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Transactions on Combinatorics
ISSN (print): 2251-8657, ISSN (on-line): 2251-8665
Vol. 7 No. 4 (2018), pp. 11-24.

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DEGREE RESISTANCE DISTANCE OF TREES WITH SOME GIVEN PARAMETERS

FANGGUO HE AND XINNONG JIANG*

Communicated by Ivan Gutman

Abstract. The degree resistance distance of a graph G is defined as $D_R(G) = \sum_{i<j}(d(v_i)+d(v_j))R(v_i, v_j)$, where $d(v_i)$ is the degree of the vertex v_i, and $R(v_i, v_j)$ is the resistance distance between the vertices v_i and v_j. Here we characterize the extremal graphs with respect to degree resistance distance among trees with given diameter, number of pendent vertices, independence number, covering number, and maximum degree, respectively.

1. Introduction

The graphs considered in this paper are simple and undirected. Let $G = (V(G), E(G))$ be a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G) = \{e_1, e_2, \ldots, e_m\}$. The distance $d(v_i, v_j)$ between the vertices v_i and v_j of the graph G is the length of the shortest path between v_i and v_j. If $S \subseteq V(G)$, let $G - S$ be the subgraph of G obtained by deleting the vertices of S and the edges incident with them. Similarly, for $E \subseteq E(G)$, we denote by $G - E$ the subgraph of G obtained by deleting the edges of E. Let $P_k = v_1v_2 \cdots v_k (k \geq 2)$ be a path of G with distinct vertices $v_1, v_2, \ldots, v_k$ and assume that $d_G(v_1) \geq 3, d_G(v_2) = \cdots = d_G(v_{k-1}) = 2$, then P_k is called a pendent path of G if $d_G(v_k) = 1$, and P_k is called a internal path of G if $d_G(v_k) \geq 3$. Other undefined notations and terminology from graph theory can be found in [3].

MSC(2010): Primary: 05C15; Secondary: 05C69.
Keywords: Trees, Degree resistance distance, Diameter, Covering number.
Received: 18 December 2017, Accepted: 11 August 2018.
∗Corresponding author.
DOI: http://dx.doi.org/10.22108/toc.2018.108656.1538

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The sum of the distances between all pairs of vertices in a graph G is the Wiener index, namely, $W(G) = \sum_{i<j} d(v_i, v_j)$. It is introduced in 1947 [29] and is one of the most thoroughly studied distance based graph invariants [6, 7, 13, 20, 24]. A modified version of the Wiener index is the degree distance defined as

$$D(G) = \sum_{i<j} (d(v_i) + d(v_j))d(v_i, v_j)$$

where $d(v_i) = d_G(v_i)$ is the degree of the vertex v_i of the graph G, For more information on degree distances of graphs, see [1, 26, 27, 9, 18, 22].

In 1993, Klein and Randić [21] introduced a distance function named resistance distance on a graph. They viewed a graph G as an electrical network such that each edge of G is assumed to be a unit resistor, then take the resistance distance between vertices v_i and v_j to be the effective resistance between them, denoted by $R(v_i, v_j)$. The Kirchhoff index $Kf(G)$ of G is defined as $Kf(G) = \sum_{i<j} R(v_i, v_j)$. The index has been extensively studied in mathematical, physical and chemical aspects (see [2, 15, 31, 12, 33, 34]).

In analogy with the degree distance of a graph, the degree resistance distance (DR-index) of a graph G was first proposed by Gutman, Feng and Yu [14] as

$$D_R(G) = \sum_{i<j} (d(v_i) + d(v_j))R(v_i, v_j)$$

Palacios [25] called this graph invariant as the additive degree-Kirchhoff index. It was systematically studied in [19, 10]. Tu and Su [28] characterized the unicyclic graphs with maximum degree resistance distance, Chen et al.[5] characterized n-vertex unicyclic graphs with given girth having maximum and second maximum DR-index, Yang and Klein gave the formula for the degree resistance distances of the subdivisions and triangulations of graphs [32], In [14] some properties of DR-index were presented and the unicyclic graphs with the minimum and the second minimum DR-index were also completely characterized. The extremal graphs on degree resistance distance of cacti are determined by Liu and Du et al. [23, 10].

If G is a tree, then $R(u, v) = d(u, v)$ for any two vertices u and v [21]. Consequently, the degree distances and degree resistance distances are equal as well [8], i,e.

$$D_R(G) = D(G) = 4W(G) - n(n-1) = 4Kf(G) - n(n-1). \quad (1.1)$$

According to (1.1), if a result holds for DR-index of trees, then it is true for degree distance of trees. Further, the extremal graphs corresponding to the indices $D_R(G), D(G), W(G)$ and $Kf(G)$ of a tree G are the same.

Motivated from [16, 30, 11, 17], we study some mathematical properties of $D_R(G)$ for a graph G. In this paper, we characterize the extremal graphs with respect to degree resistance distance among trees with given diameter, number of pendent vertices, independence number, covering number, and maximum degree, respectively.

http://dx.doi.org/10.22108/toc.2018.108656.1538

2. Preliminaries and some lemmas

Let $R_G(u,v)$ denote the resistance distance between u and v in a graph G. It is known that $R_G(u,v) = R_G(v,u)$ and $R_G(u,v) \geq 0$ with equality if and only if $u = v$.

For a vertex v in G, we define the following two functions

$$Kf_v(G) = \sum_{u \in G} R_G(u,v), \quad D_v(G) = \sum_{u \in G} d_G(u) R_G(u,v).$$

For the sake of conciseness, we write $u \in G$ instead of $u \in V(G)$. By the definition of $D_R(G)$, we also have

$$D_R(G) = \sum_{v \in G} (d_G(v) \sum_{u \in G} R_G(u,v)).$$

Lemma 2.1. [21] *Let G be a graph, x be a cut vertex of G and let u, v be vertices belonging to different components which arise upon deletion of x. Then$R_G(u,v) = R_G(u,x) + R_G(x,v)$.*

Lemma 2.2. [14] *Let G be a connected graph with a cut-vertex v such that G_1 and G_2 are two connected subgraphs of G having v as the only common vertex and $V(G_1) \cup V(G_2) = V(G)$. Let $n_1 = |V(G_1)|, n_2 = |V(G_2)|$, $m_1 = |E(G_1)|, m_2 = |E(G_2)|$. Then*

$$D_R(G) = D_R(G_1) + D_R(G_2) + 2m_2 Kf_v(G_1) + 2m_1 Kf_v(G_2) + (n_1 - 1)D_v(G_2) + (n_2 - 1)D_v(G_1).$$

Let v be a vertex of degree $p+1$ in a graph G, such that $vv_1, vv_2, \ldots, vv_p$ are pendent edges incident with v, and u is the neighbor of v distinct from $v_1, v_2, \ldots, v_p$. We form a graph $G' = \sigma(G,v)$ by deleting the edges $vv_1, vv_2, \cdots, vv_p$ and adding new edges $uv_1, uv_2, \ldots, uv_p$. We say that G' is a σ-transform of the graph G (see Figure 1).

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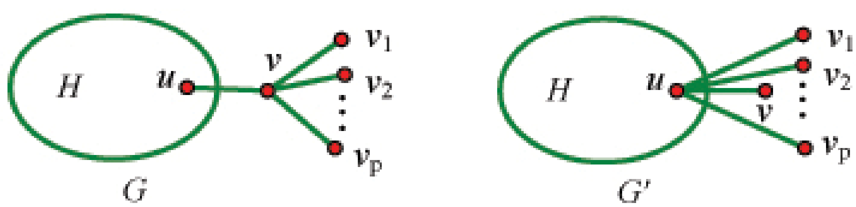

FIGURE 1. The graphs G and G'

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Lemma 2.3. [14] *Let $G' = \sigma(G,v)$ be a σ-transform of the graph G, $d_G(u) \geq 1$. Then $D_R(G) \geq D_R(G')$. Equality holds if and only if G is a star with v as its center.*

Lemma 2.4. *Suppose that G is a connected graph with a cut vertex u and $d_G(u) \geq 3$. Let the paths $P = u_1 u_2 \cdots u_k$ and $Q = v_1 v_2 \cdots v_t$ $(k \geq t)$ be the connected components of $G - u$, and let $G' = G - v_{t-1}v_t + u_k v_t$ (as shown in Fig.2). Then $D_R(G) < D_R(G')$.*

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FIGURE 2. The graphs G and G'

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Proof. ? For the sake of brevity, for any two vertices u, v of G (or G'), we let $d(v) = d_G(v)$ (or $d'(v) = d_{G'}(v)$) and $R(u,v) = R_G(u,v)$ (or $R'(u,v) = R_{G'}(u,v)$). Let $A = \{v_t, v_{t-1}, u_k\}$, $B = \{v_{t-1}, \dots v_1, u, u_1, \dots u_{k-1}\}$, $H = V(G) \setminus (A \cup B)$. In the transformation from G to G', for any pair of vertices x, y satisfying $x, y \in B \cup H$, the term $\sum_{x,y}(d(x)+d(y))R(x,y)$ does not change, and $V(G) = V(G')$. Then

$$\begin{aligned}
D_1 &= \sum_{x,y \in B\cup H} [(d'(x)+d'(y))R'(x,y) - (d(x)+d(y))R(x,y)] = 0 \\
D_2 &= \sum_{x,y\in A} [(d'(x)+d'(y))R'(x,y) - (d(x)+d(y))R(x,y)] \\
&= [2(k+t)+3+3(t-1+k)] - [3+2(t+k)+3(k+t-1)] = 0
\end{aligned}$$

Note that $R'(u_k,y) = R(u_k,y)$, $R'(v_{t-1},y) = R(v_{t-1},y)$ and $R'(v_t,y) = t+k+1-R(v_t,y)$ for $y \in B$, let $d_G(u) = d$, then

$$\begin{aligned}
D_3 &= \sum_{x\in A, y\in B} [(d'(x)+d'(y))R'(x,y) - (d(x)+d(y))R(x,y)] \\
&= (\sum_{x\in A, y\in B-u} + \sum_{x\in A, y=u})[(d'(x)+d'(y))R'(x,y) - (d(x)+d(y))R(x,y)] \\
&= \sum_{y\in B-u} [4R(u_k,y) + 3R(v_{t-1},y) + 3(t+k+1-R(v_t,y)) \\
&\quad -(3R(u_k,y)+4R(v_{t-1},y)+3R(v_t,y))] + (2+d)(k-t+1) \\
&= \sum_{y\in B-u} (R(u_k,y) - R(v_{t-1},y) - 6R(v_t,y)) + 3(t+k+1)(t+k-3) + (2+d)(k-t+1) \\
&= 4(t-k-1) + (2+d)(k-t+1) > 0.
\end{aligned}$$

Since $d'(y) = d(y)$ for $y \in H$, by lemma 2.1, we have

$$\begin{aligned} D_4 &= \sum_{x\in A, y\in H} [(d'(x)+d'(y))R'(x,y) - (d(x)+d(y))R(x,y)] \\ &= \sum_{y\in H} [(1+d(y))(k+1+R(u,y)) + (1+d(y))(t-1+R(u,y)) + (2+d(y))(k+R(u,y)) \\ &\quad -(1+d(y))(t+R(u,y)) - (2+d(y))(t-1+R(u,y)) - (1+d(y))(k+R(u,y))] \\ &= \sum_{y\in H} (k-t+1)(2+d(y)) > 0. \end{aligned}$$

Therefor, by the definition of $D_R(G)$, we get

$$D_R(G') - D_R(G) = D_1 + D_2 + D_3 + D_4 > 0.$$

This completes the proof. □

Let G be a tree, and v be a vertex of G such that $N_G(v) = \{v_1, v_2, \ldots v_{s-1}, u_1\}$. It is obvious that $G - v$ has s ($s \geq 3$) connected components, denoted by $T_1, T_2, \ldots T_s$. Let there exists a connected component $T_i (i = 1, \ldots s)$ such that T_i is a path, without loss of generality, say $T_s = P_k$. Now we construct a new graph G' obtained from G by removing the edges $vv_2, vv_3, \ldots vv_{s-1}$ and adding new edges $v_1v_2, v_1v_3, \ldots v_1v_{s-1}$, as shown in Figure ?3?. We call the process from G to G' a graph transformation.

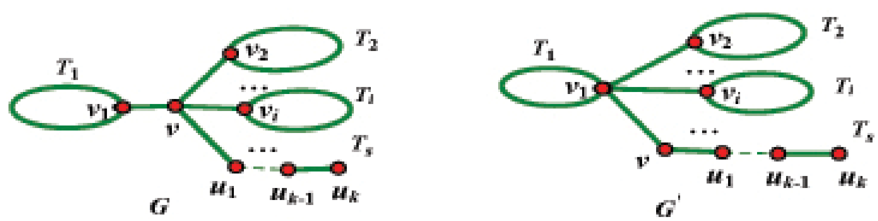

FIGURE 3. The graphs G and G'

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Lemma 2.5. *Let G and G' be the trees depicted in figure ?3. If $d_G(v, u_k) < max\{d_G(v_1, x) : x \in V(T_1)\}$ or $|V(T_1)| > k+1$, then $D_R(G') < D_R(G)$.*

Proof. Let $G_1 = G - T_1 - T_s$, $G_2 = G - T_2 - \cdots - T_{k-1}$. It is easy to see that that G_1 and G_2 are two connected subgraphs of G and G' having v and v_1 as the only common vertex, respectively. Let $n_1 = |V(G_1)|$, $n_2 = |V(G_2)|$, $m_1 = |E(G_1)|$, $m_2 = |E(G_2)|$. Using Lemma 2.2, we have

$$\begin{aligned} D_R(G) &= D_R(G_1) + D_R(G_2) + 2m_2 Kf_v(G_1) + 2m_1 Kf_v(G_2) + (n_2-1)D_v(G_1) + (n_1-1)D_v(G_2) \\ D_R(G') &= D_R(G_1) + D_R(G_2) + 2m_2 Kf_{v_1}(G_1) + 2m_1 Kf_{v_1}(G_2) + (n_2-1)D_{v_1}(G_1) + (n_1-1)D_{v_1}(G_2) \end{aligned}$$

By the definition of $Kf_v(G)$ and $D_v(G)$, we know that $Kf_v(G_2) = Kf_{v_1}(G_2)$, $D_v(G_2) = D_{v_1}(G_2)$. Let $H = V(T_1) \setminus \{v_1\}$, $|H| = h$, therefore

$$\begin{aligned} D_R(G) - D_R(G') &= 2m_2(Kf_v(G_1) - Kf_{v_1}(G_1)) + (n_2 - 1)(D_v(G_1) - D_{v_1}(G_1)) \\ &= 2m_2[(\sum_{x \in H}(R(x, v_1) + 1) + 1 + \frac{k(k+1)}{2}) - (\sum_{x \in H} R(x, v_1) + \frac{(k+2)(k+1)}{2})] \\ &\quad +(n_2 - 1)[(\sum_{x \in H} d(x)(R(x, v_1) + 1) + d(v_1) + k^2) - (\sum_{x \in H} d(x)R(x, v_1) + (k+1)^2)] \\ &= 2m_2(h - k) + (n_2 - 1)(\sum_{x \in H} d(x) + d(v_1) - 2k - 1) \end{aligned}$$

Note that $\sum_{x \in H} d(x) = 2(h - 1)$, $m_2 = n_2 - 1$, thus

$$D_R(G) - D_R(G') = m_2(4h - 4k + d(v_1) - 3)$$

Since $d_G(v, u_k) < max\{d_G(v_1, x) : x \in V(T_1)\}$, there exists a vertex $z \in V(T_1)$ such that $d_G(v, u_k) < d_G(v_1, z)$. Thus $h \geq k + 1$. Similarly, if $|V(T_1)| > k + 1$, then $h \geq k + 1$. It follows that

$$D_R(G) - D_R(G') \geq m_2(d(v_1) + 1) > 0.$$

This completes the proof. □

Let G be a connected graph and $P = u_0u_1 \cdots u_{k-1}u_k$ be an internal path of G. Suppose that $G - \{u_1, u_2, \ldots u_{k-1}\} = G_1 \cup G_2$ with $u_0 \in G_1$ and $u_k \in G_2$. We construct a new graph G' obtained from G by contracting the path P and attaching the path P to the common vertex u_0 of G_1 and G_2, as shown in Fig.4. We call the procedure of constructing G' from G the A-transformation of G.

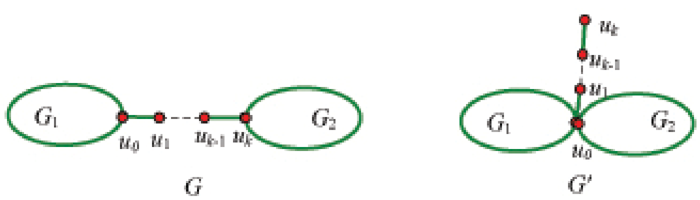
? 

FIGURE 4. The graphs G and G'

Lemma 2.6. *Assume that G and G' be two connected graphs, and G is an A-transformation of G. Then $D_R(G) > D_R(G')$ (see Fig.4).*

Proof. Let $A = \{u_0, u_1, \ldots u_{k-1}, u_k\}, B = V(G_1) \setminus u_0$ and $C = V(G2) \setminus uk$. In the transformation from G to G', the term $\sum_{i,j}(d_G(i) + d_G(j))R_G(i, j)$ does not change for any $i, j \in A$, or $i, j \in B$, or $i, j \in C$. Since $R(i, j) > R'(i, j), d(i) = d'(i), d(j) = d'(j)$ for $i \in B, j \in C$, and $R(i, j) = R'(i, j), d(i) =$

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$d'(i), d(j) = d'(j)$ for $i \in A \setminus \{u_0, u_k\}, j \in B$, then

$$\begin{aligned}
D_1 &= \sum_{i\in B, j\in C} [(d(i)+d(j))R(i,j) - (d'(i)+d'(j))R'(i,j)] > 0 \\
D_2 &= (\sum_{i\in A, j\in B} + \sum_{i\in A, j\in C})[(d(i)+d(j))R(i,j) - (d'(i)+d'(j))R'(i,j)] \\
&= \sum_{j\in B} k(d(u_k)-1) + \sum_{i\in A\setminus\{u_0,u_k\}, j\in C} (2+d(j))(k-2i) + \sum_{j\in C}[k(d(u_0)+d(j)) - (d(u_k)-1)R(uk,j)] \\
&\quad + \sum_{j\in C}[(d(u_k)+d(j))R(u_k,j) - (1+d(j))(k+R(u_k,j))] \\
&= |B|k(d(u_k)-1) + (\sum_{i\in A\setminus\{u_0,u_k\}} (k-2i))(\sum_{j\in C}(2+d(j))) + \sum_{j\in C} k(d(u_0)-1) \\
&= |B|k(d(u_k)-1) + |C|k(d(u_0)-1) > 0.
\end{aligned}$$

Therefore, $D_R(G) - D_R(G') = D_1 + D_2 > 0$, the result follows. □

Lemma 2.7. [10] *Let u be a vertex of G such that there are p pendent vertices $u_1, u_2, \ldots, u_p$ attached to u. Let v be another vertex of G such that there are q pendent vertices $v_1, v_2, \ldots, v$ attached to v. Let $G_1 = G - \{vv_1, vv_2, \ldots, vv_q\} + \{uv_1, uv_2, \ldots, uv_q\}$ and $G_2 = G - \{uu_1, uu_2, \ldots, uu_p\} + \{vu_1, vu_2, \ldots, vu_p\}$. Then either $D_R(G) > D_R(G_1)$ or $D_R(G) > D_R(G_2)$.*

As usual, let P_n and S_n denote the path and star on n vertices. By direct calculation:

$$D_R(S_n) = (n-1)(3n-4), \qquad D_R(P_n) = \frac{1}{3}n(n-1)(2n-1).$$

Let T_n be a tree of order n. In view of Lemma 2.3, Lemma 2.4 and Lemma 2.6, the following corollary immediately implies.

Corollary 2.8. *Among trees of order n, the path P_n has greatest and the star S_n has smallest degree resistance distance. i.e. $3n^2 - 7n + 4 \le D_R(T_n) \le \frac{1}{3}n(n-1)(2n+1)$.*

A spider is a tree with at most one vertex of degree more than 2, and this vertex is called the hub of the spider (if no vertex of degree more than two, then any vertex can be the hub). A leg of a spider is a path from the hub to a leaf.

Lemma 2.9. *Let $T_{P_n,k}$ be a spider with k legs, and each leg is isomorphic to P_n, then*

$$D_R(T_{P_n,k}) = \frac{1}{3}k(n-1)(2n-1)[3(n-1)(k-1)+n].$$

Proof. First of all, it is claimed that $D_R(T_{P_n,k}) = kDR(Pn) + 2mk(k1)Kfv(Pn) + k(k-1)(n-1)Dv(Pn)$, where $m = n-1$ is the number of edges in P_n .

By induction on k. If $k = 2$, by Lemma 2.2, calculation shows that $D_R(T_{P_n,2}) = 2D_R(P_n) + 4mKf_v(P_n) + 2(n-1)D_v(P_n)$, the result holds.

We assume that the equality holds for $k-1$. By Lemma 2.2 and the induction hypothesis, we get

$$\begin{aligned} D_R(T_{P_n,k}) &= D_R(T_{P_n,k-1}) + D_R(P_n) + 2m(k-1)Kf_v(P_n) + 2mKf_v(T_{P_n,k-1}) \\ &\quad +(k-1)(n-1)D_v(P_n) + (n-1)D_v(T_{P_n,k-1}) \\ &= (k-1)D_R(P_n) + 2m(k-1)(k-2)Kf_v(P_n) + (k-1)(k-2)(n-1)D_v(P_n) + D_R(P_n) \\ &\quad +4m(k-1)Kf_v(P_n) + 2(n-1)(k-1)D_v(P_n) \\ &= kD_R(P_n) + 2mk(k-1)Kf_v(P_n) + k(k-1)(n-1)D_v(P_n) \end{aligned}$$

Since $Kf_v(P_n) = \frac{n(n-1)}{2}$ and $D_v(P_n) = (n-1)^2$, applying Corollary 2.8, we have

$$D_R(T_{P_n,k}) = \frac{1}{3}k(n-1)(2n-1)[3(n-1)(k-1)+n].$$

□

3. Degree resistance distance of trees with given parameters

Let $\mathcal{T}_{n,d}$ denote the set of all n-vertex trees of diameter d, and $D_{n,d}$ be the n-vertex tree obtained by attaching the center of the star S_{n-d} to the vertex v_l ($l = \lfloor \frac{d}{2} \rfloor + 1$ or $\lceil \frac{d}{2} \rceil + 1$) of a path on $d+1$ vertices, as shown in Fig. 5.

Theorem 3.1. *Among the graphs in $\mathcal{T}_{n,d}$, the tree $D_{n,d}$ is the unique graph with the minimum degree resistance distance.*

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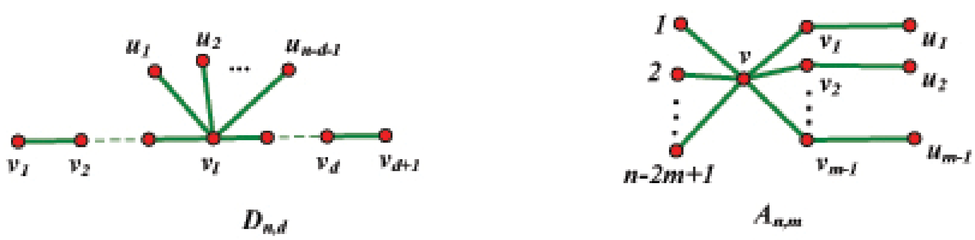

FIGURE 5. The graphs $D_{n,d}$ and $A_{n,m}$

Proof. Let T be a tree in $\mathcal{T}_{n,d}$ with minimum degree resistance distance. Choose a longest path P in T, say $P = v_1v_2\cdots v_dv_{d+1}$. It can be confirmed that T is formed by attaching $n-d-1$ pendant edges to the vertices $v_2, v_3, \ldots v_d$. If not so, there exists at least one pendant edge uw such that the vertices u, w are distinguished from $v_2, v_3, \ldots v_d$. let v be the neighbor of u different from w in T, $T' = T - uw + vw$. Note that $T' \in \mathcal{T}_{n,d}$, applying Lemma 2.3, we can get $D_R(T') < D_R(T)$. It is a contradiction.

Furthermore, we can verify that the $n-d-1$ pendant edges are all attached to the vertex v_l, that is, the tree T is $D_{n,d}$, where $l = \lfloor \frac{d}{2} \rfloor + 1$ or $l = \lceil \frac{d}{2} \rceil + 1$. If not so, we can choose a vertex $v_j (j \neq \lfloor \frac{d}{2} \rfloor + 1, \lceil \frac{d}{2} \rceil + 1)$ in $V(P)$ such that $d(v_j) \geq 3$ and the path connecting v_j and v_1 or v_{d+1} is a pendant path. Without loss of generality, let the path between v_j and v_1 be a pendant path and

$d(v_j, v_1) < d(v_{j+1}, v_{d+1})$. We construct a tree T'' by deleting the pendant edges of v_j (except v_jv_1 if $j = 2$) and adding those edges to the vertex v_{j+1}, obviously, $T'' \in \mathcal{T}_{n,d}$. By lemma 2.5, it implies that $D_R(T'') < D_R(T)$. This is a contradiction. □

Applying Lemma 2.2 and Corollary 2.8, by calculation we obtain $D_R(D_{n,d})$ as follows:

(*a*) $D_R(D_{n,d}) = \frac{1}{3}[9n^2 + (3d^2 - 6d - 21)n - (d^3 + 3d^2 - 10d) + 12]$, if d is even

(*b*) $D_R(D_{n,d}) = \frac{1}{3}[9n^2 + (3d^2 - 6d - 18)n - (d^3 + 3d^2 - 7d) + 9]$, if d is odd.

Furthermore, according to Lemma 2.4 and Theorem 3.1, we have the following orders:

$$D_R(S_n) = D_R(D_{n,2}) < D_R(D_{n,3}) < \cdots < D_R(D_{n,n-1}) = D_R(P_n). \tag{3.1}$$

Assume that $\mathcal{P}(n, k)$ is the set of all n-vertex trees with k $(2 \le k \le n-1)$ leaves. Let $H_{n,k}$ be a spider with n vertices and k legs, and the lengths of $n - 1 - kr$ legs among them are $r + 1$, the others are r, where $r = \lfloor \frac{n-1}{k} \rfloor$. It is clear that $H_{n,k} \in \mathcal{P}(n, k)$, we get the following result.

Theorem 3.2. *Among the graphs in $\mathcal{P}(n, k)$, the tree $H_{n,k}$ has smallest degree resistance distance.*

Proof. Let T be a tree in $\mathcal{P}(n, k)$ with minimum degree resistance distance. If $k = 2$ or $n - 1$, the result follows immediately. Therefore, we suppose that $3 \le k \le n - 2$.

First of all, we can confirm that T is a spider, that is, T is a tree with at most one vertex of degree more than 2. Otherwise, suppose that there exists two vertices u, v in T such that $d(u) \ge 3$ and $d(v) \ge 3$. Choose a vertex from u, v such that the length of one of its pendent paths is as small as possible. Without loss of generality, let u be the vertex, and $P_u = ux_1 \cdots x$ be the pendent path. Denote by y the neighbor of u in T lying on the path connecting u and v ($y = v$ if u and v are adjacent). Obviously, $d_T(u, x) < max\{d_T(y, w) : w \in T_v\}$, where T_v is the component of $T - y$ including the vertex v. According to Lemma 2.5, we can obtain a tree $T' \in \mathcal{P}(n, k)$ such that $D_R(T') < D_R(T)$, a contradiction.

Secondly, we can claim that $T \cong H_{n,k}$. Otherwise, in view of Lemma ?2.4?, it follows that $D_R(T) < D_R(H_{n,k})$, it is a contradiction.

This completes the proof. □

Proposition 3.3. $D_R(H_{n,k}) = \frac{kr}{3}(2r+1)(3rk-2r+1) + \frac{n-kr-1}{2}(5kr^2-6r^2+3kr+4rn-8r+4n-4)$, *where* $r = \lfloor \frac{n-1}{k} \rfloor$.

Proof. Let A be the set of pendent vertices with length of legs $r + 1$ in $H_{n,k}$, $H' = H_{n,k} - A$. By Lemma 2.9, then

$$\begin{aligned}
D_R(H_{n,k}) &= (\sum_{i,j \in H'} + \sum_{i,j \in A} + \sum_{i \in A, j \in H'})(d(i) + d(j))R(i, j) \\
&= D_R(T_{P_{r+1,k}}) + 2(r+1)(n-kr-1)(n-kr-2) + \\
&\quad (n-kr-1)\sum_{j \in H'}(1 + d(j))R(i_0, j)(i_0 \in A) \\
&= \frac{kr}{3}(2r+1)(3rk-2r+1) + \frac{n-kr-1}{2}(5kr^2-6r^2+3kr+4rn-8r+4n-4)
\end{aligned}$$

□

According to Lemma 2.2 and Lemma 2.4, we get the following orders:

$$D_R(S_n) = D_R(H_{n,n-1}) < D_R(H_{n,n-2}) < \cdots < D_R(H_{n,2}) = D_R(P_n). \tag{3.2}$$

Given a graph G, a subset S of $V(G)$ is called an independent set of G if no pair of vertices of S is adjacent. The independence number $\alpha(G)$ of G is the number of vertices in the largest independent set of G. Recall that if T is a tree of order n, then $\lfloor \frac{n}{2} \rfloor \le \alpha(T) \le n-1$?[4]?.

Let $\mathcal{Q}_{n,\alpha}$ be the set of n-vertex trees with independent number α. Denote by $I_{n,\alpha}$ the spider obtained from the star $S_{\alpha+1}$ by attaching to $n-\alpha-1$ of its pendent vertices a new pendent vertex. Obviously, $I_{n,\alpha} \in \mathcal{Q}_{n,\alpha}$.

Theorem 3.4. *Among the graphs in $\mathcal{Q}_{n,\alpha}$, the tree $I_{n,\alpha}$ has smallest degree resistance distance.*

Proof. Let $T \in \mathcal{Q}_{n,\alpha}$, and S be a largest independent set of T with $|S| = \alpha$. Let q be the number of pendent vertices in T, therefore $T \in \mathcal{P}(n,q)$. By Proposition 3.3, we have

$$D_R(T) \ge D_R(H_{n,q}) \tag{3.3}$$

For the tree T, one can easily see that $q \le \alpha$. If $q = \alpha$, considering the fact that $\lfloor \frac{n}{2} \rfloor \le \alpha \le n-1$, then $H_{n,\alpha} \cong I_{n,\alpha}$. If $q < \alpha$, applying Eq. (3.2), then we get

$$D_R(H_{n,q}) > D_R(H_{n,q+1}) > \cdots > D_R(H_{n,\alpha}) = D_R(I_{n,\alpha}) \tag{3.4}$$

According to (3.3) and (3.4), it follows that $D_R(T) \ge D_R(I_{n,\alpha})$. This completes the proof. □

Corollary 3.5. *If $T \in \mathcal{Q}_{n,\alpha}$, then $D_R(T) \ge 4n^2 - \alpha^2 + 4\alpha - 13n + 9$ with equality if and only if $T \cong I_{n,\alpha}$.*

Proof. Since $\lfloor \frac{n}{2} \rfloor \le \alpha \le n-1$, then $r = \lfloor \frac{n-1}{\alpha} \rfloor = 1$. According to Theorem 3.4, we have

$$D_R(I_{n,\alpha}) = D_R(H_{n,\alpha}) = 4n^2 - \alpha^2 + 4\alpha - 13n + 9.$$

By Corollary 3.5, the result follows. □

By using Lemma 2.3, we have the following conclusion.

Corollary 3.6. *For $\lfloor \frac{n}{2} \rfloor \le \alpha \le n-1$, $D_R(I_{n,\alpha}) < D_R(I_{n,\alpha+1})$.*

A covering of a graph G is a vertex subset $K \subseteq V(G)$ such that each edge of G has at least one end in the set K. The number of vertices in a minimum covering of a graph G is called the covering number of G. Let $\mathcal{K}_{n,m}$ be the set of all n-vertex trees with covering number m, $A_{n,m}$ the tree obtained from the star S_{n-m+1} by attaching $m-1$ pendent edges to the pendent vertices $v_1, v_2, \ldots, v_{m-1}$ of S_{n-m+1}, as shown in Fig.5. It is easy to see that $A_{n,m} \in \mathcal{K}_{n,m}$, where $1 \le m \le \lfloor \frac{n}{2} \rfloor$.

Theorem 3.7. [4] *For a bipartite graph G, the sum of the independence number and the covering number of G is equal to the number of vertices of G.*

http://dx.doi.org/10.22108/toc.2018.108656.1538

Theorem 3.8. *Among the graphs in $\mathcal{K}_{n,m}$, the tree $A_{n,m}$ has smallest degree resistance distance.*

Proof. Let $T' \in \mathcal{K}_{n,m}$, and K be a minimum covering set of T' with $|K| = m$. From Theorem 3.8, it follows that the the independence number of T' is equal to $n - m$, then $T' \in \mathcal{Q}_{n,n-m}$. By Corollary 3.5, we have

$$D_R(T') \geq D_R(I_{n,n-m}).$$

By the definition of $A_{n,m}$, $I_{n,n-m}$, we get $A_{n,m} \cong I_{n,n-m}$, thus $D_R(T') \geq D_R(A_{n,m})$. The result follows. □

In view of Corollary 3.6 and Corollary 3.9, we have the following result.

Corollary 3.9. *If $T \in \mathcal{K}_{n,m}$, then $D_R(T) \geq 3n^2 - m^2 + 2nm - 4m - 9n + 9$ with equality if and only if $T \cong A_{n,m}$.*

A subset M of $E(G)$ is called a matching of G if no two edges in M are adjacent in G. The matching number of a graph G is the maximum cardinality of a matching of G. Denote by $\mathcal{M}_{n,p}$ the set of all n-vertex trees with matching number p.

Theorem 3.10. [4] *In any bipartite graph, the number of edges in a maximum matching is equal to the number of vertices in a minimum covering.*

According to Corollary 3.9 and Corollary 3.11, we obtain the following corollary.

Corollary 3.11. *Among the graphs in $\mathcal{M}_{n,p}$, the tree $A_{n,p}$ is the unique graph with minimum degree resistance distance.*

Let the path $P_n = v_1v_2 \cdots v_n$, $T(a, n, b)$ be the tree obtained from P_n by attaching a and b pendent vertices to the end vertices v_1 and v_n, respectively. It is clear that the graph $T_{(a,2,b)}$ is the double star $S(a, b)$. We denote by $\mathcal{N}_{n,\Delta}$ the set of all n-vertex trees with maximum degree Δ.

Theorem 3.12. *If $T \in \mathcal{N}_{n,\Delta}$, and $\Delta \geq \frac{n}{2}$, then $D_R(T) \geq 3n^2 - 4\Delta^2 + 4n\Delta - 11n + 8$ with equality if and only if $T \cong T_{(\Delta-1,2,n-\Delta-1)}$.*

Proof. Let T' be the graph with minimum degree resistance distance in $\mathcal{N}_{n,\Delta}$. Let $v \in T'$ and $d(v) = \Delta$. Considering that $\Delta \geq \frac{n}{2}$, T' must satisfy the following three conditions:

(i) By Lemma 2.3 and Lemma 2.4, T' contains no pendent path with length greater than 1.

(ii) By Lemma 2.7, All pendent edges of T' are incident to v and another common vertex.

(iii) By Lemma 2.6, T' contains no internal path with length greater than 1.

Therefore $T' \cong T_{(\Delta-1,2,n-\Delta-1)}$. By calculation, we get $D_R(T') = 3n^2 - 4\Delta^2 + 4n\Delta - 11n + 8$. □

Theorem 3.13. *If $T \in \mathcal{N}_{n,\Delta}$, then $D_R(T) \leq \frac{1}{3}(n-\Delta)(2n\Delta+2n^2-4\Delta^2+15\Delta-3n-11)+(\Delta-1)(3\Delta-4)$ with equality if and only if $T \cong T(\Delta - 1, n - \Delta, 1)$.*

http://dx.doi.org/10.22108/toc.2018.108656.1538

Proof. Choose T' in $\mathcal{N}_{n,\Delta}$, such that its degree resistance distance is as large as possible, and let $v_0 \in T'$ with $d_{T'}(v_0) = \Delta$. In view of Lemma 2.3 and 2.6, any path with end vertex v_0 in T' is a pendent path, that is, $d_{T'}(x) \leq 2$ for any $x \in V(T') \setminus v_0$. According to Lemma 2.4, there are $\Delta - 1$ pendent vertices attached to v_0, thus $T' \cong T_{(\Delta-1,n-\Delta,1)}$. By direct calculation $D_R(T')$, we get the value, and the result follows. □

Theorem 3.14. *Let T be a n-vertex tree with two vertices of maximum degree Δ, then $D_R(T) \geq D_R(T_{(\Delta-1,n-2\Delta+2,\Delta-1)})$ with equality if and only if $T \cong T_{(\Delta-1,n-2\Delta+2,\Delta-1)}$.*

Proof. Assume that $\mathcal{N}_{n,2\Delta}$ is the set of all n-vertex trees with two vertices of maximum degree Δ. Let $T' \in \mathcal{N}_{n,2\Delta}$ be the tree with maximum degree resistance distance, and let $d_{T'}(v) = d_{T'}(u) = \Delta$. It can be confirmed that $d_{T'}(x) \leq 2$ for any $x \in V(T') \setminus \{v, u\}$, that is, all paths are pendent paths except that between u and v. Otherwise, Let $H \subseteq V(T') \setminus \{v, u\}$ be the set of vertices of T' with degree at least three. There exists a vertex y in H such that y is the end vertex of a pendent path, according to Lemma 2.6, we can get a tree T'' with $D_R(T'') > D_R(T')$, a contradiction.

On the other hand, we show that all pendent paths in T' are pendent edges. Otherwise, choose a shortest pendent path (except pendent edge), without loss of generality, say $vv_1v_2 \cdots v_t$. Move the other pendent paths with the end vertex v from v to v_1, by Lemma 2.5, we get a tree in $\mathcal{N}_{n,2\Delta}$ such that the degree resistance distance of the tree is greater than that of T', it is contradictory to the choice of T'. Thus, $T' \cong T_{(\Delta-1,n-2\Delta+2,\Delta-1)}$. □

Remark 1: With regard to the problem to determine the trees with maximum degree resistance distance for the given parameters such as number of pendent vertices or independence number, our approach may need to be modified, it would be interesting to continue studying the extremal graphs.

Remark 2: According to (1.1), for the given parameters mentioned above, the extremal graphs corresponding to degree distance, Wiener index and Kirchhoff index among trees are the same as that of DR-index.? ?

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Fangguo He

College of Mathematics and Physics, Huanggang Normal University, Huanggang, China

Email: hfg0118@126.com

Xinnong Jiang

College of Life Science and Techonolgy, Huazhong University of Science and Technology, Wuhan, China

Email: jiangxn88@hust.edu.cn

http://dx.doi.org/10.22108/toc.2018.108656.1538