operatorname{ht} P = n-1$.
(2) *If P is a minimal prime ideal of $I(\boldsymbol{B})$ and If $P \neq I_L$ then* $\operatorname{ht} P \geq n-1$.

Proof. (1) The required equality will be followed from Lemma 2.2, Corollary 3 of [9, Chap. 5, Sec. 14], and [6, Theorem 2.2 and Corollary 2.5], .

(2) Since $I(\mathbf{B}_i) \subseteq I(\mathbf{B}) \subseteq I_L$, then by case (1), the claim is proved.

□

Lemma 3.18. *With the same assumptions as in Lemma 3.17, let L be the lattice generated by rows of $\boldsymbol{B}$. Then* $\operatorname{ht} I_L = n-1$.

Proof. By Corollary 3 of [9, Chap. 5, Sec. 5], we have $\operatorname{codim} I_L = \operatorname{ht} I_L$. Since $\mathbf{B}$ is the incidence matrix of an oriented graph, its rank is equal to $n-1$. Hence by [6][Theorem 2.2 and Corollary 2.5], we have

$$\operatorname{ht} I_L = \operatorname{codim} I_L = n-1.$$

□

The facts proved in the above two lemmas, can be collected in the following Corollary.

Corollary 3.19. *Let G be an oriented graph with n vertices and m edges. Let $\boldsymbol{B}$ be the incidence matrix of G and let $1 \leq i \leq n$ be an arbitrary integer. Let $\boldsymbol{B}_i$ be the matrix obtained by removing i-th row of* $\mathbf{B}$. *Then* $\operatorname{ht} I(\boldsymbol{B}_i) = n-1$. *Moreover, all minimal prime ideals of $I(\boldsymbol{B}_i)$ have the same hight, i.e., $I(\boldsymbol{B}_i)$ is unmixed.*

Corollary 3.20. *With the assumptions as above,* $\operatorname{ht} I(\boldsymbol{B}) = n-1$.

Proof. Since $I(\mathbf{B}_i) \subseteq I(\mathbf{B}) \subseteq I_L$, hence $\operatorname{ht} I(\mathbf{B}) = n-1$. □

The following theorem shows that dimension of the quotient ring, depend only to the number of vertices and edges of G and does not depend on the direction of edges of the graph.

Theorem 3.21. *Let $R = \mathbb{K}[x_1, \ldots, x_m]$ and G be an oriented graph with n vertices and m edges. Let $\boldsymbol{B}$ be the incidence matrix of G and let $1 \leq i \leq n$ be an arbitrary integer. Let $\boldsymbol{B}_i$ be the matrix obtained by removing i-th row of $\mathbf{B}$, then:*

$$\dim \frac{R}{I(\boldsymbol{B}_i)} = m - n + 1.$$

Proof. By definitions we have:

$$\dim \frac{R}{I(\mathbf{B}_i)} = \max\{\dim \frac{R}{Q} | Q \in \operatorname{Spec} R, I(\mathbf{B}_i) \subset Q\}.$$

Now by Lemmas 3.17 and 3.18 the claim is proved. □

Corollary 3.22. *With the above notations*

$$\dim \frac{R}{I(\boldsymbol{B})} = m - n + 1.$$

Proof. Since I_L is a minimal prime ideal of $I(\mathbf{B})$ and heights of the other minimal prime ideals of $I(\mathbf{B})$ is greater than or equal to $n - 1$, the claim will follow. □

Corollary 3.23. *The heights and dimensions of $I(\boldsymbol{B})$ and $I(\boldsymbol{B}_i)$ are independent of the way G is directed.*

Proposition 3.24. *Let G be an i-cycleless oriented graph with incidence matrix $\boldsymbol{B}$. Then the ideal $I(\boldsymbol{B})$ is a complete intersection ideal.*

Proof. By Corollary 3.19 $\operatorname{ht} I(\mathbf{B}) = n - 1$. Since $I(\mathbf{B}) = I(\mathbf{B}_i)$ and $I(\mathbf{B}_i)$ is generated by $n - 1$ polynomials, then $I(\mathbf{B})$ is a complete intersection ideal. □

Remark 3.25. *Koszul complex provides a free complex for $\frac{R}{I(\boldsymbol{B})}$. It is proved that $I(\boldsymbol{B}_i)$ is a complete intersection ideal, then Koszul complex provides a free resolution for $\frac{R}{I(\boldsymbol{B}_i)}$. Also as mentioned above if G be a i-cycleless graph, $I(\boldsymbol{B}_i) = I(\boldsymbol{B})$, then in this case Koszul complex is a free resolution for $\frac{R}{I(\boldsymbol{B})}$.*

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