VISUAL CRYPTOGRAPHY SCHEME ON GRAPHS WITH $m^*(G) = 4$

MAHMOOD DAVARZANI

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Abstract. Let $G = (V, E)$ be a connected graph and $\Gamma(G)$ be the strong access structure where obtained from graph $G$. A visual cryptography scheme (VCS) for a set $P$ of participants is a method to encode a secret image such that any pixel of this image change to $m$ subpixels and only qualified sets can recover the secret image by stacking their shares. The value of $m$ is called the pixel expansion and the minimum value of the pixel expansion of a VCS for $\Gamma(G)$ is denoted by $m^*(G)$. In this paper we obtain a characterization of all connected graphs $G$ with $m^*(G) = 4$ and $\omega(G) = 5$ which $\omega(G)$ is the clique number of graph $G$.

1. Introduction

A secret sharing scheme is a method to share a secret among a set of participants such that only qualified subsets can reconstruct the secret from their shares, in addition non-qualified subsets can not obtain any information about the secret.

Visual cryptography scheme, (VCS), is a kind of secret sharing scheme, was introduced by Naor and Shamir [9]. They investigate the case of $((k, n) - VCS)$ where $2 \leq k \leq n$, in which the secret image is visible if and only if $k$ or more participants stack their shares, whereas any set of less than $k$ participants have no information on the secret image. In a VCS, decoder is human visual system and participants in a qualified set can see secret image without knowledge of cryptographic. Ateniese et al. [3, 4] extended this scheme to general access structures. In this model, $P$ is set of participants and $\Gamma = (Q, F)$ is access structure such that $Q \subseteq 2^P$ is the collection of qualified sets and $F \subseteq 2^P$ is the
collection of forbidden sets. We assume that image secret is collection of black and white pixels. Now in a VCS any pixel of this image is replaced by \( m \) subpixels and give to each shares. The number of \( m \) is called the pixel expansion and for a given general access structure \( \Gamma \), the minimum value of \( m \) is denoted by \( m^*(\Gamma) \), and is called the optimal pixel expansion.

If the vertex set \( V \) in a graph \( G = (V, E) \) be the set of participants and any element of \( \Gamma \) be the subset of \( V \) which contains at least one edge, then this access structure is denoted by \( \Gamma(G) \) and the optimal expansion is denoted by \( m^*(G) \). Atenies et al. in [3, 4] studied the construction of VCSs that obtained from graphs and proved that \( m^*(\Gamma) = 2 \) if and only if \( \Gamma = \Gamma(G) \) where \( G \) is a complete bipartite graph. They have also proved that \( m^*(K_n) \) is the smallest \( m \) which \( n \leq \left( \frac{m}{2} \right) \). So \( m^*(K_2) = 2 \) and \( m^*(K_3) = 3 \). In addition they proved that \( m^*(H) \leq m^*(G) \) where \( H \) is induced subgraph \( G \). Since \( m^*(K_6) = 4 \) and \( m^*(K_7) = 5 \) thus if \( m^*(G) = 4 \) and \( H \) be the induced subgraph \( G \) then the biggest induced complete subgraph \( G \) is \( K_6 \).

Arumugam et al. in [1, 2] obtained a characterization of all connected graphs \( G \) for which \( m^*(G) = 2 \) and \( 3 \). In this paper, we study the graphs with \( m^*(G) = 4 \). We give a characterization of all connected graphs \( G \) which \( m^*(G) = 4 \) and \( \omega(G) = 5 \) that \( \omega(G) \) is the clique number of graph \( G \).

2. Preliminaries

Let \( P = \{1, 2, \ldots, n\} \) be a set of participants and let \( 2^P \) denote the set of all subsets of \( P \). If \( Q \subseteq 2^P \) and \( F \subseteq 2^P \) such that \( Q \cap F = \emptyset \), then the pair \( \Gamma = (Q, F) \) is called an access structure on \( P \). We refer elements of \( Q \) as qualified sets and to elements of \( F \) as forbidden sets. We say \( \Gamma \) is strong access structure whenever \( Q \) is monotone increasing and \( F \) is monotone decreasing and \( Q \cup F = 2^P \). Throughout this paper we consider only strong access structures. Define \( \Gamma_0 \) to consist of all the minimal qualify sets : \( \Gamma_0 = \{ A \in Q : A' \notin Q \text{ for all } A' \subseteq A \} \).

Let \( S \) be an \( n \times m \) boolean matrix. If \( X \subseteq P = \{1, 2, \ldots, n\} \) then \( S[X] \) denotes the \( |X| \times m \) matrix obtained from \( S \) by considering its restriction to rows corresponding to the elements in \( X \), further \( S_X \) denotes the vector obtained by applying the boolean OR operation to the rows of \( S[X] \) and \( w(S_X) \) is Hamming weight of \( S_X \).

**Definition 2.1.** [3] Let \( \Gamma = (Q, F) \) be a strong access access structure on a set of \( n \) participants. Two \( n \times m \) boolean matrices \( S^0 \) and \( S^1 \) construct a VCS if there exist a positive real number \( \alpha \) and the set \( \{t_X | X \in Q\} \) satisfying the following conditions:

1. Any qualified set \( X = \{i_1, i_2, \ldots, i_q\} \in Q \) can recover the shared image by stacking their transparencies. Formally \( w(S^0_X) \leq t_X - \alpha m \), whereas \( w(S^1_X) \geq t_X \).
2. Any forbidden set \( X = \{i_1, i_2, \ldots, i_q\} \in F \) has no information on the shared image. Formally the two \( q \times m \) matrices \( S^0[X] \) and \( S^1[X] \) are equal up to a column permutation.

The first property is attributed to the contrast of the image and the second property is related to security. We assume that the message consist of a collection of black and white pixels. Let \( \pi \) be a random permutation of \( \{1, 2, \ldots, m\} \). Now a VCS is used to encrypt an image as follows. If a pixel
in the secret image is white (resp. black), then \( \pi \) is applied to the columns of \( S^0 \) (resp. \( S^1 \)) and row \( i \) of the permuted matrix is the share of \( i \)th participant. Therefore each share is a collection of \( m \) black and white subpixels. The value of \( m \) is called the pixel expansion and the value of \( \alpha \) is called relative contrast that measure clarity of reconstructed image.

One problem in a VCS is to minimize the pixel expansion and maximize the relative contrast. Several results on these two concepts can be found in \([10, 11]\). The minimum value of the pixel expansion \( m \) of a VCS for \( \Gamma = (Q, F) \) is denoted by \( m^*(\Gamma) \).

**Definition 2.2.** Let \( \Gamma = (Q, F) \) be an access structure on a set \( P \) of participants. Then \( \Gamma' = (Q', F') \) is the induced access structure on \( P' \subseteq P \) that \( Q' = Q \cap 2^P' \) and \( F' = F \cap 2^P' \).

Let \( G = (V, E) \) be a graph, then we can define a VCS on \( G \) such that a subset \( X \) of \( V \) is qualified if and only if the induced subgraph \( G[X] \) contains at least one edge of \( G \). The access structure based on graph \( G \) is denoted by \( \Gamma(G) \) and \( m^*(G) \) is the minimum value of pixel expansion \( m \) a VCS that \( \Gamma(G) \) is the access structures.

**Theorem 2.3.** \([3]\) Let \( \Gamma = (Q, F) \) be an access structure on a set \( P \) of participants and let \( \Gamma' = (Q|P', F|P') \) be the induced access structure on the subset of participants \( P' \). Then \( m^*(\Gamma') \leq m(\Gamma) \).

**Remark 2.4.** If \( H = (V', E') \) be an induced subgraph of \( G = (V, E) \), then \( \Gamma(H) \) is an induced access structure of \( \Gamma(G) \) and by Theorem 2.3, \( m^*(H) \leq m^*(G) \).

Ateniese et al. \([3, 4]\) studied the construction of VCSs on general access structures and graph access structures. They showed in \([3]\) that how can obtain basis matrices \( S^0 \) and \( S^1 \) of a VCS on a complete graph.

**Theorem 2.5.** \([3]\) Let \( \Gamma = (Q, F) \) be an access structure on a set \( P \) of participants. Let \( X, Y \subseteq P \) be two nonempty subsets of participants such that \( X \cap Y = \emptyset \), \( X \in F \) and \( X \cup Y \in Q \). Then in any \((\Gamma, m)\) - VCS for this access structure, we have \( w(S^1_{X \cup Y}) - w(S^0_Y) \geq \alpha \cdot m \) where \( S^0 \) and \( S^1 \) are basis matrices, \( m \) is the pixel expansion and \( \alpha \) is the relative contrast.

**Remark 2.6.** \([2]\) By Theorem 2.5, if \( Y = \{y\} \), then \( S^1[X \cup \{y\}] \) has at least one column with 1 in the row corresponding to \( y \) and with zero in all other entries. such a column in \( S^1[X \cup \{y\}] \) is called an unavoidable pattern.

For complete graph \( K_n \) and complete bipartite graph, we have the following theorems.

**Theorem 2.7.** \([3]\) Let \( G = K_n \) be complete graph. Then the value \( m^*(K_n) \) is the smallest integer \( m \) such that \( n \leq \binom{m}{\frac{m}{2}} \).

**Theorem 2.8.** \([3]\) Let \( \Gamma \) be a strong access structure on a set of participants \( P \). Then \( m^*(\Gamma) = 2 \) if and only if \( \Gamma = \Gamma(G) \) where \( G \) is a complete bipartite graph with \( V(G) = P \).

An clique, \( C \), in a graph \( G = (V, E) \) is a subset of the vertices such that every two distinct vertices are adjacent. This is equivalent to the condition that the induced subgraph \( G[C] \) is complete. A maximum clique of a graph \( G \) is a clique such that there is no clique with more vertices. The clique
number \( \omega(G) \) of a graph \( G \) is the number of vertices in a maximum clique in \( G \). An independent set, \( I \), in a graph is a subset of vertices such that no two vertices in \( I \) are adjacent. A maximal independent set is an independent set containing the largest possible number of vertices in graph. The following theorem gives a relation between \( m^*(G) \) and number of maximal independent sets in \( G \) that proved by Dehkordi and Cheraghi in [7].

**Theorem 2.9.** [7] Let \( G \) be a graph with the number of maximal independent sets \( l \), then \( m^*(G) \geq t \) where \( t \) is the smallest integer such that \( l \leq \left( \frac{t}{\lfloor \frac{1}{4} \rfloor} \right) \).

3. Main results

Let \( G = (V, E) \) be a connected graph with \( m^*(G) = 4 \). Then by Remark 2.4 for any induced subgraph \( H \) of \( G \) having no isolated vertices, we have \( m^*(H) \leq 4 \). We have from Theorem 2.9 that the number of maximal independent sets in \( H \) is at most 6. Further by Theorem 2.7, we have \( \omega(G) \leq 6 \). In [6] we characterized all of graphs which \( m^*(G) = 4 \) and \( \omega(G) = 6 \). In this paper, we consider case of \( \omega(G) = 5 \). If \( \omega(G) = 5 \), then \( K_5 \) is induced subgraph of graph \( G \). We first prove the following lemma.

**Proposition 3.1.** Let \( G = K_5 \) be a complete graph and \( V = \{ v_1, v_2, \ldots, v_5 \} \) is vertices set. Then one of the pairs of base matrices a VCS for \( \Gamma(G) \) is

\[
S^1[V] \sim \begin{bmatrix}
1 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 \\
1 & 0 & 0 & 1 \\
1 & 0 & 1 & 0
\end{bmatrix}, \quad S^0[V] \sim \begin{bmatrix}
1 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 \\
1 & 1 & 0 & 0
\end{bmatrix}.
\]

**Proof.** By Theorem 2.7, \( m^*(K_5) = 4 \). From Theorem 6.6 and corollary 6.7 in [3], a \( (\Gamma(K_5), 4) \)-VCS implies the existence of a sperner family of size 5 over a ground set of size 4. Let ground set is \( P = \{ a_1, a_2, a_3, a_4 \} \), now only sperner family of size 5 over \( P \) is \( B_1 = \{ a_1, a_2 \} \), \( B_2 = \{ a_2, a_3 \} \), \( B_3 = \{ a_3, a_4 \} \), \( B_4 = \{ a_1, a_4 \} \), \( B_5 = \{ a_1, a_3 \} \). From Theorem 7.2 in [3], we obtain basis matrices for a VCS with strong access structure \( \Gamma(K_5) \) from following definitions.

\[
S^1(i, j) = \begin{cases}
1 & a_j \in B_i \\
0 & a_j \notin B_i
\end{cases}, \quad S^0(i, j) = \begin{cases}
1 & 1 \leq j \leq |B_i| \\
0 & |B_i| + 1 \leq j \leq 4
\end{cases}.
\]

Hence \( S^1[V] \sim \begin{bmatrix}
1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 \\
1 & 0 & 0 & 1
\end{bmatrix} \) and \( S^0[V] \sim \begin{bmatrix}
1 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 \\
1 & 0 & 0 & 1 \\
1 & 0 & 0 & 0
\end{bmatrix} \). \( \square \)

**Lemma 3.2.** Let \( G \) be a connected graph with \( m^*(G) = 4 \). If \( \omega(G) = 5 \) where \( \omega(G) \) is the clique number of graph \( G \), then \( G \) is \( (K_5 \cup K_1) \)-free.

**Proof.** Assume that \( G \) is not \( (K_5 \cup K_1) \)-free, thus \( G \) contains \( K_5 \cup K_1 \) as an induced subgraph. So if \( Z = V(K_5) = \{ v_1, \ldots, v_5 \} \) and \( V(K_1) = \{ x \} \), then the vertex \( x \) is not connected to any of the
vertices of the $Z$. Given that $G[Z] = K_5$ and by using Proposition 3.1, we have $S^1[Z] \sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$.

Without loss of generality assume that the rows of $S^1[Z]$ corresponds to $v_1, v_2, \ldots, v_5$ respectively. Using Remark 2.6 with $X = \{x, v_2\}$ and $Y = \{v_1\}$, which $X \in F$ and $X \cup Y \in Q$, then $S^1[X \cup Y]$ has at least one column with 1 in the row corresponding to $v_1$ and with zero in all other entries. Therefore, the row corresponding to $x$ in $S^1[Z \cup \{x\}]$ must be $[0 \ ? \ ? \ ?]$ where ? represents the presence of either 0 or 1. So the first entry is zero and the following table shows that other entries of the row corresponding to $x$ are also zero.

<table>
<thead>
<tr>
<th>$X$</th>
<th>$Y$</th>
<th>row of $x$ in $S^1[Z \cup {x}]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x, v_2$</td>
<td>$v_1$</td>
<td>$[0 \ ? \ ? \ ?]$</td>
</tr>
<tr>
<td>$x, v_3$</td>
<td>$v_2$</td>
<td>$[0 \ 0 \ ? \ ?]$</td>
</tr>
<tr>
<td>$x, v_4$</td>
<td>$v_3$</td>
<td>$[0 \ 0 \ 0 \ ?]$</td>
</tr>
<tr>
<td>$x, v_5$</td>
<td>$v_4$</td>
<td>$[0 \ 0 \ 0 \ 0]$</td>
</tr>
</tbody>
</table>

Thus $S^1[Z \cup \{x\}] \sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$. This gives a contradiction since no row of $S^1$ can have weight zero.

Hence $G$ is $(K_5 \cup K_1)$-free. □

Since graph $G$ is $(K_5 \cup K_1)$-free, then for any vertex of $x \in G$, we have $N(x) \cap Z \neq \emptyset$, where $N(x)$ is the open neighborhood of $x$ consisting of all vertices which are adjacent to $x$. Also, since $\omega(G) = 5$, it follows that $1 \leq |N(x) \cap Z| \leq 4$. For any nonempty proper subset $X \subseteq \{1, 2, \ldots, 5\}$, we define the set $V_X$ as follows:

$$V_X := \{x \in V \setminus Z, N(x) \cap Z = \{v_i : i \in X\}\}$$

We now determine the properties of above sets in following lemmas.

**Lemma 3.3.** Let $G$ be a connected graph with $m^*(G) = 4$ and $\omega(G) = 5$. Then with above definition, we have

(i) If $|X| = 1$, then $V_X = \emptyset$,

(ii) If $|X| = 2$, then $V_X = \emptyset$ except probably $V_{14}$ and $V_{23}$,

(iii) If $|X| = 3$, then $V_X = \emptyset$ except probably $V_{125}$ and $V_{345}$,

(iv) If $|X| = 4$, then $V_X$ can be available.

**Proof.** Let $G[Z] = K_5$ and $Z = \{v_1, v_2, \ldots, v_5\}$. By Lemma 3.1, $S^1[Z \cup \{x\}] \sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$. Without loss of generality we assume that rows of $S^1[Z]$ corresponds to $v_1, v_2, \ldots, v_5$ respectively.

(i) Suppose $V_1 \neq \emptyset$ and let $x \in V_1$. Using Remark 2.6, we have the following table:

http://dx.doi.org/10.22108/toc.2019.113671.1599
Therefore row of $x$ in $S^1[Z \cup \{x\}]$ is $[0 \ 0 \ 0 \ 0]$. This gives a contradiction, hence $V_1 = \emptyset$. A similar proof shows that other $V_i$'s are empty.

(ii). According to $S^1[Z]$, $v_2$ and $v_3$ are zero in first column and other entries in this column are nonzero, so $V_{23} \neq \emptyset$ and if $x \in V_{23}$ then we have

$$S^1[Z \cup \{x\}] \sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad S^0[Z \cup \{x\}] \sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

Similarly if $y \in V_{14}$ then we have

$$S^1[Z \cup \{y\}] \sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad S^0[Z \cup \{y\}] \sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

The other $V_{ij}$'s are empty. For example let $x \in V_{12}$, then by Remark 2.6, we have the following table:

<table>
<thead>
<tr>
<th>$X$</th>
<th>$Y$</th>
<th>row of $x$ in $S^1[Z \cup {x}]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x, v_2$</td>
<td>$v_1$</td>
<td>$\begin{bmatrix} 0 &amp; ? &amp; ? &amp; ? \end{bmatrix}$</td>
</tr>
<tr>
<td>$x, v_3$</td>
<td>$v_2$</td>
<td>$\begin{bmatrix} 0 &amp; 0 &amp; ? &amp; ? \end{bmatrix}$</td>
</tr>
<tr>
<td>$x, v_4$</td>
<td>$v_3$</td>
<td>$\begin{bmatrix} 0 &amp; 0 &amp; 0 &amp; ? \end{bmatrix}$</td>
</tr>
<tr>
<td>$x, v_5$</td>
<td>$v_4$</td>
<td>$\begin{bmatrix} 0 &amp; 0 &amp; 0 &amp; 0 \end{bmatrix}$</td>
</tr>
</tbody>
</table>

Therefore row of $x$ in $S^1[Z \cup \{x\}]$ is $[0 \ 0 \ 0 \ 0]$. This gives a contradiction, hence $V_{12} = \emptyset$. A similar proof shows that other $V_{ij}$'s are empty.

(iii). In $S^1[Z]$, $v_1, v_2$ and $v_5$ are zero in last column and other entries in this column are nonzero, so $V_{125} \neq \emptyset$. If $x \in V_{125}$, then we have

$$S^1[Z \cup \{x\}] \sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad S^0[Z \cup \{x\}] \sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

Similarly if $y \in V_{345}$, then we have

$$S^1[Z \cup \{y\}] \sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad S^0[Z \cup \{y\}] \sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

By the (ii), graph $G$ can contains $V_{14}$ and $V_{23}$. From each one, we can make three $V_{ijk}$, where $1 \leq i,j,k \leq 5$ and $i,j,k$ are different as follows:

http://dx.doi.org/10.22108/toc.2019.113671.1599
<table>
<thead>
<tr>
<th>$V_{ij}$</th>
<th>$V_{23}$</th>
<th>$V_{14}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V_{123}$</td>
<td>$V_{124}$</td>
<td></td>
</tr>
<tr>
<td>$V_{123}$</td>
<td>$V_{234}$</td>
<td>$V_{134}$</td>
</tr>
<tr>
<td>$V_{123}$</td>
<td>$V_{234}$</td>
<td>$V_{134}$</td>
</tr>
</tbody>
</table>

All of $V_{ijk}$’s in above table are $\emptyset$. Let $V_{123}$ that is obtained from $V_{23}$ is not empty and let $x \in V_{123}$. Then by (ii) row of $x$ in $S^1[Z \cup \{x\}]$ is $[1 \; 0 \; 0 \; 0]$. However $\{x, v_1\} \in F$, Thus in $S^1[Z \cup \{x\}]$ we must have the unavoidable patterns of $[0\;\text{ and }\;1\;\text{ while the first pattern doesn’t exist. Hence }$ $V_{123} = \emptyset$. Similarly all of $V_{ijk}$’s in above table are empty. Now, it is sufficient that we show $V_{135}$ and $V_{245}$ are empty. Let $V_{135}$ and $V_{245}$ are not empty and $x \in V_{245}, y \in V_{135}$. Using Remark 2.6, we have the following tables:

<table>
<thead>
<tr>
<th>$X$</th>
<th>$Y$</th>
<th>row of $x$ in $S^1[Z \cup {x}]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x, v_3$</td>
<td>$v_4$</td>
<td>$[0 ; ? ; ? ; ?]$</td>
</tr>
<tr>
<td>$x, v_3$</td>
<td>$v_2$</td>
<td>$[0 ; 0 ; ? ; ?]$</td>
</tr>
<tr>
<td>$x, v_1$</td>
<td>$v_2$</td>
<td>$[0 ; 0 ; 0 ; ?]$</td>
</tr>
<tr>
<td>$x, v_1$</td>
<td>$v_4$</td>
<td>$[0 ; 0 ; 0 ; 0]$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$X$</th>
<th>$Y$</th>
<th>row of $y$ in $S^1[Z \cup {y}]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y, v_2$</td>
<td>$v_1$</td>
<td>$[0 ; ? ; ? ; ?]$</td>
</tr>
<tr>
<td>$y, v_4$</td>
<td>$v_1$</td>
<td>$[0 ; 0 ; ? ; ?]$</td>
</tr>
<tr>
<td>$y, v_4$</td>
<td>$v_5$</td>
<td>$[0 ; 0 ; 0 ; ?]$</td>
</tr>
<tr>
<td>$y, v_2$</td>
<td>$v_3$</td>
<td>$[0 ; 0 ; 0 ; 0]$</td>
</tr>
</tbody>
</table>

Therefore rows of $x$ and $y$ in $S^1[Z \cup \{x, y\}]$ are $[0 \; 0 \; 0 \; 0]$. This gives a contradiction, hence $V_{245}$ and $V_{135}$ are empty.

(iv). Let $x \in V_{1234}$. By Remark 2.6, we have the following table:

<table>
<thead>
<tr>
<th>$X$</th>
<th>$Y$</th>
<th>row of $x$ in $S^1[Z \cup {x}]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x, v_5$</td>
<td>$v_1$</td>
<td>$[? ; 0 ; ? ; ?]$</td>
</tr>
<tr>
<td>$x, v_5$</td>
<td>$v_3$</td>
<td>$[? ; 0 ; ? ; 0]$</td>
</tr>
<tr>
<td>$v_2$</td>
<td>$x$</td>
<td>$[1 ; 0 ; ? ; 0]$</td>
</tr>
<tr>
<td>$v_4$</td>
<td>$x$</td>
<td>$[1 ; 0 ; 1 ; 0]$</td>
</tr>
</tbody>
</table>

Hence $S^1[Z \cup \{x\}] \sim \begin{bmatrix}
1 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 \\
1 & 0 & 1 & 0
\end{bmatrix}$. Also, since $\{x, v_5\}$ is forbidden set and $w(S^1_{\{x\}}) = 2$, then $S^0[Z \cup \{x\}] \sim \begin{bmatrix}
1 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 \\
1 & 1 & 0 & 0
\end{bmatrix}$. Let $y \in V_{1235}$. By Remark 2.6, we have the following table:
<table>
<thead>
<tr>
<th>$X$</th>
<th>$Y$</th>
<th>row of $y$ in $S^1[Z \cup {y}]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y, v_4$</td>
<td>$v_1$</td>
<td>$[? \ 0 \ ? \ ?]$</td>
</tr>
<tr>
<td>$y, v_4$</td>
<td>$v_3$</td>
<td>$[? \ 0 \ 0 \ ?]$</td>
</tr>
<tr>
<td>$v_3$</td>
<td>$y$</td>
<td>$[1 \ 0 \ 0 \ ?]$</td>
</tr>
<tr>
<td>$v_5$</td>
<td>$y$</td>
<td>$[1 \ 0 \ 0 \ 1]$</td>
</tr>
</tbody>
</table>

Hence $S^1[Z \cup \{y\}] \sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$.

Also, since $\{y, v_4\}$ is forbidden set and $w(S^1_{\{y\}}) = 2$, then $S^0[Z \cup \{y\}] \sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$. Let $z \in V_{1245}$. By Remark 2.6, we have the following table:

<table>
<thead>
<tr>
<th>$X$</th>
<th>$Y$</th>
<th>row of $z$ in $S^1[Z \cup {z}]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$z, v_3$</td>
<td>$v_4$</td>
<td>$[0 \ ? \ ? \ ?]$</td>
</tr>
<tr>
<td>$z, v_3$</td>
<td>$v_2$</td>
<td>$[0 \ 0 \ ? \ ?]$</td>
</tr>
<tr>
<td>$v_2$</td>
<td>$z$</td>
<td>$[0 \ 0 \ ? \ 1]$</td>
</tr>
<tr>
<td>$v_4$</td>
<td>$z$</td>
<td>$[0 \ 0 \ 1 \ 1]$</td>
</tr>
</tbody>
</table>

Hence $S^1[Z \cup \{z\}] \sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$.

Also, since $\{z, v_3\}$ is forbidden set and $w(S^1_{\{z\}}) = 2$, then $S^0[Z \cup \{z\}] \sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$. Let $s \in V_{1345}$. By Remark 2.6, we have the following table:

<table>
<thead>
<tr>
<th>$X$</th>
<th>$Y$</th>
<th>row of $s$ in $S^1[Z \cup {s}]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s, v_2$</td>
<td>$v_1$</td>
<td>$[0 \ ? \ ? \ ?]$</td>
</tr>
<tr>
<td>$s, v_2$</td>
<td>$v_3$</td>
<td>$[0 \ ? \ ? \ 0]$</td>
</tr>
<tr>
<td>$v_3$</td>
<td>$s$</td>
<td>$[0 \ 1 \ ? \ 0]$</td>
</tr>
<tr>
<td>$v_1$</td>
<td>$s$</td>
<td>$[0 \ 1 \ 1 \ 0]$</td>
</tr>
</tbody>
</table>

Hence $S^1[Z \cup \{s\}] \sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$.

Also, since $\{s, v_2\}$ is forbidden set and $w(S^1_{\{s\}}) = 2$, then $S^0[Z \cup \{s\}] \sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$. Let $t \in V_{2345}$. By Remark 2.6, we have the following table:
Lemma 3.4. Let $G$ be a connected graph with $m^*(G) = 4$ and $\omega(G) = 5$. Also, suppose that $Z = \{v_1, v_2, \ldots, v_5\}$ is a clique of $G$. Then $V_X$ is an independent set for any $X \subseteq \{1, 2, \ldots, 5\}$ with $2 \leq |X| \leq 4$.

Proof. Let $|X| = 2$. Then by Lemma 3.3, we consider the cases $X = \{1, 4\}$ and $X = \{2, 3\}$. Let $x, y \in V_{14}$. If $\{x, y\} \in Q$, then we have 8 maximal independent sets as follows: $\{x, v_2\}$, $\{x, v_3\}$, $\{x, v_5\}$, $\{y, v_2\}$, $\{y, v_3\}$, $\{y, v_5\}$, $\{v_1\}$, $\{v_4\}$. By Thorem 2.9, $G$ has at most 6 maximal independent sets, thus this gives a contradiction. Hence $V_{14}$ is an independent set. Similarly, the set of $V_{23}$ is independent. Now let $x, y \in V_{125}$. If $\{x, y\} \in Q$, then we have 7 maximal independent sets as follows: $\{x, v_3\}$, $\{x, v_4\}$, $\{y, v_3\}$, $\{y, v_4\}$, $\{v_1\}$, $\{v_2\}$, $\{v_5\}$ and $\{v_3\}$. By Theorem 2.9, this gives a contradiction, hence $V_{125}$ is an independent set. Similarly, the set of $V_{345}$ is independent.

Now we show that $V_{1234} = \emptyset$. If $V_{1234} \neq \emptyset$, let $x, y \in V_{1234}$. Then by Lemma 3.3, we have

$$S^1[Z \cup \{x, y\}] \sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}, \quad S^0[Z \cup \{x, y\}] \sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}.$$

So $w(S^1_{\{x,y\}}) = w(S^0_{\{x,y\}}) = 2$, thus $\{x, y\} \in F$. Hence $V_{1234}$ is a independent set and this complete proof. \(\square\)

Lemma 3.5. Let $G$ be a connected graph with $m^*(G) = 4$ and $\omega(G) = 5$. Then $V_{14} \cup V_{23}$ is an independent set.

Proof. From lemma 3.3, we have

$$S^1[Z \cup \{x, y\}] \sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}, \quad S^0[Z \cup \{x, y\}] \sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}.$$

http://dx.doi.org/10.22108/toc.2019.113671.1599
If $S^0[Z \cup \{x,y\}] \sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$ then $w(S^1_{\{x,y\}}) > w(S^0_{\{x,y\}})$, thus $\{x,y\} \in Q$. However $w(S^1_{\{x,y,v_5\}}) = w(S^0_{\{x,y,v_5\}})$, which is contradiction since $\Gamma(G)$ is strong access structure. Hence $S^0[Z \cup \{x,y\}] \sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$. In this case $w(S^1_{\{x,y\}}) = w(S^0_{\{x,y\}})$ and this show that $\{x,y\} \in F$. Hence $x$ and $y$ are nonadjacent. Thus $V_{14} \cup V_{23}$ is an independent set.

\[\square\]

**Remark 3.6.** Suppose that $V_{125}, V_{345} \neq \emptyset$. Let $x \in V_{125}$, $y \in V_{345}$. From Lemma 3.3, we have

$$S^1[Z \cup \{x,y\}] \sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}, S^0[Z \cup \{x,y\}] \sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

In first form of $S^0[Z \cup \{x,y\}]$, we have $w(S^1_{\{x,y\}}) > w(S^0_{\{x,y\}})$, hence $\{x,y\}$ is qualified set and this show that $x$ and $y$ are adjacent. In second form of $S^0[Z \cup \{x,y\}]$, we have $w(S^1_{\{x,y\}}) = w(S^0_{\{x,y\}})$, hence $\{x,y\}$ is forbidden set, thus $x$ and $y$ are nonadjacent.

**Lemma 3.7.** Let $G$ be a connected graph with $\mu^*(G) = 4$ and $\omega(G) = 5$. Also, $X$, $Y$ and $W$ be nonempty proper subsets of $\{1,2,\ldots,5\}$. If $|X| = 2$, $|Y| = 3$ and $|W| = 4$, then with previous definitions,

(i) The sets of $V_X \cup V_Y$ are independent sets,

(ii) The sets of $V_X \cup V_W$ are independent sets,

(iii) The induced subgraphs $G[V_Y \cup V_W]$ are complete bipartite graphs.

**Proof.** (i). Let $V_{14}, V_{125} \neq \emptyset$. If $x \in V_{14}$ and $y \in V_{125}$, then by Lemma 3.3, we have

$$S^1[Z \cup \{x,y\}] \sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}, S^0[Z \cup \{x,y\}] \sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}.$$

Hence $w(S^1_{\{x,y\}}) > w(S^0_{\{x,y\}})$, thus $\{x,y\} \in Q$. However $w(S^1_{\{x,y,v_1\}}) = w(S^0_{\{x,y,v_1\}})$, which is contradiction since $\Gamma(G)$ is strong access structure. So $\{x,y\} \in F$ and hence $V_{14} \cup V_{125}$ is independent set. A similar proof shows that $V_{14} \cup V_{345}, V_{23} \cup V_{125}$ and $V_{23} \cup V_{345}$ are independent sets.

(ii). Let $V_{14}, V_{1234} \neq \emptyset$. If $x \in V_{14}$ and $y \in V_{1234}$, then by Lemma 3.3, we have

$$S^1[Z \cup \{x,y\}] \sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}, S^0[Z \cup \{x,y\}] \sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}.$$

Hence $w(S^1_{\{x,y\}}) = w(S^0_{\{x,y\}})$, thus $\{x,y\} \in F$. A similar proof shows that $V_X \cup V_W$ is an independent set.
(iii). Let \( V_{125}, V_{1234} \neq \emptyset \). If \( x \in V_{125} \) and \( y \in V_{1234} \), then by Lemma 3.3, we have

\[
S^1[Z \cup \{x, y\}] \sim \begin{bmatrix}
1 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 \\
0 & 0 & 1 & 1
\end{bmatrix}, \quad S^0[Z \cup \{x, y\}] \sim \begin{bmatrix}
1 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 \\
1 & 0 & 1 & 0
\end{bmatrix}.
\]

Hence \( w(S^1_{\{x,y\}}) > w(S^0_{\{x,y\}}) \), thus \( \{x, y\} \in Q \). A similar proof shows that the induced subgraph \( G[Y \cup W] \) is a complete bipartite graph. \( \square \)

**Lemma 3.8.** Let \( G \) be a connected graph with \( m^*(G) = 4 \) and \( \omega(G) = 5 \). If \( V_{ij} \)'s are not empty, then \( V_{ijk} \)'s are empty.

**Proof.** Let \( V_{ij} \)'s are not empty and \( x \in V_{23} \) and \( y \in V_{14} \). If \( V_{125} \neq \emptyset \), then by Lemma 3.3 we have

\[
S^1[Z \cup \{x, y, z\}] \sim \begin{bmatrix}
1 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 \\
0 & 0 & 1 & 1
\end{bmatrix}, \quad S^0[Z \cup \{x, y, z\}] \sim \begin{bmatrix}
1 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 \\
1 & 1 & 0 & 0
\end{bmatrix}.
\]

Hence \( w(S^1_{\{x,y,z\}}) > w(S^0_{\{x,y,z\}}) \), therefore \( \{x, y\} \in Q \) and this is contradiction with Lemma 3.5. \( \square \)

We now proceed to characterize connected graphs \( G \) such that \( m^*(G) = 4 \) and \( \omega(G) = 5 \).

Let \( F \) be the family of graphs that obtained from complete graph \( K_5 \) with \( V(K_5) = \{v_1, v_2, \ldots, v_5\} \) by adding nine independent sets \( V_{14}, V_{23}, V_{125}, V_{345}, V_{1234}, V_{1235}, V_{1245}, V_{1345}, V_{2345} \) and \( V_{2345} \) where \( |V_{14}| = n_1, |V_{23}| = n_2, |V_{125}| = n_3, |V_{345}| = n_4, |V_{1234}| = n_5, |V_{1235}| = n_6, |V_{1245}| = n_7, |V_{1345}| = n_8, |V_{2345}| = n_9 \) and these sets satisfy in Lemma 3.4 to Lemma 3.8. According to the definition of \( V_{1234} \), if \( x \in V_{1234} \), then \( \{x, v_5\} \) is independent, so we can replace set of \( v_5 \) with a set of independent vertices, with the name \( V'_5 \) instead of set \( V_{1234} \). Similarly, the vertices \( v_1, v_2, v_3 \) and \( v_4 \) can replace by independent sets \( V'_1, V'_2, V'_3 \) and \( V'_4 \). A few graphs in the family \( F \) are given in Figure 1.

![Figure 1. Graphs in family F](http://dx.doi.org/10.22108/toc.2019.113671.1599)
Theorem 3.9. Let $G$ be a connected graph with $\omega(G) = 5$. Then $m^*(G) = 4$ if and only if for specified values $n_1, n_2, \ldots, n_9$, $G$ is isomorphic to a graph $H$ in $F$.

Proof. Let $G = (V, E)$ be a connected graph containing the complete graph $V(K_5) = \{v_1, v_2, \ldots, v_5\}$ and $m^*(G) = 4$. Let $Z = \{v_1, v_2, \ldots, v_5\}$. It follows from Lemma 3.2 that every vertex $u \in V - Z$ is adjacent to at least one vertex in $Z$. Further since $G$ is $K_6$ free, $u$ is adjacent to at most four vertices in $Z$. Therefore by Lemma 3.3 to Lemma 3.8, we conclude that $G \in F$. To prove the converse, consider $H \in F$. By Lemma 3.8, if $|V_{14}| = n_1$ and $|V_{23}| = n_2$, then $V_{125}$ and $V_{345}$ are empty sets. Given that $|V_{1234}| = n_5, |V_{1235}| = n_6, |V_{1245}| = n_7, |V_{1345}| = n_8$ and $|V_{2345}| = n_9$, then the basis matrices for VCS of the access structure $\Gamma(H)$ are:

$$
S^1 \sim \begin{bmatrix}
1 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 \\
1 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 \\
0_{n_1} & 0_{n_1} & 1_{n_1} & 0_{n_1} \\
1_{n_2} & 0_{n_2} & 0_{n_2} & 0_{n_2} \\
1_{n_5} & 0_{n_5} & 1_{n_5} & 0_{n_5} \\
1_{n_6} & 0_{n_6} & 0_{n_6} & 1_{n_6} \\
0_{n_7} & 0_{n_7} & 1_{n_7} & 1_{n_7} \\
0_{n_8} & 1_{n_8} & 1_{n_8} & 0_{n_8} \\
1_{n_9} & 1_{n_9} & 0_{n_9} & 0_{n_9}
\end{bmatrix}
$$

and

$$
S^0 \sim \begin{bmatrix}
1 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 \\
0_{n_1} & 0_{n_1} & 0_{n_1} & 0_{n_1} \\
1_{n_2} & 1_{n_2} & 0_{n_2} & 0_{n_2} \\
1_{n_5} & 1_{n_5} & 0_{n_5} & 0_{n_5} \\
0_{n_6} & 0_{n_6} & 0_{n_6} & 0_{n_6} \\
1_{n_7} & 1_{n_7} & 0_{n_7} & 0_{n_7} \\
0_{n_8} & 0_{n_8} & 0_{n_8} & 0_{n_8} \\
1_{n_9} & 1_{n_9} & 0_{n_9} & 0_{n_9}
\end{bmatrix}
$$

where $1_n (0_n)$ denotes the $n \times 1$ column matrix with all entries one (zero). It is simple work that $S^0$ and $S^1$ are basis matrices for a VCS of the access structure $\Gamma(H)$. Hence $m^*(H) \leq 4$. However $H$ contains $K_5$, thus $m^*(H) = 4$. Now if $G \in F$, then $G$ is an induced subgraph of $H$ and since $G$ contains $K_5$ as a subgraph, we have $m^*(G) = 4$.

Further if $V_{125}$ and $V_{345}$ are not empty and $|V_{125}| = n_3$ and $|V_{345}| = n_4$, then by Lemma 3.8 dont exist $V_{14}$ and $V_{23}$ simultaneously. Let $V_{14} \neq \emptyset$, in this case the basis matrices for VCS of the access structure $\Gamma(H)$ are:

http://dx.doi.org/10.22108/toc.2019.113671.1599
In this case, similar to discussion of above, we have $m^*(G) = 4$. □

4. Conclusion

Ateniese et al. [3] have proved that $m^*(\Gamma) = 2$ if and only if $\Gamma = \Gamma(G)$ where $G$ is a complete bipartite graph. Also Arumugam et al. [1] have obtained a characterization of all connected graphs $G$ where $m^*(G) = 3$. If $m^*(G) = 4$ then $\omega(G) \leq 6$. We have obtained previously in [6] a characterization of all connected graphs $G$ for which $\omega(G) = 6$. In this paper, we obtained a characterization of all connected graphs $G$ for which $\omega(G) = 5$. The next problem is to characterize all graphs where $m^*(G) = 4$ and $\omega(G) \leq 4$.

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REFERENCES


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**M. Davarzani**

Faculty of Mathematics and Computer Science, Kharazmi University, P.O.Box 1561836314, Tehran, Iran

*Email:* mahmood.davarzani@gmail.com

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