SIZE RAMSEY NUMBER OF BIPARTITE GRAPHS
AND BIPARTITE RAMANUJAN GRAPHS

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ABSTRACT. Given a graph $G$, a graph $F$ is said to be Ramsey for $G$ if in every edge coloring of $F$ with two colors, there exists a monochromatic copy of $G$. The minimum number of edges of a graph $F$ which is Ramsey for $G$ is called the size-Ramsey number of $G$ and is denoted by $\hat{r}(G)$. In 1983, Beck gave a linear upper bound (in terms of $n$) for $\hat{r}(P_n)$, where $P_n$ is a path on $n$ vertices, giving a positive answer to a question of Erdős. After that, different approaches were attempted by several authors to reduce the upper bound for $\hat{r}(P_n)$ for sufficiently large $n$ and most of these approaches are based on the classic models of random graphs. Also, Haxell and Kohayakama in 1994 proved that the size Ramsey number of the cycle $C_n$ is linear in terms $n$, however the Szemerédi’s regularity lemma is used in their proof and so no specific constant coefficient is provided.

Here, we provide a method to obtain an upper bound for the size Ramsey number of a graph using good expander graphs such as Ramanujan graphs. In particular, we give an alternative proof for the linearity of the size Ramsey number of paths and cycles. Our method has two privileges in compare to the previous ones. Firstly, it proves the upper bound for every positive integer $n$ in comparison to the random graph methods which needs $n$ to be sufficiently large. Also, due to the recent explicit constructions for bipartite Ramanujan graphs by Marcus, Spielman and Srivastava, we can constructively find the graphs with small sizes which are Ramsey for a given graph. We also obtain some results about the bipartite Ramsey numbers.

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1. Introduction

In this note, we only consider undirected simple finite graphs. For a given graph $G$, we use $V(G)$, $E(G)$ and $e(G)$ for the vertex set, edge set and the number of edges of $G$, respectively. For every $v \in V(G)$, by $N_G(v)$ we mean the set of all neighbors of $v$ and for a subset $S$ of vertices, $N_S(v)$ stands for the set of all neighbors of $v$ in $S$. As usual, a path and a cycle on $n$ vertices are denoted by $P_n$ and $C_n$, respectively. Also the complete bipartite graph with partite sets $(X, Y)$ is denoted by $K[X,Y]$ and briefly by $K_{m,n}$, if $|X| = m$ and $|Y| = n$. For a given graph $G$ and two subsets $A, B \subseteq V(G)$, we mean by $e_G(A, B)$ the number of all edges with one endpoints in $A$ and another in $B$. Also, for $X \subseteq V(G)$, the graph $G[X]$ is the induced subgraph of $G$ on the set $X$.

Let $G, G_1, G_2$ be given graphs. We say that $G$ is Ramsey for $G_1$ and $G_2$ and we write $G \rightarrow (G_1, G_2)$, if in each edge coloring of $G$ with colors red and blue, there is either a red copy of $G_1$ or a blue copy of $G_2$. In this context, the Ramsey number of two graphs $G_1, G_2$, is defined as the minimum number of vertices of a graph $G$ which is Ramsey for $G_1, G_2$. In other words,

$$r(G_1, G_2) = \min\{|V(G)| : G \rightarrow (G_1, G_2)\}.$$  

Also, the size Ramsey number of graphs $G_1, G_2$, denoted by $\hat{r}(G_1, G_2)$, introduced by Erdős et al. [8] in 1978, is defined as the minimum number of edges of a graph $G$ which is Ramsey for $G_1, G_2$, i.e.

$$\hat{r}(G_1, G_2) = \min\{|E(G)| : G \rightarrow (G_1, G_2)\}.$$  

In the diagonal case, where $G = G_1 = G_2$, we may write $r(G)$ and $\hat{r}(G)$ for $r(G, G)$ and $\hat{r}(G, G)$, respectively.

As an easy observation, one can verify that

$$\hat{r}(G_1, G_2) \leq \left(\frac{r(G_1, G_2)}{2}\right).$$

Erdős et al. in [8] proved that the equality holds in the above upper bound when $G_1, G_2$ are the complete graphs. Also, they conjectured that this upper bound is far from being sharp when $G_1, G_2$ are some sparse graphs like $P_n$. In particular, Erdős offered $100 for a proof or disproof of the following fact.

Is it true that $\hat{r}(P_n)/n \rightarrow \infty$ or $\hat{r}(P_n)/n^2 \rightarrow 0$

In 1983 Beck [2] gave an answer to the question of Erdős. In fact, he showed that $\hat{r}(P_n) \leq 900n$ for sufficiently large $n$. This result verifies the linearity of the size-Ramsey number of paths in terms of the number of vertices. From that time, different approaches and methods were attempted by several authors to reduce the upper bound for $\hat{r}(P_n)$ (see [3, 5] and the references therein). Currently, the best known upper bound is due to Dudek et al. [6] which proved that $\hat{r}(P_n) \leq 74n$, for sufficiently large $n$. All of these results proves the existence of a graph $F$ with $O(n)$ number of edges which is Ramsey for $P_n$ when $n$ is sufficiently large. Beside the weakness of having no upper bound for the

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small numbers \( n \), these existence methods give no insight about the structure of such a graph \( F \) and the way one can explicitly construct it.

Here, we provide an alternative algebraic method which uses bipartite Ramanujan graphs to provide some upper bounds for the size Ramsey numbers of graphs. As a matter of fact, \( d \)-regular Ramanujan graphs are the graphs with a linear number of edges (in terms of the number of vertices) which have some good expanding properties. So, they are good candidates for a graph which is Ramsey for some sparse graphs. Our results, in contrast to the previous ones, are hold for every \( n \) (it does not need to be sufficiently large) and the method is constructive.

2. Main results

In this section, we will use the bipartite Ramanujan graphs and their expanding properties to give an upper bound for the size-Ramsey number of a graph. In particular, we give a linear upper bound for the size Ramsey number of the path \( P_n \) and the cycle \( C_{2n} \), for every positive integer \( n \). First, we define bipartite Ramanujan graphs and explore some of their expanding properties.

Let \( G \) be a \( d \)-regular graph. The eigenvalues of \( G \) is defined as the eigenvalues of its adjacency matrix. It is evident that \( d \) is the largest eigenvalue of \( G \) and \( d \) is an eigenvalue of \( G \) if and only if \( G \) is bipartite. When \( G \) is bipartite, \( d \) and \( -d \) are called the trivial eigenvalues of \( G \). Now, we give the definition of a bipartite Ramanujan graph.

**Definition 2.1.** Let \( G \) be a \( d \)-regular bipartite graph on \( n \) vertices with eigenvalues \( d = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n = -d \). Define

\[
\lambda(G) = \max\{|\lambda_i| : 2 \leq i \leq n - 2\}.
\]

\( G \) is said to be **bipartite Ramanujan** if \( \lambda(G) \leq 2\sqrt{d-1} \).

Trivial examples of Ramanujan graphs are complete graphs and regular complete bipartite graphs. The following lemma is a generalization of [1, Theorem 9.2.4] and show that bipartite Ramanujan graphs are good expanders. The proof is similar to the classic arguments. For the completeness, we give a proof here.

**Lemma 2.2.** Let \( G = (V, E) \) be a \( d \)-regular bipartite graph with two color classes \( X \) and \( Y \) such that \( |X| = |Y| = n \) and suppose that \( \lambda = \lambda(G) \) is defined as in Definition 2.1. Then, for every subset \( B \) of \( X \), with \( |B| = bn \), we have

\[
\sum_{v \in Y} (|N_B(v)| - bd)^2 \leq \lambda^2 b(1 - b)n.
\]

**Proof.** Let \( A \) be the adjacency matrix of \( G \) and define a vector \( f : V \to \mathbb{R} \) by \( f(v) = 1 - b \) for \( v \in B \), \( f(v) = -b \) for \( v \in X - B \) and \( f(v) = 0 \) for \( v \in Y \). Clearly, \( \sum_{v \in X} f(v) = \sum_{v \in Y} f(v) = 0 \). So, \( f \) is orthogonal to the eigenvectors corresponding to the eigenvalues \( d \) and \( -d \). Therefore,

\[
\langle Af, Af \rangle \leq \lambda^2 \langle f, f \rangle.
\]

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Also, on the one hand,
\[ \langle f, f \rangle = bn(1-b)^2 + (1-b)n(-b)^2 = b(1-b)n, \]
and the other hand,
\[ \langle Af, Af \rangle = \sum_{v \in Y} (|N_B(v)|(1-b) + (d-|N_B(v)|)(-b))^2 = \sum_{v \in Y} (|N_B(v)| - bd)^2. \]
So, the desired result follows.

Also, the following recent result due to Marcus, Spielman, and Srivastava guarantees the existence of regular bipartite Ramanujan graphs of all orders and all degrees. Also, their method gives an explicit construction for such graphs.

**Theorem 2.3.** [10, 11] For all integers \( n, d, n \geq d \geq 3 \), there is a \( d \)-regular \( n \times n \) bipartite Ramanujan graphs.

We also need the definition of the bipartite Ramsey number. Given two bipartite graphs \( G_1, G_2 \), the *bipartite Ramsey number* of \( G_1, G_2 \), denoted by \( br(G_1, G_2) \), is the smallest integer \( n \) such that for any edge coloring of \( K_{n,n} \) with two colors blue and red, there is either a blue copy of \( G_1 \), or a red copy of \( G_2 \). In other words, it is the smallest integer \( n \) such that \( K_{n,n} \rightarrow (G_1, G_2) \). The existence of such a positive integer is guaranteed by a result of Erdős and Rado [7]. It is easy to see that for bipartite graphs \( G_1, G_2 \), we have \( R(G_1, G_2) \leq 2br(G_1, G_2) \).

Now, we are ready to prove our main result.

**Theorem 2.4.** Let \( G_1, G_2 \) be two bipartite graphs, \( N, N', d \) be positive integers and \( 0 < b < 1 \) be a number. Also suppose that

(i) \( br(G_1, K_{N',N'}) \leq N \),
(ii) \( br(G_2, K_{[bN],[bN]}) \leq N' \), and
(iii) \( d^2/(d-1) > 4(1/b - 1)^2 \).

Then, every \( d \)-regular \( N \times N \) bipartite Ramanujan graph is Ramsey for \( G_1, G_2 \) and thus, \( r(G_1, G_2) \leq dN \).

**Proof.** Let \( F \) be a \( d \)-regular \( N \times N \) bipartite Ramanujan graph (which exists due to Theorem 2.3) with bipartite sets \( X, Y \). We are going to prove that \( F \rightarrow (G_1, G_2) \). Consider an edge coloring of \( F \) by two colors red and blue. Extend this coloring to an edge coloring of the graph \( K_{N,N} \) by coloring non-edges of \( F \) between \( X \) and \( Y \) by green. So, by (i), there is either a red copy of \( G_1 \), or a blue-green copy of \( K_{N',N'} \) in \( K_{N,N} \). In the former case, we are done (since \( F \) contains a red \( G_1 \)). In the latter case, we have a graph \( K_{N',N'} \) whose edges are colored by blue and green. So, by (ii), there is either a blue copy of \( G_2 \) or a green copy of \( K_{[bN],[bN]} \). Again, in the former case, we are done. Suppose that

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the latter case occurs. Thus, there are two sets \( B \subseteq X \) and \( C \subseteq Y \) with \( |B| = |C| = \frac{b'}{b}N \), for some \( b' \geq b \), such that \( e_F(B, C) = 0 \). Thus, by Lemma 2.2, we have

\[
\lambda^2 b'(1-b')N \geq \sum_{v \in Y} (|N_B(v)| - b'd)^2 = \sum_{v \in C} (b'd)^2 + \sum_{v \in Y \setminus C} (|N_B(v)| - b'd)^2 \\
\geq b'N(b'd)^2 + \frac{1}{|Y \setminus C|} \left( \sum_{v \in Y \setminus C} (|N_B(v)| - b'd) \right)^2 \\
= b'^2d^2N + \frac{1}{(1-b')N} (db'N - b'd(1-b')N)^2 \\
= b'^2d^2N + \frac{1}{(1-b')}b'^4d^2N,
\]

where the second inequality holds due to Cauchy-Schwartz inequality. Now, since \( F \) is a bipartite Ramanujan graph, \( \lambda^2 \leq 4(d-1) \). This implies that \( 4(d-1)(1-b')^2 \geq \lambda^2(1-b')^2 \geq b'^2d^2(1-b') + b'^4d^2 = b'^2d^2 \). Thus, \( d^2/(d-1) \leq 4(1/b' - 1)^2 \leq 4(1/b - 1)^2 \) which is in contradiction with (iii). This completes the proof.

Now, we show how one may apply Theorem 2.4 to prove an upper bound for the size Ramsey number of paths. First, using the following result from [5], we can give an upper bound for the bipartite Ramsey number of a path versus a complete bipartite graph.

**Lemma 2.5.** [5] Let \( G \) be a graph on \( N \) vertices and assume that \( G \) does not contain the path \( P_n \) as a subgraph. Then \( V(G) \) can be partitioned into three sets \( R, S, T \) such that \( |R| \leq n - 1 \), \( G[R] \) contains a Hamiltonian path, \( e_G(S, T) = 0 \) and \( S, T \) have almost equal sizes, i.e. \(|\|S\| - |T|\|\leq 1\|\).

**Lemma 2.6.** Let \( m, n \) be positive integers. Then, we have \( br(P_n, K_{m,m}) \leq 2m + \lceil (n-1)/2 \rceil - 1 \).

**Proof.** Let \( H \) be the complete bipartite graph with bipartition \( (X, Y) \), where \(|X| = |Y| = N = 2m + \lceil (n-1)/2 \rceil - 1\). Also, suppose that the edges of \( H \) are colored by red and blue and there is no red \( P_n \) in \( H \). We prove that \( H \) contains a blue copy of \( K_{m,m} \). Let \( H_r \) be the subgraph of \( H \) induced by red edges. By applying Lemma 2.5 to the graph \( H_r \), since \( H_r \) contains no \( P_n \) as a subgraph, there are subsets \( R, S, T \) having the properties mentioned there. Let \( X_1 = S \cap X, X_2 = T \cap X, Y_1 = S \cap Y \) and \( Y_2 = T \cap Y \) and without loss of generality assume that \(|X_1| \geq |X_2|\) and \(|Y_2| \geq |Y_1|\) (this can be done since \(|\|S\| - |T|\|\leq 1\|\)). Since \( e_{H_r}(S, T) = 0 \), all edges between \( X_1 \) and \( Y_2 \) are blue. So, it is enough to show that \(|X_1|, |Y_2| \geq m\). Since \( G[R] \) contains a Hamiltonian path, we have \(|R \cap X| \leq \lceil (n-1)/2 \rceil\). Thus, \(|X_1| \geq \lceil (N - \lceil (n-1)/2 \rceil) / 2 \rceil = m\). Similarly, \(|Y_2| \geq m\) and therefore, \( H \) contains a blue copy of \( K_{m,m} \). This completes the proof. \( \square \)

**Remark 2.7.** Note that the upper bound in Lemma 2.6 is tight, in the sense that there are infinitely many integers \( m, n \) for which equality holds in the upper bound. To see this, let \( n \geq 3 \) be an odd integer and let \( r = (n-1)/2 \). Also, suppose that \( m \) is an integer such that \( r \) divides \( m-1 \). Let \( N = 2m+r-2 \), DOI: http://dx.doi.org/10.22108/toc.2019.111317.1573
so r divides N. Considering the graph $F = K_{N,N}$ with bipartition $(X,Y)$, we provide an edge coloring for $F = K_{N,N}$ by two colors such that there is neither a red $P_n$, nor a blue $K_{m,m}$. For this, partition $X$ into $N/r$ sets $X_1, \ldots, X_{N/r}$ all of equal sizes $r$. Analogously, take the sets $Y_1, \ldots, Y_{N/r}$ of equal sizes partitioning $Y$. Now, for every $1 \leq i \leq N/r$, color all edges between $X_i$ and $Y_i$ by red. Also, color remaining edges by blue. Every red $P_n$ should be contained in $X_i \cup Y_i$, for some $i$. Nevertheless, $|X_i \cup Y_i| = 2r \leq n - 1$, so there is no red copy of $P_n$. Suppose that $F$ contains a blue copy of $K_{m,m}$ with bipartition $(X',Y')$, where $X' \subset X$ and $Y' \subset Y$. Then, clearly the set $I = \{i : X_i \cap X' \neq \emptyset\}$ has no intersection with the set $J = \{i : Y_i \cap Y' \neq \emptyset\}$. Therefore, either $|I| \leq N/2r$, or $|J| \leq N/2r$. On the other hand, since $|X'| = |Y'| = m$, we have $|I|, |J| \geq \lfloor m/r \rfloor = (m-1)/r+1$. So, $(m-1)/r+1 \leq N/2r$ which is a contradiction. Hence, $br(P_n, K_{m,m}) \geq 2m + r - 2$ and equality holds in Lemma 2.6.

As a result of Theorem 2.4 and Lemma 2.6, we can prove a linear upper bound for the size Ramsey number of $P_n$.

**Corollary 2.8.** Let $n \geq 13$ be an integer and let $G$ be a 78–regular $6n \times 6n$ bipartite Ramanujan graph. Then, we have $G \rightarrow (P_n, P_n)$ and thus, $\hat{r}(P_n) \leq 468n$.

**Proof.** Set $b = .184$, $d = 78$, $N = 6n$ and $N' = 2\lceil bn \rceil + \lfloor (n - 1)/2 \rfloor - 1$. Thus, by Lemma 2.6, $br(P_n, K_{\lceil bn \rceil, \lceil bn \rceil}) \leq 2\lceil bn \rceil + \lfloor (n - 1)/2 \rfloor - 1 = N'$. Also, again by Lemma 2.6,

$$br(P_n, K_{N',N'}) \leq 2N' + \left\lceil \frac{n-1}{2} \right\rceil - 1 \leq 4nN + \frac{3n}{2} + 1 \leq 6n = N.$$ 

Also, $d^2/(d-1) > 4(1/b - 1)^2$. Hence, the result follows from Theorem 2.4. \hfill \Box

For the next result, we need the following result from [9].

**Lemma 2.9.** Let $n, m_1, m_2$ be positive integers such that $n$ is even with $m_2 \geq n \geq (\lceil \log m_1 \rceil + \lceil \log m_2 \rceil + 1)$. Then, $br(C_n, K_{m_1, m_2}) \leq 18m_1 + 25m_2$.

Now, we can prove an upper bound for the size Ramsey number of an even cycle.

**Corollary 2.10.** Let $n \geq 7390$ be an even integer and $d = 13660415$. Also, let $G$ be a $d$–regular $43^2n \times 43^2n$ bipartite Ramanujan graph. Then, we have $G \rightarrow (C_n, C_n)$ and thus, $\hat{r}(C_n) \leq 43^2dn$.

**Proof.** Let $b = 1/43^2$, $N = 43^2n$, $N' = 43n$ and $d = 13660415$. Then, by Lemma 2.9, $br(C_n, K_{N',N'}) \leq 43N' \leq N$. Also, we have $br(C_n, K_{\lceil bn \rceil, \lceil bn \rceil}) \leq 43bN \leq N'$ and $d^2/(1 - d) > 4(1/b - 1)^2$. Therefore, the result follows from Theorem 2.4. \hfill \Box
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