A NOTE ON 1-FACTORIZABILITY OF QUARTIC SUPERSOLVABLE CAYLEY GRAPHS

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ABSTRACT. Alspach et al. conjectured that every quartic Cayley graph on an even solvable group is 1-factorizable. In this paper, we verify this conjecture for quartic Cayley graphs on supersolvable groups of even order.

1. Introduction and Preliminary Results

Let $G$ be a finite group with identity 1 and $S \subseteq G \setminus \{1\}$. A Cayley graph with respect to the set $S$, denoted by $\text{Cay}(G, S)$, is a graph whose vertex set is the set of elements of $G$ with adjacency defined by

$$g \sim h \text{ if and only if } g^{-1}h \in S \cup S^{-1}$$

for every $g, h \in G$, where $S^{-1} = \{s^{-1} \mid s \in S\}$. We see at once that if $S$ generates $G$, then $\text{Cay}(G, S)$ is connected.

A group $G$ is supersolvable if there exists a normal series

$$\{1\} = H_0 \lhd H_1 \lhd \cdots \lhd H_n = G$$

such that each quotient group $H_i/H_{i-1}$ is cyclic. Note that every supersolvable group is a solvable group.

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A \textit{j-factor} of a graph is a spanning subgraph which is regular of valence \( j \). In particular, a 1-factor of a graph is a collection of edges such that each vertex is incident with exactly one edge. A 1-factorization of a regular graph is a partition of the edge set of the graph into disjoint 1-factors. A 1-factorization of a regular graph of valence \( v \) is equivalent to a coloring of the edges in \( v \) colors (coloring each 1-factor a different color).

Strong investigated about 1-factorizability of Cayley graphs in 1985 \cite{strong}. After that, Alspach et al. \cite{alspach} studied the factorization of quartic Cayley graphs on some solvable groups of even order. They posed the following conjecture:

\textbf{Conjecture 1.1.} \cite{alspach} \textit{Every quartic Cayley graph on an even solvable group is 1-factorizable.}

They proved their conjecture for an even solvable group \( G \) such that the commutator subgroup \( G' \) is an elementary abelian \( p \)-group. Also Abdollahi showed that every Cayley graph on a nilpotent group of even order is 1-factorizable \cite{abdollahi}.

In this paper, we verify Conjecture 1.1 for an even supersolvable group. In fact we prove the following theorem:

\textbf{Theorem 1.2.} \textit{Every quartic Cayley graph on an even supersolvable group is 1-factorizable}

To prove the above theorem, we need several lemmas.

\textbf{Lemma 1.3.} \cite{strong} \textit{Every cubic Cayley graph on a solvable group is 3-edge-colorable.}

The following lemmas are due to Strong \cite{strong}.

\textbf{Lemma 1.4.} Let \( S_1, S_2 \subseteq G \setminus \{1\}, \) not necessarily generating sets. Suppose \( \text{Cay}(G, S_2) \) is 1-factorizable and \( S_2 \subseteq S_1 \). If every element in \( S_1 \setminus S_2 \) has even order then \( \text{Cay}(G, S_1) \) is 1-factorizable.

\textbf{Lemma 1.5.} If \( G \) is a 2-generated group of even order with a cyclic commutator subgroup, then \( \text{Cay}(G, \{a, a^{-1}, b, b^{-1}\}) \) is 1-factorizable.

\textbf{Lemma 1.6.} Suppose that \( N \) is a normal subgroup of \( G \) and \( S \) is a generating set of \( G \) disjoint from \( N \). Assume that when \( s_i \neq s_j^\pm 1 \), neither \( s_is_j \) nor \( s_is_j^{-1} \) belongs to \( N \). If \( \text{Cay}(G/N, S/N) \) is 1-factorizable, then so is \( \text{Cay}(G, S) \).

2. The proof of Theorem 1.2

The proof falls naturally into three parts. First, assume that all elements of \( S \) have order 2. Since the edges generated by an element of order 2 form a 1-factor, it follows that \( \text{Cay}(G, S) \) is 1-factorizable. If \( S = \{a, a^{-1}, b, c\} \), where \( O(a) > 2 \) and \( O(b) = O(c) = 2 \), then by Lemma 1.3, \( \text{Cay}(G, S \setminus \{c\}) \) is 1-factorizable and hence, Lemma 1.4 completes the proof. Now, assume that \( S = \{a, a^{-1}, b, b^{-1}\} \) where \( O(a), O(b) > 2 \) and let \( N \) be a minimal normal subgroup of \( G \). Suppose that the theorem is false and let \( G \) be the smallest supersolvable group in the question such that \( \text{Cay}(G, S) \) is not 1-factorizable.
1-factorizable. Suppose that \( \langle S \rangle \neq G \). Since \( \langle S \rangle \) is a proper supersolvable subgroup of \( G \), we can see that \( \text{Cay}(\langle S \rangle, S) \) is 1-factorizable by our assumption. Now let \( T = \{x_1, \ldots, x_t\} \), where \( t \in \mathbb{N} \), be a left transversal set of \( \langle S \rangle \) in \( G \). Thus \( \{x_i \text{Cay}(\langle S \rangle, S) : 1 \leq i \leq t \} \) is the set of the connected components of \( \text{Cay}(G, S) \) where for every \( 1 \leq i \leq t \), \( x_i \text{Cay}(\langle S \rangle, S) \) is a graph which its vertex set is \( x_i(S) \) and two vertices \( x_iy_j \) and \( x_iy_k \) are adjacent if and only if \( (x_iy_j)^{-1}(x_iy_k) \in S \). Therefore for every \( 1 \leq i \leq t \), \( x_i \text{Cay}(\langle S \rangle, S) \) and \( \text{Cay}(\langle S \rangle, S) \) are isomorphic and hence \( x_i \text{Cay}(\langle S \rangle, S) \) is 1-factorizable. So \( \text{Cay}(G, S) \) is 1-factorizable which is a contradiction. Thus let \( \langle S \rangle = G \). We continue the proof in two cases:

**Case 1.** Suppose that \( N \cap S \neq \emptyset \). If \( a, b \in N \), then \( N = G \). Moreover, \( |G| \) is even and \( N \) is a cyclic group of prime order. So, \( N = G = \mathbb{Z}_2 \) and the proof is complete. If \( a \in N \), then \( G/N = \langle bN \rangle \) is abelian and hence, \( G' \leq N \). From this we have \( G' \) is cyclic. Lemma 1.5 shows \( \text{Cay}(G, S) \) is 1-factorizable. This is a contradiction.

**Case 2.** Let \( N \cap S = \emptyset \). The proof will be divided into two subcases.

**Subcase (a).** Let \( |N| \) be odd. Since \( |N| \) is odd and \( |G| \) is even, \( |G/N| \) is even. If \( aN \in \{bN, b^{-1}N\} \), then \( G/N = \langle aN, bN \rangle = \langle aN \rangle \). So \( O(aN) = O(bN) \) is even and therefore \( O(a), O(b) \) are even. Thus \( \text{Cay}(G, \{a, a^{-1}\}) \) is an union of cycles of even lengths which is 1-factorizable. Lemma 1.4 shows that \( \text{Cay}(G, S) \) is 1-factorizable. This is impossible. Let \( aN \notin \{bN, b^{-1}N\} \). Since \( |S| = |S/N| \), we conclude that \( \text{Cay}(G/N, S/N) \) is 1-factorizable by our assumption, and so is \( \text{Cay}(G, S) \) from Lemma 1.6, a contradiction.

**Subcase (b).** Suppose that \( |N| \) is even. So \( |N| = 2 \). If \( |G/N| \) is even, then the same argument as used in Subcase (a) shows that \( \text{Cay}(G, S) \) is 1-factorizable. This is a contradiction. Now, let \( |G/N| = m \) be odd. Since \( |N| = 2 \), \( N \leq Z(G) \), and hence we have \( G = M \times N \), where \( M \) is a Hall subgroup of order \( m \). Note that \( G \) is a supersolvable group, so is \( M \). Thus \( M \) has a minimal normal subgroup \( M_1 \) of prime order \( p \). Since \( G = M \times N \) and \( p \mid m \), we get \( M_1 \) is a minimal normal subgroup of \( G \) of odd order. Hence, by substituting \( N \) with \( M_1 \) in Subcase (a), we see that \( \text{Cay}(G, S) \) is 1-factorizable, which is a contradiction. This completes the proof.

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