ON THE ZERO FORCING NUMBER OF GENERALIZED SIERPIŃSKI GRAPHS

EBRAHIM VATANDOOST*, FATEMEH RAMEZANI AND SAEID ALIKHANI

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Abstract. In this article we study the Zero forcing number of Generalized Sierpiński graphs $S(G, t)$. More precisely, we obtain a general lower bound on the Zero forcing number of $S(G, t)$ and we show that this bound is tight. In particular, we consider the cases in which the base graph $G$ is a star, path, a cycle or a complete graph.

1. Introduction

Let $G = (V, E)$ be a non-empty graph of order $n$, vertex set $V$ and edge set $E$. For a given graph $G$ and $S \subseteq V(G)$, we denote by $\langle S \rangle$ the subgraph induced by $S$. For a vertex $v \in V(G)$, the set $N_G(v) = \{u : uv \in E(G)\}$ is the open neighborhood of $v$, and the degree of a vertex $v \in V(G)$ is $\deg_G(v) = |N_G(v)|$.

The letters of a word $u$ of length $t$ are denoted by $u_1u_2 \ldots u_t$ and the concatenation of two words $u$ and $v$ is denoted by $uv$. Let $V^t$ be the set of words of size $t$ on alphabet $V$. In [12], Klavzar and Milutinović introduced the graph $S(K_n, t)$ whose vertex set is $V^t$, where $u$ is adjacent to $v$ if and only if there exists $1 \leq i \leq t$ such that:

(i) $u_j = v_j$, if $j < i$; (ii) $u_i \neq v_i$; (iii) $u_j = v_i$ and $v_j = u_i$ if $j > i$.

When $n = 3$, those graphs are isomorphic to the Tower of Hanoi graphs. In [13], those graphs have been called Sierpiński graphs. This construction was generalized in [10] for any graph $G = (V, E)$, by defining the generalized Sierpiński graph, $S(G, t)$, as the graph with vertex set $V^t$ and edge set defined

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*Corresponding author.

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These motivated us to consider the zero forcing number of Generalized Sierpiński graphs. For this aim and is denoted by $G$ is a family of induced disjoint paths in the graph that cover (or include) all vertices of the graph. Let each vertex of a graph $G$ be given one of two colors “black” and “white”. Let $Z$ denote the (initial) set of black vertices of $G$. If the white vertex $u_2$ is the only white neighbor of a black vertex $u_1$, then $u_1$ changes the color of $u_2$ to black (color-change rule) and we say “$u_1$ forces $u_2$” which we denote by $u_1 \rightarrow u_2$. A sequence, $u_1 \rightarrow u_2 \rightarrow \cdots \rightarrow u_i \rightarrow u_{i+1} \rightarrow \cdots \rightarrow u_t$, obtained through repetitious applications of the color-change rule is called a forcing chain. The set $Z$ is said to be a zero forcing set of $G$ if all vertices of $G$ will be turned black after nitely many applications of the color-change rule. The zero forcing number, $Z(G)$, of $G$ is the minimum cardinality among all zero forcing sets. In [1] it is shown that for any graph $G$, $M(G) \leq Z(G)$. A path covering of a graph is a family of induced disjoint paths in the graph that cover (or include) all vertices of the graph. The minimum number of such paths that cover the vertices of a graph $G$ is the path cover number of $G$ and is denoted by $P(G)$. Since the forcing chains form a set of covering paths we have $P(G) \leq Z(G)$. These motivated us to consider the zero forcing number of Generalized Sierpiński graphs. For this aim

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we obtain the lower bound for \( Z(S(G, t)) \) for any graph \( G \) and we discuss the tightness of this bound. Also, the zero forcing numbers of Generalized Sierpiński graph of path, cycle, star and complete graph are determined.

![Figure 2. The Sierpiński graph \( S(G, 3) \) for the graph \( G \) of Figure 1.]

2. Preliminaries

First we give some facts that we need in later sections.

**Theorem 2.1.** \([1][5][17]\) Let \( G \) be a connected graph of order \( n \geq 2 \). Then

i. \( Z(G) = 1 \) if and only if \( G \simeq P_n \).

ii. \( Z(G) = n - 1 \) if and only if \( G \simeq K_n \).

iii. If \( G \) is a tree, then \( Z(G) = P(G) \).

iv. For any integer \( n \geq 2 \), \( Z(K_{1,n}) = n - 1 \).

**Theorem 2.2.** \([4]\) Let \( G \) be any graph. Then

i. For \( v \in V(G) \), \( Z(G) - 1 \leq Z(G \setminus \{v\}) \leq Z(G) + 1 \).

ii. For \( e \in E(G) \), \( Z(G) - 1 \leq Z(G \setminus \{e\}) \leq Z(G) + 1 \).

**Theorem 2.3.** \([16]\) For any tree \( T \) and any positive integer \( t \), \( S(T, t) \) is a tree.

3. Main Results

In this section we obtain a lower bound for \( Z(S(G, t)) \) and then we show that this bound is tight.

**Theorem 3.1.** Let \( G \) be a graph of order \( n \) and size \( m \). Then for any integer \( t \geq 2 \),

\[
Z(S(G, t)) \geq n^{t-1}Z(G) - m \frac{n^{t-1} - 1}{n - 1}.
\]

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Proof. Let \( e^t_{ij} = \{ij \ldots j, ji \ldots i\} \in E(S(G,t)) \) where \( \{i, j\} \in E(G) \). Notice that in this notation \( e^t_{ij} = e^t_{ji} \). Thus

\[
S(G,t) \setminus \{e^t_{ij} : \{i, j\} \in E(G)\} \simeq nS(G, t - 1).
\]

Now, we use Theorem 2.2 to achieve the result. Let \( \{i, j\} \) be an edge in \( G \). Then \( Z(S(G,t)) \geq Z \left( S(G,t) \setminus \{e^t_{ij}\} \right) - 1 \). By using this structure for all \( e^t_{ij} \in E(S(G,t)) \), we have

\[
Z(S(G,t)) \geq nZ(S(G, t - 1)) - |\{e^t_{ij} : \{i, j\} \in E(G)\}| = nZ(S(G, t - 1)) - m.
\]

Again

\[
Z(S(G,t - 1)) \geq nZ(S(G, t - 1)) - |\{e^{t-1}_{ij} : \{i, j\} \in E(G)\}| = nZ(S(G, t - 2)) - m.
\]

Therefore, \( Z(S(G,t)) \geq n^2Z(G, t - 2) - nm - m \). With similar argument we have

\[
Z(S(G,t)) \geq n^{t-1}Z(G) - m\frac{n^{t-1} - 1}{n - 1}.
\]

\( \square \)

In the next Theorem, the zero forcing number of Generalized Sierpiński graph of \( K_n \) is obtained and we will see that the lower bound in Theorem 3.1 is tight.

**Theorem 3.2.** For any positive integers \( n \) and \( t \),

\[
Z(S(K_n,t)) = \frac{n^t - 2n^{t-1} + n}{2}
\]

Proof. To obtain the upper bound we define the following sets.

For \( t = 2 \), \( Z_2 = \{ij : 1 \leq i \leq n - 1 \text{ and } i \leq j \leq n - 1\} \).

For \( t \geq 3 \), \( Z_t = \{iz : 1 \leq i \leq n \text{ and } z \in Z_{t-1}\} \setminus \{ij \ldots j : 2 \leq i \leq n \text{ and } 1 \leq j < i\} \).

By induction on \( t \geq 2 \) we show that \( Z_t \) is a forcing set of \( S(K_n,t) \). For \( t = 2 \), use the following instructions from \( i = 1 \) to \( i = n - 1 \) to make all vertices black.

\[
i i \rightarrow in
\]

\[
i j \rightarrow ji \text{ for } i + 1 \leq j \leq n,
\]

and at the end, \( n1 \rightarrow nn \). Hence, \( Z_2 \) is a forcing set of \( S(K_n,2) \). Now suppose that for any \( t = k \), \( Z_t \) is a forcing set of \( S(K_n,t) \) and we show that \( Z_{k+1} \) is a forcing set of \( S(K_n, k+1) \). Since \( Z_k \) is a forcing set, all vertices in \( V_1 \) will be forced by \( Z_{k+1} \). Use the following structures from \( i = 1 \) to \( i = n - 1 \).

\[
i n \ldots n \rightarrow ni \ldots i
\]

\[
i j \ldots j \rightarrow ji \text{ for } i + 1 \leq j \leq n,
\]

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and at the end, \( n \cdot n_1 \rightarrow n \ldots n \). So \( Z_{k+1} \) is a forcing set of \( S(K_n, k + 1) \). Therefore, for any \( t \geq 2 \)
\[
Z(S(K_n, t)) \leq |Z_t| = n|Z_{t-1}| - \frac{n(n-1)}{2} \\
= n^2|Z_{t-2}| - \frac{n^2(n-1)}{2} - \frac{n(n-1)}{2} \\
\vdots \\
= n^{t-2}|Z_2| - \frac{n^{t-2}(n-1)}{2} - \ldots - \frac{n^2(n-1)}{2} - \frac{n(n-1)}{2} \\
= \frac{n(n-1)}{2} \left( n^{t-2} - \frac{n^{t-2} - 1}{n-1} \right) \\
= n^t - 2n^{t-1} + n.
\]

Theorems 3.1 and 2.1 complete the proof.

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**Zero Forcing Number On** \( S(C_n, t) \)

Here we will give the similar result for \( Z(S(C_n, t)) \).

**Theorem 3.3.** For any integers \( n \geq 4 \) and \( t \geq 2 \), \( Z(S(C_n, t)) = \frac{n^t - 2n^{t-2} + n}{n-1} \).

**Proof.** Let \( t = 2 \). Let \( Z_2 = \{ ii : 1 \leq i \leq n \} \cup \{12, 13\} \). Follow this structure to force all vertices. Notice that the addition is taken modulo \( n \).

\[
11 \rightarrow 1n \rightarrow n1, 13 \rightarrow 14 \rightarrow \ldots \rightarrow 1(n-1) \\
\text{for } 1 \leq i \leq n-1 \\
i(i+1) \rightarrow (i+1)i \rightarrow (i+1)(i+n-1) \rightarrow (i+1)(i+n-2) \rightarrow \ldots \rightarrow (i+1)(i+2)
\]

Hence, \( Z(S(C_n, 2)) \leq |Z_2| = n + 2 \). Now we show that \( Z(S(C_n, 2)) = n + 2 \). Let \( Z \) be a forcing set of minimum cardinality. Since for each \( 1 \leq i \leq n \), \( V_i \) has two vertices of degree three, \( |V_i \cap Z| \geq 1 \). We can assume that the starting forcing chain starts in \( V_1 \).

**Case I.** \( \{12, 1n\} \cap Z \neq \emptyset \). Let \( 12 \in Z \). Since \( \text{deg}(12) = 3 \), \( |V_1 \cap Z| = 3 \) or \( |V_1 \cap Z| = 2 \) and \( 21 \in Z \).

If \( |V_1 \cap Z| = 3 \), then \( |Z| \geq n + 2 \). Otherwise, since \( \text{deg}(21) = 3 \), \( \{22, 2n\} \cap Z \neq \emptyset \). Thus \( |V_2 \cap Z| \geq 2 \).

**Case II.** \( \{12, 1n\} \cap Z = \emptyset \). If \( 11 \in Z \), then \( |V_1 \cap Z| = 3 \). Otherwise, \( |V_1 \cap Z| = 2 \) and \( \{21, 1n\} \cap Z \neq \emptyset \).

Let \( 21 \in Z \). Since \( \text{deg}(21) = 3 \), \( 22 \in Z \) or \( 2n \in Z \). So \( |V_2 \cap Z| \geq 2 \).

In both cases, \( |Z| \geq n + 2 \), so that \( Z(S(C_n, 2)) = n + 2 \). Therefore, \( Z(S(C_n, 2)) = n + 2 \). For \( t \geq 3 \), let
\[
Z_t = \{ iz : z \in Z_{t-1} \text{ and } 1 \leq i \leq n \} \setminus \{(i(i-1) \cdots (i-1)) : 2 \leq i \leq n\} \cup \{n1 \cdots 1\}.
\]

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By induction, we see that $Z_t$ is a forcing set of $S(C_n, t)$. First suppose that $t = 3$. Since $V_1 \cap Z_3 = \{1z : z \in Z_2\}$, so all vertices in $V_1$ will be blacked by $Z_3$. Now for omitted vertices of $\{iz : z \in Z_2, 1 \leq i \leq n\}$, use the following forcing chain:

$$
\begin{align*}
122 & \rightarrow 211 \\
233 & \rightarrow 322 \\
\vdots \\
(n-1)nn & \rightarrow n(n-1)(n-1) \\
1nn & \rightarrow n11
\end{align*}
$$

Hence, $Z_3$ is a forcing set of $S(C_n, 3)$. Suppose that for any $t \geq k$, $Z_t$ is a forcing set of $S(C_n, t)$. Now we show that this is true for $t = k + 1$. Since, $Z_k$ is a forcing set, all vertices in $V_1$ will be black. Also $1n \cdots n \rightarrow n1 \cdots 1$. For $2 \leq i \leq n-1$, $(i-1)i \cdots i \rightarrow i(i-1) \cdots (i-1)$ and since, $(V_i) \simeq S(C_n, k)$, all vertices in $V_i$ will be forced by $Z_{k+1} \cap V_i$.

Hence, $Z_t$ is a forcing set for $S(C_n, t)$ for each $t \geq 3$ and so

$$
Z(S(C_n, t)) \leq |Z_t| = n|Z_{t-1}| - n = \cdots = n^{t-2}|Z_2| - \sum_{i=1}^{t-2} n^i = \frac{n^t - 2n^{t-2} + n}{n-1}.
$$

To obtain the lower bound, we use the similar argument as in proof of Theorem 3.1.

$$
Z(S(C_n, t)) \geq n^{t-2}|Z(S(C_n, 2))| - \frac{n(n^{t-2} - 1)}{n-1}
= n^{t-2}(n + 2) - \frac{n(n^{t-2} - 1)}{n-1}
= \frac{n^t - 2n^{t-2} + n}{n-1}.
$$

This completes the proof. \hfill \square

**Zero Forcing Number On $S(K_{1,n}, t)$**

Let $V(K_{1,n}) = \{0, 1, \ldots, n\}$ where $\deg(0) = n$. Let $S(K_{1,n}, 0) = K_1$. For any positive integer $t$, we use the notation

$$
S(K_{1,n}, t) \sim S(K_{1,n}, t-1)
$$

when vertex $00 \cdots 00$ of $S(K_{1,n}, t)$ is adjacent to an extreme of $V_i$ in $S(K_{1,n}, t-1)$ for some $1 \leq i \leq n$.

Now let $G_2 : S(K_{1,n}, 1) \sim K_1$ and $G_t : S(K_{1,n}, t-1) \sim G_{t-1}$. With this notations in mind we will prove the following results.

**Lemma 3.4.** For any positive integer $n$ and $t \geq 2$, $S(K_{1,n}, t) \setminus \{00 \cdots 00\} \simeq nG_t$ and also for $t \geq 3$ we have $G_t \setminus \{i0 \cdots 0\} \simeq (n+1)G_{t-1}$ for some $1 \leq i \leq n$.
Proof. We use induction on \( t \) to reach the result. For \( t = 2 \), we have
\[
S(K_{1,n}, 2) \setminus \{00\} \simeq \bigcup_{i=1}^{n} (V_i \sim K_1)
\]
where \( V_i \simeq S(K_{1,n}, 1) \) for \( 1 \leq i \leq n \). Thus \( V_i \sim K_1 \simeq S(K_{1,n}, 1) \sim K_1 \) for any \( 1 \leq i \leq n \) and so \( S(K_{1,n}, 2) \setminus \{00\} \simeq nG_2 \). Now, suppose that for any \( k \geq t \),
\[
S(K_{1,n}, k) \setminus \{0 \cdots 0\} \simeq nG_k
\]
and we will prove the result for \( t = k+1 \). For any \( 1 \leq i \leq n \) there is the following path in \( S(K_{1,n}, k+1) \):
\[
0 \cdots 0 - 0 \cdots 0i - 0 \cdots 0i0 - 0 \cdots 0i2 - 0 \cdots 0i00 - \cdots - 0i \cdots i0 - 0i \cdots i - i0 \cdots 0.
\]
Hence,
\[
S(K_{1,n}, k+1) \setminus \{0 \cdots 0\} \simeq \bigcup_{i=1}^{n} (V_i \sim V_{0i} \sim V_{00i} \sim \cdots \sim V_{0 \cdots 0i}).
\]
As we know \( V_i \simeq S(K_{1,n}, k), V_{0i} \simeq S(K_{1,n}, k-1) \) and so on. Therefore,
\[
S(K_{1,n}, k+1) \setminus \{0 \cdots 0\} \simeq n(S(K_{1,n}, k) \sim S(K_{1,n}, k-1) \sim \cdots S(K_{1,n}, 1) \sim K_1)
\]
\[
\simeq n(S(K_{1,n}, k) \sim G_k)
\]
\[
\simeq nG_{k+1}.
\]
Since \( N_{G_t}(\{i0 \cdots 0\}) = \{i0 \cdots 0j : 1 \leq j \leq n\} \cup \{i0 \cdots i\} \) for some \( 1 \leq i \leq n \),
\[
G_t \setminus \{i0 \cdots 0\} \simeq nG_{t-1} \cup G_{t-1} \simeq (n+1)G_{t-1}.
\]
This completes the proof. \( \square \)

**Theorem 3.5.** For any positive integers \( n \) and \( t \),
\[
Z(S(K_{1,n}, t)) = (n - 1)(n + 1)^{t-1}.
\]

**Proof.** First, we use Lemma 3.4 and Theorem 2.2 to obtain the lower bound.
\[
Z(S(K_{1,n}, t)) \geq Z(S(K_{1,n}, t) \setminus \{0 \cdots 0\}) - 1
\]
\[
= nZ(G_t) - 1
\]
\[
\geq n(Z(G_t \setminus \{i0 \cdots 0\}) - 1) - 1
\]
\[
= n(n + 1)Z(G_{t-1}) - (n + 1)
\]
\[
\geq n(n + 1) (Z(G_{t-1} \setminus \{0i0 \cdots 0\}) - 1) - (n + 1)
\]
\[
= n(n + 1)^2Z(G_{t-2}) - (n + 1)^2
\]
\[\vdots\]
\[
\geq n(n + 1)^{t-2}Z(G_2) - (n + 1)^{t-2}
\]

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But $Z(G_2) = Z(S(K_{1,n}, 1) \sim K_1) = Z(K_{1,n} \sim K_1) = Z(K_{1,n+1}) = n$. Hence,

$$Z(S(K_{1,n}, t)) \geq n^2(n+1)^{t-2} - (n+1)^{t-2} = (n+1)^{t-1}(n-1).$$

To obtain the upper bound we define the following sets. For $t = 2$, let

$$Z_2 = \{ij : 1 \leq i, j \leq n\} \setminus \{nn\},$$

for $t = 3$, let

$$Z_3 = \{iz : 0 \leq i \leq n \text{ and } z \in Z_2\} \cup \{inn : 1 \leq i \leq n\} \setminus \{0ii, 0n(n-1) : 1 \leq i \leq n\}$$

and for $t \geq 4$, let

$$Z_t = \{iz : 0 \leq i \leq n \text{ and } z \in Z_{t-1}\} \cup \{i0\cdots n(n-1) : 1 \leq i \leq n\} \setminus \{0\cdots i : 1 \leq i \leq n\}.$$ 

By induction on $t \geq 2$ we show that $Z_t$ is a forcing set of $S(K_{1,n}, t)$. Let $t = 2$. For any $1 \leq i, j \leq n-1$, $\deg(ij) = 1$ and $ij \in Z_2$. So $ij$ forces $i0$. Since $N(i0) \setminus Z_2 = \{0i\}$, $i0$ forces $0i$ for $1 \leq i \leq n-1$. With following the path

$$i0 \to 00 \to 0n \to n0 \to nn$$

all vertices will be black. Hence, $Z_2$ is a forcing set of $Z(S(K_{1,n}, 2))$. Let $t = 3$. All vertices in

$\{iz : 0 \leq i \leq n \text{ and } z \in Z_2\} \cup \{inn : 1 \leq i \leq n\}$

force $ij0$ for $1 \leq i, j \leq n$ and $ij0$ forces $i0j$ and as following it forces $i00$. Now, $\{0ii : 1 \leq i \leq n\}$ will be forced by $\{i00 : 1 \leq i \leq n\}$. Since

$\{0ii : 1 \leq i \leq n\} \cup \{0ij : 1 \leq i \leq n-1 \text{ and } 1 \leq j \leq n\}$

are black, $\{0i0 : 1 \leq i \leq n-1\}$ and then $\{00i : 1 \leq i \leq n-1\}$ will be black. Now, by following the path

$$00i \to 000 \to 00n \to n00 \to 0n(n-1)$$

all vertices will get black. We suppose that for $t = k-1$, $Z_{k-1}$ is a forcing set and we will prove it for $Z_k$. Since $Z_{k-1}$ is a forcing set, $\{iz : 0 \leq i \leq n \text{ and } z \in Z_{k-1}\} \cup \{i0\cdots 0n(n-1) : 1 \leq i \leq n\}$ force all vertices in $V_i$ for $1 \leq i \leq n$. By similar argument as in $t = 3$, we see that the set $\{0i\cdots i : 1 \leq i \leq n\} \cup \{0\cdots 0\text{ }(n(n-1)) \}$ will be forced and so $Z_k$ is a forcing set. Hence,

$$Z(S(K_{1,n}, t)) \leq |Z_t| = (n+1)|Z_{t-1}| = (n+1)^2|Z_{t-2}| = \cdots = (n+1)^{t-2}|Z_2|$$

$$= (n+1)^{t-2}(n^2-1) = (n+1)^{t-1}(n-1).$$

This completes the proof. $\square$

**Zero Forcing Number On $S(P_n, t)$**

Let $V = \{1, 2, \ldots, n\}$ be the vertex set of $P_n$, and $\langle V_{wu} \rangle$ be a copy of $P_n$ in $S(P_n, t)$ for $w \in V^{t-2}$ and $u \in V$. Also we say $\langle V_{wu} \rangle$ and $\langle V_{w'v} \rangle$ are two consecutive paths when $\{x, y\}$ is an edge in $S(P_n, t)$ for $x \in V_{wu}$ and $y \in V_{w'v}$ where $w, w' \in V^{t-2}$ and $u, v \in V$. Also we use $V_{wu} \sim V_{w'v}$ for induced subgraph on $V_{wu} \cup V_{w'v}$. With these notations in mind we will prove the following results.

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Lemma 3.6. Let $t \geq 2$, $w, w' \in V^{t-2}$ and $u, v \in V$. If $\langle w_u \rangle$ and $\langle w_{u'} \rangle$ are two consecutive paths in $S(P_n, t)$, then the path cover number of $V_{wu} \sim V_{w'v}$ is two.

Proof. Since $\langle w_u \rangle$ and $\langle w_{u'} \rangle$ are two consecutive paths, there are $x \in V_{wu}$ and $y \in V_{w'v}$ such that $\{x, y\} \in E(S(P_n, t))$. Hence, $\{u, v\} \in E(P_n)$ and so $u = v + 1$ or $v = u + 1$. Also $\deg(x) = 3$ or $\deg(y) = 3$. Thus $V_{wu} \sim V_{w'v}$ is not a path and by Theorem 2.1, $P(V_{wu} \sim V_{w'v}) \geq 2$. On the other hand, $P(V_{wu} \sim V_{w'v}) \leq P(\langle w_u \rangle) + P(\langle w_{u'} \rangle) = 2$. This completes the proof. □

Theorem 3.7. For any positive integers $n$ and $t$, $Z(S(P_n, t)) = n^{t-1}$.

Proof. As we know there are $n^{t-1}$ copies of $P_n$ in $S(P_n, t)$ and by Lemma 3.6, the path cover number of each pair of consecutive paths is two. Hence, $P(S(P_n, t)) = n^{t-1}$ and by Theorems 2.1 and 2.3 we have $Z(S(P_n, t)) = n^{t-1}$. □

Question 1. As we see the zero forcing number of $S(K_n, t)$ is equal to lower bound in Theorem 3.1. For which other family of graphs the zero forcing number is exactly the lower bound given in this work?

Question 2. Let $G$ be a universal graph of order $n$ with exactly one vertex of degree $n - 1$. Which is the relation between $Z(S(G, t))$ and $Z(G)$?

A tree is called Starlike if it has exactly one vertex of degree more than two. It is denoted by $S(\ell_1, \ell_2, \ldots, \ell_r)$ such that $S(\ell_1, \ell_2, \ldots, \ell_r) \setminus \{v\} = P_{\ell_1} \cup P_{\ell_2} \cup \cdots \cup P_{\ell_r}$, where $v$ is the vertex of degree more than two. One can see that $Z(S(\ell_1, \ell_2, \ldots, \ell_r)) = r - 1$.

Question 3. What is the zero forcing number of Generalized Sierpiński graph of Starlike?

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Ebrahim Vatandoost
Department of Basic Science, Imam Khomeini International University, Qazvin, Iran.
Email: vatandoost@sci.ikiu.ac.ir

Fatemeh Ramezani
Department of Mathematics, Yazd University, P.O.Box 89195-741, Yazd, Iran.
Email: f.ramezani@yazd.ac.ir

Saeid Alikhani
Department of Mathematics, Yazd University, P.O.Box 89195-741, Yazd, Iran.
Email: alikhani@yazd.ac.ir

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