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A NOTE ON SOME LOWER BOUNDS OF THE LAPLACIAN ENERGY OF A GRAPH

IGOR MILOVANOVIĆ, MARJAN MATEJIĆ, PREDRAG MILOŠEVIĆ, EMINA MILOVANOVIĆ AND AKBAR ALI*

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Abstract. For a simple connected graph G of order n and size m, the Laplacian energy of G is defined as $LE(G) = \sum_{i=1}^{n} |\mu_i - \frac{2m}{n}|$ where $\mu_1, \mu_2, \ldots, \mu_{n-1}, \mu_n$ are the Laplacian eigenvalues of G satisfying $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1} > \mu_n = 0$. In this note, some new lower bounds on the graph invariant $LE(G)$ are derived. The obtained results are compared with some already known lower bounds of $LE(G)$.

1. Introduction and Preliminaries

Let G be a simple connected graph with n vertices, m edges and let $(d_1, d_2, \ldots, d_n)$ be the degree sequence of G satisfying $\Delta = d_1 \geq d_2 \geq \cdots \geq d_n = \delta > 0$. Denote by $\mu_1, \mu_2, \ldots, \mu_{n-1}, \mu_n$ the Laplacian eigenvalues of G such that $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1} > \mu_n = 0$. Some well known properties of these eigenvalues are given [3] below

$$\sum_{i=1}^{n-1} \mu_i = 2m \quad \text{and} \quad \sum_{i=1}^{n-1} \mu_i^2 = \sum_{i=1}^{n} d_i^2 + \sum_{i=1}^{n} d_i = M_1(G) + 2m,$$

where $M_1(G) = \sum_{i=1}^{n} d_i^2$ is the first Zagreb index [6].

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*Corresponding author.

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In [7], Gutman and Zhou proposed the Laplacian energy of a graph G, which is denoted as $LE(G)$ and is defined as

$$LE(G) = \sum_{i=1}^{n} \left| \mu_i - \frac{2m}{n} \right|.$$

In this note, we establish some lower bounds on the Laplacian energy $LE(G)$ of G. We compare the obtained bounds with some existing ones. In what follows, we recall some inequalities involving $LE(G)$ that will be used in the remaining part of the paper.

In [13], the following inequality was proven

$$LE(G) \geq \frac{4m}{n}. \tag{1.1}$$

The equality sign in (1.1) holds if and only if G is regular k-partite graph, where $1 \leq k \leq n-1$.

In [4], (see also [5, 10]) it was shown that

$$LE(G) \geq 2\left(1 + \Delta - \frac{2m}{n}\right), \tag{1.2}$$

with equality if and only if $G \cong K_{1,n-1}$ (that is, the star graph on n vertices). Also, it was proven than bounds (1.1) and (1.2) are not comparable.

The following lower bound for $LE(G)$ was derived in [7]

$$LE(G) \geq \sqrt{2\left(M_1(G) + 2m - \frac{4m^2}{n}\right)}. \tag{1.3}$$

The equality sign in (1.3) holds if and only if n is even and $G \cong K_{\frac{n}{2},\frac{n}{2}}$ (that is, the complete bipartite graph of order n with the same number of vertices in its two partite sets).

2. Main Results

In the following theorem, we establish the lower bound for $LE(G)$ in terms of M_1, n, m and k, where $k \geq \mu_1$.

Theorem 2.1. *Let G be a non-trivial simple connected graph with n vertices and m edges. Then, for any real k satisfying $k \geq \mu_1$, it holds that*

$$LE(G) \geq \frac{2\left(M_1(G) + 2m - \frac{4m^2}{n}\right)}{k}, \tag{2.1}$$

where M_1 is the first Zagreb index. Equality sign in (2.1) holds if and only if either $k = n$ and $G \cong K_n$ (that is, the complete graph on n vertices), or n is even and $G \cong K_{\frac{n}{2},\frac{n}{2}}$ (that is, the complete bipartite graph of order n with the same number of vertices in its two partite sets).

Proof. Let $x = (x_i)$ and $a = (a_i)$, $i = 1, 2, \ldots, n$, be real number sequences with the properties

$$\sum_{i=1}^{n} |x_i| = 1 \qquad \text{and} \qquad \sum_{i=1}^{n} x_i = 0. \tag{2.2}$$

For such sequences, the following inequality was proven in the monograph [12, pp. 346]

$$\left|\sum_{i=1}^{n} a_i x_i\right| \le \frac{1}{2}\left(\max_{1\le i\le n} a_i - \min_{1\le i\le n} a_i\right). \tag{2.3}$$

Let $a_i = \mu_i - \frac{2m}{n}$ and $x_i = \dfrac{\mu_i - \frac{2m}{n}}{\sum_{i=1}^{n}\left|\mu_i - \frac{2m}{n}\right|}$, $i = 1, 2, \dots, n$. Since

$$\sum_{i=1}^{n} x_i = \frac{\sum_{i=1}^{n}\left(\mu_i - \frac{2m}{n}\right)}{\sum_{i=1}^{n}\left|\mu_i - \frac{2m}{n}\right|} = 0 \qquad \text{and} \qquad \sum_{i=1}^{n} |x_i| = \frac{\sum_{i=1}^{n}\left|\mu_i - \frac{2m}{n}\right|}{\sum_{i=1}^{n}\left|\mu_i - \frac{2m}{n}\right|} = 1,$$

the conditions for inequality (2.3) are satisfied. Hence, we have

$$\left|\frac{\sum_{i=1}^{n}\left(\mu_i - \frac{2m}{n}\right)^2}{\sum_{i=1}^{n}\left|\mu_i - \frac{2m}{n}\right|}\right| \le \frac{1}{2}\left(\max_{1\le i\le n}\left(\mu_i - \frac{2m}{n}\right) - \min_{1\le i\le n}\left(\mu_i - \frac{2m}{n}\right)\right),$$

that is

$$\frac{M_1(G) + 2m - \frac{4m^2}{n}}{LE(G)} \le \frac{1}{2}\left(\mu_1 - \frac{2m}{n} + \frac{2m}{n}\right),$$

wherefrom we obtain

$$LE(G) \ge \frac{2\left(M_1(G) + 2m - \frac{4m^2}{n}\right)}{\mu_1} \ge \frac{2\left(M_1(G) + 2m - \frac{4m^2}{n}\right)}{k}.$$

□

If we take $k = n$ in Theorem 2.1, then

$$LE(G) \ge \frac{2\left(M_1(G) + 2m - \frac{4m^2}{n}\right)}{n}, \tag{2.4}$$

with equality if and only if either $G \cong K_n$, or $G \cong K_{\frac{n}{2},\frac{n}{2}}$ and n is even.

Remark 2.2. *The inequality (2.4) is not comparable with (1.3); for example, if $G \cong K_n$ then the inequality (2.4) is stronger than (1.3), but if $G \cong P_n$ then (2.4) is weaker than (1.3).*

In [11], the following bound for $M_1(G)$ was established

$$M_1(G) \ge \frac{4m^2}{n} + \frac{1}{2}(\Delta - \delta)^2, \tag{2.5}$$

with equality if and only if $d_2 = d_3 = \cdots = d_{n-1} = \frac{\Delta+\delta}{2}$. Let us note that equality case in (2.5) was proven in [9] (see also [1, 2]). From (2.4), (1.3) and (2.5) the following result is obtained.

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Corollary 2.3. *If G is a non-trivial simple connected graph with order n, size m, minimum degree δ and maximum degree Δ, then*

$$LE(G) \geq \frac{4m + (\Delta - \delta)^2}{n} \tag{2.6}$$

and

$$LE(G) \geq \sqrt{4m + (\Delta - \delta)^2}\,. \tag{2.7}$$

The equality sign in (2.6) holds if either $G \cong K_n$ or n is even and $G \cong K_{\frac{n}{2},\frac{n}{2}}$. While, the equality sign in (2.7) holds if n is even and $G \cong K_{\frac{n}{2},\frac{n}{2}}$.

Remark 2.4. *Since $(\Delta - \delta)^2 \geq 0$, the inequality (2.6) is stronger than (1.1). The inequalities (2.7) and (1.1) are incomparable; for example, if $G \cong P_n$ or $G \cong K_{1,n-1}$ then the inequality (2.7) is stronger than (1.1), but if $G \cong K_n$ then (1.1) is stronger than (2.7).*

Remark 2.5. *In [7], it was proven that the inequalities (1.3) and (1.2) are incomparable. The inequalities (2.6) and (2.7) are not comparable with (1.2). Therefore, we have the following lower bound for $LE(G)$*

$$LE(G) \geq \max\left\{ \frac{4m + (\Delta - \delta)^2}{n}, \sqrt{4m + (\Delta - \delta)^2}, 2\left(1 + \Delta - \frac{2m}{n}\right) \right\}. \tag{2.8}$$

with equality if and only if $G \cong K_n$ or $G \cong K_{1,n-1}$, or n is even and $G \cong K_{\frac{n}{2},\frac{n}{2}}$.

As a special case of a general inequality derived in [8], the following lower bound on $M_1(G)$ was proven

$$M_1(G) \geq \Delta^2 + \frac{(2m - \Delta)^2}{n-1} + \frac{(\Delta_2 - \delta)^2}{2}\,, \tag{2.9}$$

where $\Delta_2 = d_2$. It can be easily verified that the equality sign in (2.9) holds if and only if $d_3 = d_4 = \cdots = d_{n-1} = \frac{\Delta_2 + \delta}{2}$. From (2.9), (2.4) and (1.3), the next result follows.

Corollary 2.6. *If G is a non-trivial simple connected graph with order n, size m, minimum degree δ, maximum degree Δ and second maximum degree Δ_2 then*

$$LE(G) \geq \frac{2\Delta^2 + \frac{2(2m-\Delta)^2}{n-1} + 2m + (\Delta_2 - \delta)^2 - \frac{4m^2}{n}}{n}$$

and

$$LE(G) \geq \sqrt{2\Delta^2 + \frac{2(2m - \Delta)^2}{n-1} + 2m + (\Delta_2 - \delta)^2 - \frac{4m^2}{n}}.$$

Lower bounds for the Laplacian energy $LE(G)$ of G, given in inequalities (1.2), (2.6) and (2.7) as well as the exact values of $LE(G)$ when G is the complete graph ($G \cong K_n$), path graph ($G \cong P_n$) and star graph ($G \cong K_{1,n-1}$), for several values of n are presented in Tables 1, 2 and 3, respectively.

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Table 1. Obtained values for LE for the complete graph $(LE(K_n))$

n	Exact value	Inequality (1.2)	Inequality (2.6)	Inequality (2.7)
5	8	2	8	6.325
10	18	2	18	13.416
20	38	2	38	27.568
50	98	2	98	70
60	118	2	118	84.143
100	198	2	198	140.712

Table 2. Obtained values for LE for the path graph $(LE(P_n))$

n	Exact value	Inequality (1.2)	Inequality (2.6)	Inequality (2.7)
5	6.07214	2.8	3.4	4.123
10	12.697	2.4	3.7	6.083
20	25.412	2.2	3.85	8.775
50	63.641	2.08	3.94	14.0357
60	76.377	2.067	3.95	15.4
100	127.313	2.04	3.97	19.925

As it can be seen that the lower bounds for $LE(G)$ given in (1.2), (2.6) and (2.7) are incomparable, so the utilization of the lower bound given in (2.8) (which is the maximum of the lower bounds given in (1.2), (2.6) and (2.7)) is purposeful. In spite of that, the question is whether (2.8) gives satisfactory results for all classes of graphs? According to Table 1, one can see that (2.8) gives satisfactory results when m is large ($m \gg n$), that is, when $G \cong K_n$, $G \cong K_{n-1} + e$, $G \cong K_n - e$, $\dots$. Similarly, according to Table 3, when Δ is large and m is small, that is, when $G \cong K_{1,n-1}$, $G \cong K_{1,n-2} + e, \dots$, the inequality (2.8) gives satisfactory results. However, according to Table 2, when both Δ and m are small, that is, when $G \cong P_n$, $G \cong P_{n-1} + e, \dots$, the results obtained by (2.8) are not good enough. Therefore, in this case the problem of finding the inequality that gives better lower bound for $LE(G)$ remains open.

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TABLE 3. Obtained values for LE for the star graph ($LE(K_{1,n-1})$)

n	Exact value	Inequality (1.2)	Inequality (2.6)	Inequality (2.7)
5	6.8	6.8	5	5
10	16.4	16.4	10	10
20	36.2	36.2	20	20
50	96.08	96.08	50	50
60	116.067	116.067	60	60
100	196.04	196.04	100	100

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Igor Milovanović

Faculty of Electronic Engineering, University of Niš, P. O. Box 18000, Niš, Serbia

Email: `Igor.Milovanovic@elfak.ni.ac.rs`

Marjan Matejić

Faculty of Electronic Engineering, University of Niš, P. O. Box 18000, Niš, Serbia

Email: `marjan.matejic@elfak.ni.ac.rs`

Predrag Milošević

Faculty of Electronic Engineering, University of Niš, P. O. Box 18000, Niš, Serbia

Email: `predrag.milosevic@elfak.ni.ac.rs`

Emina Milovanović

Faculty of Electronic Engineering, University of Niš, P. O. Box 18000, Niš, Serbia

Email: `Emina.Milovanovic@elfak.ni.ac.rs`

Akbar Ali

Knowledge Unit of Science, University of Management and Technology, P. O. Box 51310, Sialkot, Pakistan

Email: `akbarali.maths@gmail.com`

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