ON THE DOUBLE BONDAGE NUMBER OF GRAPHS PRODUCTS

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Abstract. A set $D$ of vertices of graph $G$ is called double dominating set if for any vertex $v$, $|N[v] \cap D| \geq 2$. The minimum cardinality of double domination of $G$ is denoted by $\gamma_d(G)$. The minimum number of edges $E'$ such that $\gamma_d(G \setminus E) > \gamma_d(G)$ is called the double bondage number of $G$ and is denoted by $b_d(G)$. This paper determines that $b_d(G \_ H)$ and exact values of $b(P_n \_ P_2)$, and generalized corona product of graphs.

1. Introduction

Throughout this paper, all graphs are finite, undirected with neither loops nor multiple edges. We use [14] for any terminology and notation not defined here. Let $G$ be a graph with and vertex set $V(G)$ edge set $E(G)$. Suppose that $x, y \in V(G)$. We recall that a walk between $x$ and $y$ is a sequence $x = v_0, e_1, v_1, e_2, \ldots, e_k, v_k = y$ of vertices and edges of $G$ such that for every $i$ with $1 \leq i \leq k$, the edge $e_i$ has endpoints $v_{i-1}$ and $v_i$. Also a path between $x$ and $y$ is a walk between $x$ and $y$ without repeated vertices. A cycle of a graph is a path such that the start and end vertices are the same. A cycle graph is a graph that consists of a single cycle. We denote the cycle graph with $n$ vertices by $C_n$. Also we write $P_n$ for the path on $n$ vertices. For a graph $G$ and a nonempty subset $S \subseteq V(G)$, the vertex-induced subgraph, denoted $< S >$, is the subgraph of $G$ with vertex-set $S$ and edges incident to members of $S$. A graph $G$ is called connected if for any vertices $x$ and $y$ of $G$ there is a path between $x$ and $y$. Otherwise, $G$ is called disconnected. The maximal connected subgraphs of $G$ are its connected

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components. For a graph $G$ and vertices $x$ and $y$ of $G$, the distance between $x$ and $y$, denoted $d(x, y)$, is the number of edges in a shortest path between $x$ and $y$. If there is no any path between $x$ and $y$, then we write $d(x, y) = \infty$. Also we recall that the largest distance among all distances between pairs of the vertices of a graph $G$ is called the diameter of $G$ and is denoted by $\text{diam}(G)$. A complete graph is a graph in which each pair of distinct vertices is joined by an edge. We denote the complete graph with $n$ vertices by $K_n$. A clique of a graph $G$ is a complete subgraph of $G$.

For a graph $G$, the girth of $G$ is the length of a shortest cycle in $G$ and is denoted by $\text{girth}(G)$. If $G$ has no cycles, we define the girth of $G$ to be infinite.

For every vertex $v \in V$, the open neighborhood $N(v)$ is the set $\{u \in V(G) : uv \in E(G)\}$ and the closed neighborhood of $v$ is the set $N[v] = N(v) \cup \{v\}$. The degree of a vertex $v \in V$ is $\deg_G(v) = d_G(v) = |N(v)|$. The minimum degree and the maximum degree of a graph $G$ are denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively. A vertex of degree one is called a leaf and its neighbor is called a support vertex. A matching of a graph $G$ is a set $M$ of edges of $G$ such that each vertex of $G$ is incident to at most one edge of $M$.

A vertex $v \in V(G)$ is said to dominate itself and all its neighbors. A set $D \subseteq V(G)$ is called a dominating set of $G$ if every vertex $v \in V(G)$ is dominated by at least one vertex of $D$, and it is a double dominating set, abbreviated DDS, of $G$ if every vertex $v \in V(G)$ is dominated by at least two vertices of $D$. The domination number, $\gamma(G)$, and double domination number, $\gamma_d(G)$, are equal to the minimum cardinalities of a dominating set and double dominating set of $G$, respectively. A dominating (double dominating) set of $G$ minimum cardinality is called a $\gamma(G)$--set ($\gamma_d(G)$--set).

Note that an isolated vertex cannot be dominated by two vertices. Therefore, while considering double domination, we always assume that a graph has no isolated vertices. Double dominating sets were introduced by F. Harary and T.W. Haynes [5] and and studied further in [1, 2, 3, 6] and elsewhere.

In 1990, Fink et al. [4] formally introduced the bondage number and then continued by others, for example see [7, 8, 9, 12, 13].

The bondage number, $b(G)$, of a nonempty graph $G$ equals the minimum cardinality among all sets of edges $X$ for which $\gamma(G \setminus X) > \gamma(G)$ holds.

In 2012, Krzywowski [11] introduced the concept of double bondage number. The double bondage number, $b_d(G)$, of a nonempty graph $G$ to be the minimum cardinality among all sets of edges $X$ for which $\delta(G \setminus X) \geq 1$ and $\gamma_d(G \setminus X) > \gamma_d(G)$ holds. If for every $X \subseteq E$, either $\gamma_d(G \setminus X) = \gamma_d(G)$ or $\delta(G \setminus X) = 0$, then we define $b_d(G) = 0$.

The join of two graphs $G$ and $H$, $G \vee H$, is the graph with vertex-set $V(G \vee H) = V(G) \cup V(H)$ and edge-set $E(G \vee H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$.

Let $G$ be a graph of order $n$ and $H_1, H_2, \ldots, H_n$ be $n$ graphs. The generalized corona product, is the graph obtained by taking one copy of graphs $G, H_1, H_2, \ldots, H_n$ and joining the $i$th vertex of $G$ to
every vertex of $H_i$. This product is denoted by $G \circ \bigwedge_{i=1}^{n} H_i$. If each $H_i$ is isomorphic to graph $H$, then generalized corona product is called the corona product of $G$ and $H$ and is denoted by $G \circ H$.

For two graphs, $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, the cartesian product, $G_1 \Box G_2$, is the graph with vertex- set $V_1 \times V_2$ and $(x_1, y_1)(x_2, y_2) \in E(G_1 \Box G_2)$ if and only if $x_1 = y_1$ and $x_2y_2 \in E_2$, or $x_2 = y_2$ and $x_1y_1 \in E_1$. For further information about graph products see [10].

In section 2, we state some results and bounds for double bondage number of graphs and state a result about the double bondage number of graphs, $G$, with $\gamma_d(G) = n - 1$. In section 3, we study the double bondage number of graphs products and obtain the double bondage number of join, and generalized corona products of graphs.

2. Some bounds

In this section we state some results and some bounds for bondage number of graphs.

Lemma 2.1. [5]

i) If $v$ is a leaf of a graph $G$, then $v$ is an element of every $\gamma_d(G)$-set.

ii) If $v$ is a support vertex of a graph $G$, then $v$ is an element of every $\gamma_d(G)$-set.

iii) If $G' = (V, E \setminus E')$ is a spanning subgraph of a graph $G = (V, E)$, then $\gamma_d(G') \geq \gamma_d(G)$.

Proposition 2.2. [5] Let $G$ be a graph with no isolated vertices and two vertex disjoint $\gamma$- sets. Then $\gamma(G) + 1 \leq \gamma_d(G) \leq 2\gamma(G)$.

Theorem 2.3. [5] A graph $G$ has $V$ as its unique $\gamma_d(G)$- set if and only if for each $v \in V$, there is a vertex with degree one in $N[v]$.

Proposition 2.4. Let $G$ be a graph of order $n$. If $G$ has $t \geq 3$ vertices of degree $n - 1$, then $b_d(G) = \lfloor \frac{t}{2} \rfloor$.

Proof. Clearly, $\gamma_d(G) = 2$. Set

$$W = \{x \in V(G) : \text{deg}(x) = n - 1\}.$$

Since $W$ is a clique in $G$, then we can choose a matching, $M$, in $< W >$ of size $\lfloor \frac{t}{2} \rfloor$. By removing the edges of $M$ of $G$, we obtain a subgraph $G'$ which $G'$ has at most one vertex of degree $n - 1$. Hence $\gamma_d(G') > 2$, and $b_d(G) \leq \lfloor \frac{t}{2} \rfloor$.

On the other hand if we remove $l$ edges of $G$ with $l < \lfloor \frac{t}{2} \rfloor$, then the remaining subgraph, $H$, has at least two vertices of degree $n - 1$, and $\gamma_d(H) = 2$. Hence $b_d(G) \geq \lfloor \frac{t}{2} \rfloor$. Therefore $b_d(G) = \lfloor \frac{t}{2} \rfloor$. □

Proposition 2.5. Let $H$ be a spanning subgraph obtained by removing $k$ edges from a graph $G$. Then $b_d(G) \leq b_d(H) + k$.

Proof. Let $E' = E(G) \setminus E(H)$ and $B$ be a double bondage set of $H$. Then $\gamma_d((G \setminus E') \setminus B) = \gamma_d(H \setminus B) > \gamma_d(H) \geq \gamma_d(G)$ and so $b_d(G) \leq |B| + |E'| = b_d(H) + k$. □
Theorem 2.6. Let $G$ be a graph. Suppose that $\{x, y, z\}$ is a clique of $G$ and $x, y$ are not support vertices of $G$. Then

$$b_d(G) \leq d(x) + d(y) - 3.$$ 

Proof. Let

$$X = \{xv : v \in N(x)\} \cup \{yv : v \in N(y)\} \setminus \{xz, yz\},$$

and $H = G - X$. Since $x, y$ are not support vertices of $G$, then $H$ has not isolated vertex and therefore has a double domination set. Suppose that $D$ is a $\gamma_d$-domination set of $H$. Since the degree of $y$ and $x$ in $H$ is equal to 1 and $z$ is the unique support vertex of $y$ and $x$ in $H$, we conclude that $x, y, z \in D$. It is not difficult to see that $D \setminus \{x\}$ is a double domination for $G$. Hence $\gamma_d(G) < \gamma_d(H)$ and we conclude that

$$b_d(G) \leq |X| = d(x) + d(y) - 3.$$ 

□

Theorem 2.7. Let $G$ be a graph. Suppose that $x - y - z - w - x$ is a cycle of order 4 in $G$ and $x, y$ are not support vertices of $G$. Then

$$b_d(G) \leq d(x) + d(y) - 3.$$ 

Proof. The proof is similar to the proof of Theorem 2.6. □

Theorem 2.8. Let $G$ be a graph with $\gamma_d(G) = n - 1$ and $S$ be the vertices of degree at least 2, which are not support vertices. Suppose that $H = \langle S \rangle$. Then

i) $H \in \{P_1, P_2, P_3, C_3, C_4, C_5\}$,

ii) If $H \in \{P_2, C_4\}$, then $b_d(G) = 1$,

iii) If $H = C_5$, then $G = C_5$ and therefore $b_d(G) = 2$,

v) If $S = \{x\}$, then $b_d(G) = \text{deg}(x) - 1$.

Proof. i) Note that $\not= \emptyset$, since otherwise, we conclude that every vertex of $G$ is leaf or support vertex and hence $\gamma_d(G) = n$, which is a contradiction. It is not difficult to see that $\gamma_d(G) \leq \gamma_d(H) + |V(G) \setminus S|$. Let $H_1$ and $H_2$ be two connected components of $H$ and consider $x_i \in V(H_i)$ for $1 \leq i \leq 2$. Then $V \setminus \{x_1, x_2\}$ is a double dominating set for $G$ and hence $\gamma_d(G) \leq n - 2$, which is a contradiction. Hence the graph $H$ is a connected graph. We claim that $\text{girth}(G) \in \{3, 4, 5, \infty\}$. Suppose that $H$ is a tree with $\text{diam}(H) \geq 3$. Consider two leaves $x, y$ of $H$. Hence $V(G) \setminus \{x, y\}$ is a double domination of $G$, which is a contradiction. Hence $\text{diam}(H) \leq 2$ and therefore $H$ is an star. Let $x$ be the center of this star. If $\text{deg}_H(x) \geq 3$ and $z, y$ are two adjacent vertices of $x$, then $V(G) \setminus \{z, y\}$ is double domination of $G$, which is a contradiction. Hence $\text{deg}_H(x) \leq 2$ and therefore $H \in \{P_1, P_2, P_3\}$.

Now suppose that $H$ has a cycle, $C_t$. If $t \geq 6$, we can choose two vertices $x$ and $y$ of cycle $C_t$, with $d(x, y) = 3$. Then $V(G) \setminus \{x, y\}$ is a double domination of $G$, which is contradiction. Hence $\text{girth}(H) \in \{3, 4, 5\}$. At first suppose that $\text{girth}(H) = 3$ and $x - y - z - x$ is a cycle of length 3 in $H$. If $\text{deg}_H(x), \text{deg}_H(y), \text{deg}_H(z) \geq 3$, then $V(G) \setminus \{x, y\}$ is a double domination set for $G$, which is
a contradiction. Suppose that \( \text{deg}_H(y) = 2 \). If \( \text{deg}_H(x) \geq 3 \), then there exists \( w \in N(x) \cap S \) with \( w \neq z, y \). Hence \( V \setminus \{y, w\} \) is a double dominating set for \( G \) and this is a contradiction. Hence \( \text{deg}_H(x) = \text{deg}_H(z) = 2 \) and we conclude that \( H = C_3 \). By the same argument, we have \( H = C_4 \) or \( H = C_5 \) where \( \text{girth}(H) = 4 \) or \( \text{girth}(H) = 5 \), respectively.

ii) Suppose that \( H = P_2 \) and \( S = \{x, y\} \). If \( \text{deg}_G(x), \text{deg}_G(y) \geq 3 \), then \( V \setminus \{x, y\} \) is a double dominating set of \( G \), which is a contradiction. Hence \( \text{deg}_G(x) = 2 \) or \( \text{deg}_G(y) = 2 \). Suppose that \( \text{deg}_G(x) = 2 \). Let \( z \) be the unique support vertex which is adjacent to \( x \). Hence all vertices of \( G \setminus \{xz\} \) are leaves or support vertices. Hence \( \gamma_d(G \setminus \{xz\}) = n \) and therefore \( \text{bd}(G) = 1 \). If \( H = C_4 \), then by the same argument, \( H \) has at least 3 vertices of degree 2 (in \( G \)). Suppose that \( x \) and \( y \) are two adjacent vertices of \( H \) of degree 2. Hence \( \gamma_d(G \setminus \{xy\}) = n \) and therefore \( \text{bd}(G) = 1 \).

iii) Let \( H = C_5 \). If \( x \) is a vertex of \( H \) of degree at least 3 in \( G \), then consider two vertices \( y \) and \( z \) of \( H \) which are adjacent to \( x \). Hence \( V \setminus \{y, z\} \) is a double domination of \( G \), which is a contradiction. Hence \( G = C_5 \) and \( \text{bd}(G) = 2 \).

v) The result is obvious.

\[ \square \]

3. Bondage number of graph products

In this section we study the double bondage number of some products of graphs. In [3], studied the double domination number of join of two graph. We summarized their results in the following proposition.

**Proposition 3.1.** Let \( G \) and \( H \) be any non-trivial graphs. Then,

\[
\gamma_d(G \vee H) = \begin{cases} 
4 & \gamma(G), \gamma(H) > 2 \\
3 & \gamma(G) = 2, \gamma(H) \geq 2 \text{ or } \gamma(G) = 1, \gamma_d(G) \geq 3, \gamma(H) \geq 2, \\
2 & \gamma(G) = \gamma(H) = 1 \text{ or } \gamma_d(G) = 2, \gamma(H) \geq 2.
\end{cases}
\]

**Remark 3.2.** Let \( G \) and \( H \) be two non-trivial graphs of orders \( n \) and \( m \), respectively. Let \( \gamma(G) = \gamma(H) = 1 \). If \( G \) has \( r \) vertices of degree \( n - 1 \) and \( H \) has \( s \) vertices of degree \( m - 1 \), then \( \text{bd}(G \vee H) = \lfloor \frac{r+s}{2} \rfloor \).

**Proof.** It follows from proposition 2.4.

**Lemma 3.3.** Let \( G \) and \( H \) be two non-trivial graphs of orders \( n \) and \( m \), respectively. If \( \gamma_d(G) = 2 \) and \( \gamma(H) > 1 \), then \( \text{bd}(G \vee H) = \text{bd}(G) \).

**Proof.** By Proposition 3.1, we have \( \gamma_d(G \vee H) = 2 \). Since \( \gamma_d(G) = 2 \), we conclude that, \( G \) has at least two vertices of degree \( n - 1 \). Suppose that \( G \) has \( t \) vertices \( x_1, x_2, \ldots, x_t \), of degree \( n - 1 \). Then \( \text{bd}(G) = \lfloor \frac{t}{2} \rfloor \). Let \( E_0 \subseteq E(G \vee H) \) such that \( \gamma_d(G \vee H \setminus E_0) < \gamma_d(G \vee H) \). If \( E_0 \subseteq E(G) \), then \( G \vee H \setminus E_0 = (G \setminus E_0) \vee H \). Hence \( \gamma_d(G \setminus E_0) \geq 3 \), which means that \( \text{bd}(G \vee H) \geq b_d(G) \). Suppose that \( E_0 \not\subseteq E(G) \). Let \( E_1 = E_0 \cap E(G) \). If \( \gamma(G) < \gamma_d(G \setminus E_1) \), then \( |E_0| \geq b_d(G) \) and hence \( \text{bd}(G \vee H) \geq \text{bd}(G) \). Suppose that \( |E_1| < b_d(G) = \lfloor \frac{t}{2} \rfloor \) and let \( l = t - |V(E_1)| \). Hence \( l \geq 2 \). Suppose that \( \{x_1, x_2, \ldots, x_l\} \subseteq \{x_1, x_2, \ldots, x_t\} \setminus V(E_1) \).
There are at least \( l - 1 \) edge with an end vertex in \( H \) and other end vertex in \( \{x_1, x_2, \ldots, x_l\} \), since otherwise

\[
(G \vee H) \setminus E_0 \cong K_2 \vee L,
\]
for some graph \( L \) and therefore \( \gamma_d((G \vee H) \setminus E_0) = 2 \), which is a contradiction. Hence

\[
|E_0| \geq l - 1 + |E_1| = t - |V(E_1)| - 1 + |E_1| \geq t - 1 - |E_1| > t - 1 - \left\lfloor \frac{t}{2} \right\rfloor
\]
and hence \( |E_0| \geq \left\lfloor \frac{t}{2} \right\rfloor \). Therefore \( b_d(G \vee H) \geq b_d(G) \).

Now choose a maximum matching, \( M \), of size \( \left\lfloor \frac{t}{2} \right\rfloor \) in the set \( \{x_1, x_2, \ldots, x_l\} \). Hence

\[
\gamma_d(G \vee H \setminus M) = \gamma_d((G \setminus M) \vee H) \geq 3,
\]

since \( \gamma_d(G \setminus M) \geq 3 \). Hence \( b_d(G \vee H) \leq b_d(G) \).

\textbf{Lemma 3.4.} Let \( G \) and \( H \) be two non-trivial graphs of orders \( n \) and \( m \), respectively. If \( \gamma(G) = 1 \), \( \gamma_d(G) \geq 3 \) and \( \gamma(H) \geq 2 \), then \( b_d(G \vee H) \leq b_d(G) + b(H) \).

\textit{Proof.} Clearly, \( \gamma_d(G \vee H) = 3 \). Let \( E_1 \subseteq E(G) \) and \( E_2 \subseteq E(H) \) with \( |E_1| = b_d(G) \) and \( |E_2| = b_d(H) \).

By removing the edges of \( E_1 \) from \( G \) and \( E_2 \) from \( H \), we obtain subgraphs \( G' \) and \( H' \) which \( \gamma_d(G') \geq 4 \) and \( \gamma(H') \geq 3 \). Hence \( \gamma_d(G' \vee H') = 4 \) and we conclude that \( b_d(G) \leq b_d(G) + b(H) \).

\textbf{Lemma 3.5.} Let \( G \) and \( H \) be two non-trivial graphs of orders \( n \) and \( m \), respectively. Let \( \gamma(G) = 2 \),

\( i) \) If \( \gamma(H) = 2 \), then \( b_d(G \vee H) \leq b(G) + b(H) \).

\( ii) \) If \( \gamma(H) > 2 \), then \( b_d(G \vee H) = b(G) \).

\textit{Proof.} \( i) \) Clearly, \( \gamma_d(G \vee H) = 3 \). Let \( E_1 \subseteq E(G) \) and \( E_2 \subseteq E(H) \) with \( |E_1| = b(G) \) and \( |E_2| = b(H) \).

By removing the edges of \( E_1 \) from \( G \) and \( E_2 \) from \( H \), we obtain subgraphs \( G' \) and \( H' \) which \( \gamma(G') \geq 3 \) and \( \gamma(H') \geq 3 \). Hence \( \gamma_d(G' \vee H') = 4 \), and \( b_d(G) \leq b(G) + b(H) \).

\( ii) \) Clearly, \( \gamma_d(G \vee H) = 3 \). Let \( E_1 \subseteq E(G) \) with \( |E_1| = b(G) \). By removing the edges \( E_1 \) from \( G \), we obtain subgraphs \( G' \) which \( \gamma(G') > 2 \) and hence \( \gamma_d(G' \vee H') = 4 \), which implies that \( b_d(G \vee H) \leq b(G) \).

On the other hand if we remove \( l \) edges of \( G \vee H \) with \( l < b(G) \), then we obtain subgraph, \( G' \), with \( \gamma(G') = 2 \) and \( \gamma(H') \geq 3 \) and \( \gamma_d(G' \vee H') = 3 \).

So \( b_d(G \vee H) \geq b(G) \). Therefore \( b_d(G \vee H) = b(G) \).

\textbf{Example 3.6.} Let \( G = 2K_2 \) and \( H = 2K_2 \), therefore \( \gamma(G) = 2 \) and \( \gamma(H) = 2 \). Then Theorem 3.5 implies that \( b_d(G \vee H) \leq 2 \) and it is easy to see that \( b_d(G \vee H) = 2 \).

\textbf{Theorem 3.7.} Let \( G \) be a non-trivial graph of order \( n \).

\[
b_d(G \vee K_1) = \begin{cases} \left\lfloor \frac{n}{2} \right\rfloor & \text{if } G \text{ has t-1 vertices of degree n-1,} \\ \leq b(G) & \text{if } \gamma(G) > 1, \end{cases}
\]
Proof. a) Set \( W_1 = \{ x \in V(G) : \deg_G(x) = n - 1 \} \) and \( W = W_1 \cup \{ u \} \) such that \( \{ u \} = V(K_1) \). Clearly, \( \gamma_d(G \vee K_1) = 2 \). Since \( W \) is an clique in \( G \vee K_1 \), then we can choose a matching, \( M \), in \( \langle W \rangle \) from size \( \lfloor \frac{t}{2} \rfloor \). By removing the edges of \( M \) from \( G \vee K_1 \), we obtain a subgraph \( K \) which \( K \) has at most one vertex of degree \( n \). Hence \( \gamma_d(K) > 2 \) and so \( b_d(G \vee K_1) \leq \lfloor \frac{t}{2} \rfloor \).

On the other hand if we remove \( l \) edges of \( G \vee K_1 \) with \( l < \lfloor \frac{t}{2} \rfloor \), then the remaining subgraph, \( K \), has at least two vertices of degree \( n \) and \( \gamma_d(K) = 2 \). Hence \( b_d(G \vee K_1) \geq \lfloor \frac{t}{2} \rfloor \). Therefore \( b_d(G \vee K_1) = \lfloor \frac{t}{2} \rfloor \).

b) Let \( \gamma(G) > 1 \), then \( \gamma_d(G \vee K_1) = 1 + \gamma(G) \). Suppose that \( E \subseteq E(G) \) with \( |E| = b(G) \). By removing the edges of \( E \) from \( G \), we obtain a subgraph \( G' \) of \( G \), which \( \gamma(G') > \gamma(G) \). Hence \( \gamma_d(G' \vee K_1) > \gamma_d(G \vee K_1) \) and So \( b_d(G \vee K_1) = b(G) \).

\[ \square \]

Example 3.8. Theorem 3.7 is sharp for arbitrarily many graphs. Let \( W_6 := C_5 \vee K_1 \), then \( b_d(W_6) = 2 = b(C_5) \).

For the cartesian product of graphs we have following lemma.

Proposition 3.9. Let \( G \) and \( H \) be two graphs,

\[ \gamma_d(G \square H) \leq \min \{ |V(G)|\gamma_d(H), |V(H)|\gamma_d(G) \}. \]

Proof. Let \( V(G) = \{ v_1, v_2, \ldots, v_n \} \) and \( V(H) = \{ u_1, u_2, \ldots, u_m \} \). Suppose that \( \gamma_d(H) = \{ u_1, u_2, \ldots, u_k \} \). It is not difficult to see that the set

\[ S = \{ (v_i, u_j) : 1 \leq i \leq n, 1 \leq j \leq k \} \]

is a double dominating set of \( G \square H \). Hence \( \gamma_d(G \square H) \leq |V(G)|\gamma_d(H) \). Similarly \( \gamma_d(G \square H) \leq |V(H)|\gamma_d(G) \). \[ \square \]

Corollary 3.10. \( \gamma_d(G \square K_2) \leq 2\gamma_d(G) \).

Proposition 3.11. Let \( n \geq 2 \) be an integer. Then \( \gamma_d(P_n \square P_2) = n + 1 \).

Proof. Let \( G = P_n \square P_2 \) and \( D \) be a minimum double dominating set for \( G \). Let

\[ T = \{ (x, y) : x \in D, y \in N[x] \}. \]

By counting \( |T| \) into two ways, we conclude that \( |D| \geq \frac{2V(G)}{\Delta(G)+1} \). Clearly \( D \) contains at least one vertex of degree two. Since \( \Delta(G) = 3 \), we find that \( |D| > \frac{2V(G)}{\Delta(G)+1} \). Hence \( \gamma_d(P_n \square P_2) \geq n + 1 \). On the other hand it is not difficult to see that \( P_n \square P_2 \) has a double dominating set of size \( n + 1 \). \[ \square \]

Proposition 3.12. Let \( n \geq 2 \) be an integer. Then \( b_d(P_n \square P_2) = 1 \)

Proof. This is a simple results of Theorem 2.6. \[ \square \]

Now we study the generalized corona product of graphs. The proof of the following lemma is easy.
Lemma 3.13. Let $G$ and $H_1, H_2, \ldots, H_n$ be graphs without isolated vertices. Then
\[
\gamma_d(G \circ \bigwedge_{i=1}^{n} H_i) = \sum_{i=1}^{n}(1 + \gamma(H_i)).
\]

Theorem 3.14. Let $G$ and $H_1, H_2, \ldots, H_n$ be graphs without isolated vertices. Then
\[
b_d(G \circ \bigwedge_{i=1}^{n} H_i) = \min_{1 \leq i \leq n} b(H_i)
\]

Proof. Suppose that $\min_{1 \leq i \leq n} b(H_i) = b(H_j) = l$. Consider $E \subseteq E(H_j)$ of size $l$ such that $\gamma(H_i \setminus E) > \gamma(H_i)$. Hence
\[
\gamma_d(G \circ \bigwedge_{i=1}^{n} H_i \setminus E) = \gamma_d(G \circ \bigwedge_{i=1}^{n} H_i''),
\]
where $H_i''$ is isomorphic to $H_i$ for any $i \neq j$ and $H_j''$ is isomorphic to $H_j \setminus E$. Hence $\gamma(H_j'') > \gamma(H_j)$. Therefore
\[
\gamma_d(G \circ \bigwedge_{i=1}^{n} H_i \setminus E) > \gamma_d(G \circ \bigwedge_{i=1}^{n} H_i).
\]

Hence
\[
b_d(G \circ \bigwedge_{i=1}^{n} H_i) \leq \min_{1 \leq i \leq n} b(H_i).
\]

It is obvious that
\[
b_d(G \circ \bigwedge_{i=1}^{n} H_i) \geq \min_{1 \leq i \leq n} b(H_i)
\]
and the result is obtained.

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