www.combinatorics.ir

Transactions on Combinatorics
ISSN (print): 2251-8657, ISSN (on-line): 2251-8665
Vol. 7 No. 4 (2018), pp. 7-10.

www.ui.ac.ir

A NOTE ON 1-FACTORIZABILITY OF QUARTIC SUPERSOLVABLE CAYLEY GRAPHS

MILAD AHANJIDEH AND ALI IRANMANESH*

Communicated by Tommy R. Jensen

Abstract. Alspach et al. conjectured that every quartic Cayley graph on an even solvable group is 1-factorizable. In this paper, we verify this conjecture for quartic Cayley graphs on supersolvable groups of even order.

1. Introduction and Preliminary Results

Let G be a finite group with identity 1 and $S \subset G \setminus \{1\}$. A *Cayley graph* with respect to the set S, denoted by $\mathrm{Cay}(G, S)$, is a graph whose vertex set is the set of elements of G with adjacency defined by

$$g \sim h \text{ if and only if } g^{-1}h \in S \cup S^{-1}$$

for every $g, h \in G$, where $S^{-1} = \{s^{-1} \mid s \in S\}$. We see at once that if S generates G, then $\mathrm{Cay}(G, S)$ is connected.

A group G is *supersolvable* if there exists a normal series

$$\{1\} = H_0 \lhd H_1 \lhd \cdots \lhd H_n = G$$

such that each quotient group H_i/H_{i-1} is cyclic. Note that every supersolvable group is a solvable group.

MSC(2010): Primary: 05C25; Secondary: 05C70.
Keywords: Cayley graph, 1-factorization, supersolvable group.
Received: 04 June 2017, Accepted: 29 May 2018.
*Corresponding author.
DOI: http://dx.doi.org/10.22108/toc.2018.104578.1500

.

A *j-factor* of a graph is a spanning subgraph which is regular of valence j. In particular, a 1-factor of a graph is a collection of edges such that each vertex is incident with exactly one edge. A 1-*factorization* of a regular graph is a partition of the edge set of the graph into disjoint 1-factors. A 1-factorization of a regular graph of valence v is equivalent to a coloring of the edges in v colors (coloring each 1-factor a different color).

Stong investigated about 1-factorizability of Cayley graphs in 1985 [4]. After that, Alspach et al. [3] studied the factorization of quartic Cayley graphs on some solvable groups of even order. They posed the following conjecture:

Conjecture 1.1. [3] *Every quartic Cayley graph on an even solvable group is* 1*-factorizable.*

They proved their conjecture for an even solvable group G such that the commutator subgroup G' is an elementary abelian p-group. Also Abdollahi showed that every Cayley graph on a nilpotent group of even order is 1-factorizable [1].

In this paper, we verify Conjecture 1.1 for an even supersolvale group . In fact we prove the following theorem:

Theorem 1.2. *Every quartic Cayley graph on an even supersolvable group is* 1*-factorizable*

To prove the above theorem, we need several lemmas.

Lemma 1.3. [2] *Every cubic Cayley graph on a solvable group is* 3*-edge-colorable.*

The following lemmas are due to Stong [4].

Lemma 1.4. *Let $S_1, S_2 \subseteq G\backslash\{1\}$, not necessarily generating sets. Suppose* $\mathrm{Cay}(G, S_2)$ *is* 1*-factorizable and $S_2 \subseteq S_1$. If every element in $S_1 \setminus S_2$ has even order then* $\mathrm{Cay}(G, S_1)$ *is* 1*-factorizable.*

Lemma 1.5. *If G is a* 2*-generated group of even order with a cyclic commutator subgroup, then* $\mathrm{Cay}(G, \{a, a^{-1}, b, b^{-1}\})$ *is* 1*-factorizable*

Lemma 1.6. *Suppose that N is a normal subgroup of G and S is a generating set of G disjoint from N. Assume that when $s_i \neq s_j^{\pm 1}$, neither s_is_j nor $s_is_j^{-1}$ belongs to N. If* $\mathrm{Cay}(G/N, S/N)$ *is* 1*-factorizable, then so is* $\mathrm{Cay}(G, S)$.

2. The proof of Theorem 1.2

The proof falls naturally into three parts. First, assume that all elements of S have order 2. Since the edges generated by an element of order 2 form a 1-factor, it follows that $\mathrm{Cay}(G, S)$ is 1-factorizable. If $S = \{a, a^{-1}, b, c\}$, where $O(a) > 2$ and $O(b) = O(c) = 2$, then by Lemma 1.3, $\mathrm{Cay}(G, S \setminus \{c\})$ is 1-factorizable and hence, Lemma 1.4 completes the proof. Now, assume that $S = \{a, a^{-1}, b, b^{-1}\}$ where $O(a), O(b) > 2$ and let N be a minimal normal subgroup of G. Suppose that the theorem is false and let G be the smallest supersolvable group in the question such that $\mathrm{Cay}(G, S)$ is not

DOI: http://dx.doi.org/10.22108/toc.2018.104578.1500

1-factorizable. Suppose that $\langle S\rangle \neq G$. Since $\langle S\rangle$ is a proper supersolvable subgroup of G, we can see that $\mathrm{Cay}(\langle S\rangle, S)$ is 1-factorizable by our assumption. Now let $T = \{x_1, \ldots, x_t\}$, where $t \in \mathbb{N}$, be a left transversal set of $\langle S\rangle$ in G. Thus $\{x_i\mathrm{Cay}(\langle S\rangle, S) : 1 \leq i \leq t\}$ is the set of the connected components of $\mathrm{Cay}(G, S)$ where for every $1 \leq i \leq t$, $x_i\mathrm{Cay}(\langle S\rangle, S)$ is a graph which its vertex set is $x_i\langle S\rangle$ and two vertices x_iy_j and x_iy_k are adjacent if and only if $(x_iy_j)^{-1}(x_iy_k) \in S$. Therefore for every $1 \leq i \leq t$, $x_i\mathrm{Cay}(\langle S\rangle, S)$ and $\mathrm{Cay}(\langle S\rangle, S)$ are isomorphic and hence $x_i\mathrm{Cay}(\langle S\rangle, S)$ is 1-factorizable. So $\mathrm{Cay}(G, S)$ is 1-factorizable which is a contradiction. Thus let $\langle S\rangle = G$. We continue the proof in two cases:

Case1. Suppose that $N \cap S \neq \emptyset$. If $a, b \in N$, then $N = G$. Moreover, $|G|$ is even and N is a cyclic group of prime order. So, $N = G = \mathbb{Z}_2$ and the proof is complete. If $a \in N$, then $G/N = \langle bN\rangle$ is abelian and hence, $G' \leq N$. From this we have G' is cyclic. Lemma 1.5 shows $\mathrm{Cay}(G, S)$ is 1-factorizable. This is a contradiction.

Case2. Let $N \cap S = \emptyset$. The proof will be divided into two subcases.

Subcase (a). Let $|N|$ be odd. Since $|N|$ is odd and $|G|$ is even, $|G/N|$ is even. If $aN \in \{bN, b^{-1}N\}$, then $G/N = \langle aN, bN\rangle = \langle aN\rangle$. So $O(aN) = O(bN)$ is even and therefore $O(a), O(b)$ are even. Thus $\mathrm{Cay}(G, \{a, a^{-1}\})$ is an union of cycles of even lengths which is 1-factorizable. Lemma 1.4 shows that $\mathrm{Cay}(G, S)$ is 1-factorizable. This is impossible . Let $aN \notin \{bN, b^{-1}N\}$. Since $|S| = |S/N|$, we conclude that $\mathrm{Cay}(G/N, S/N)$ is 1-factorizable by our assumption, and so is $\mathrm{Cay}(G, S)$ from Lemma 1.6, a contradiction.

Subcase (b). Suppose that $|N|$ is even. So $|N| = 2$. If $|G/N|$ is even, then the same argument as used in Subcase (a) shows that $\mathrm{Cay}(G, S)$ is 1-factorizable. This is a contradiction. Now, let $|G/N| = m$ be odd. Since $|N| = 2$, $N \leq Z(G)$, and hence we have $G = M \times N$, where M is a Hall subgroup of order m. Note that G is a supersolvable group, so is M. Thus M has a minimal normal subgroup M_1 of prime order p. Since $G = M \times N$ and $p \mid m$, we get M_1 is a minimal normal subgroup of G of odd order. Hence, by substituting N with M_1 in Subcase (a), we see that $\mathrm{Cay}(G, S)$ is 1-factorizable, which is a contradiction. This completes the proof.

Acknowledgments

The authors wish to thank to referee for reading the paper carefully and making the useful suggestions.

References

[1] A. Abdollahi, 1-factorizations of Cayley graphs, *Ars Combin.*, **86** (2008) 129–131.

[2] B. Alspach, Y. Liu and C. Zhang, Nowhere-zero 4-flows and Cayley graphs on solvable groups, *SIAM J. Discrete Math.*, **9** (1996) 151–154.

[3] B. Alspach, M. Morton and Y. Qin, 1-factorizations of Cayley graphs on solvable groups, Report, The University of Auckland Library, https://researchspace.auckland.ac.nz/bitstream/handle/2292/5039/393.pdf.

[4] R. Stong, On 1-factorizability of Cayley graphs, *J. Combin. Theory Ser. B*, **39** (1985) 298–307.

DOI: http://dx.doi.org/10.22108/toc.2018.104578.1500

Milad Ahanjideh

Department of Mathematics, Faculty of Mathematical Sciences, Tarbiat Modares University, P.O.Box 14115-137, Tehran, Iran

Email: ahanjidm@gmail.com

Ali Iranmanesh

Department of Mathematics, Faculty of Mathematical Sciences, Tarbiat Modares University, P.O.Box 14115-137, Tehran, Iran

Email: iranmanesh@modares.ac.ir

DOI: http://dx.doi.org/10.22108/toc.2018.104578.1500