Transactions on Combinatorics
ISSN (print): 2251-8657, ISSN (on-line): 2251-8665
Vol. 8 No. 2 (2019), pp. 45-51.

www.combinatorics.ir
www.ui.ac.ir

SIZE RAMSEY NUMBER OF BIPARTITE GRAPHS AND BIPARTITE RAMANUJAN GRAPHS

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Communicated by Gholamreza Omidi

Abstract. Given a graph G, a graph F is said to be Ramsey for G if in every edge coloring of F with two colors, there exists a monochromatic copy of G. The minimum number of edges of a graph F which is Ramsey for G is called the size-Ramsey number of G and is denoted by $\hat{r}(G)$. In 1983, Beck gave a linear upper bound (in terms of n) for $\hat{r}(P_n)$, where P_n is a path on n vertices, giving a positive answer to a question of Erdős. After that, different approaches were attempted by several authors to reduce the upper bound for $\hat{r}(P_n)$ for sufficiently large n and most of these approaches are based on the classic models of random graphs. Also, Haxell and Kohayakama in 1994 proved that the size Ramsey number of the cycle C_n is linear in terms n, however the Szemeredi's regularity lemma is used in their proof and so no specific constant coefficient is provided.

Here, we provide a method to obtain an upper bound for the size Ramsey number of a graph using good expander graphs such as Ramanujan graphs. In particular, we give an alternative proof for the linearity of the size Ramsey number of paths and cycles. Our method has two privileges in compare to the previous ones. Firstly, it proves the upper bound for every positive integer n in comparison to the random graph methods which needs n to be sufficiently large. Also, due to the recent explicit constructions for bipartite Ramanujan graphs by Marcus, Spielman and Srivastava, we can constructively find the graphs with small sizes which are Ramsey for a given graph. We also obtain some results about the bipartite Ramsey numbers.

MSC(2010): Primary: 05D10; Secondary: 05C55.

Keywords: size Ramsey number, bipartite Ramsey number, bipartite Ramanujan graph.

Received: 30 May 2018, Accepted: 02 May 2019.

? [1]This research is partially supported by INSF grant no. 95844679.

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DOI: http://dx.doi.org/10.22108/toc.2019.111317.1573

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1. Introduction

In this note, we only consider undirected simple finite graphs. For a given graph G, we use $V(G)$, $E(G)$ and $e(G)$ for the vertex set, edge set and the number of edges of G, respectively. For every $v \in V(G)$, by $N_G(v)$ we mean the set of all neighbors of v and for a subset S of vertices, $N_S(v)$ stands for the set of all neighbors of v in S. As usual, a path and a cycle on n vertices are denoted by P_n and C_n, respectively. Also the complete bipartite graph with partite sets (X, Y) is denoted by $K[X, Y]$ and briefly by $K_{m,n}$, if $|X| = m$ and $|Y| = n$. For a given graph G and two subsets $A, B \subseteq V(G)$, we mean by $e_G(A, B)$ the number of all edges with one endpoints in A and another in B. Also, for $X \subseteq V(G)$, the graph $G[X]$ is the induced subgraph of G on the set X.

Let G, G_1, G_2 be given graphs. We say that G is Ramsey for G_1 and G_2 and we write $G \to (G_1, G_2)$, if in each edge coloring of G with colors red and blue, there is either a red copy of G_1 or a blue copy of G_2. In this context, the Ramsey number of two graphs G_1, G_2, is defined as the minimum number of vertices of a graph G which is Ramsey for G_1, G_2. In other words,

$$r(G_1, G_2) = \min\{|V(G)| : G \to (G_1, G_2)\}.$$

Also, the size Ramsey number of graphs G_1, G_2, denoted by $\hat{r}(G_1, G_2)$, introduced by Erdős et al. [8] in 1978, is defined as the minimum number of edges of a graph G which is Ramsey for G_1, G_2, i.e.

$$\hat{r}(G_1, G_2) = \min\{|E(G)| : G \to (G_1, G_2)\}.$$

In the diagonal case, where $G = G_1 = G_2$, we may write $r(G)$ and $\hat{r}(G)$ for $r(G, G)$ and $\hat{r}(G, G)$, respectively.

As an easy observation, one can verify that

$$\hat{r}(G_1, G_2) \leq \binom{r(G_1, G_2)}{2}. \tag{1.1}$$

Erdős et al. in [8] proved that the equality holds in the above upper bound when G_1, G_2 are the complete graphs. Also, they conjectured that this upper bound is far from being sharp when G_1, G_2 are some sparse graphs like P_n. In particular, Erdős offered \$100 for a proof or disproof of the following fact.

$$\text{Is it true that } \hat{r}(P_n)/n \to \infty \text{ or } \hat{r}(P_n)/n^2 \to 0$$

In 1983 Beck [2] gave an answer to the question of Erdős. In fact, he showed that $\hat{r}(P_n) < 900n$ for sufficiently large n. This result verifies the linearity of the size-Ramsey number of paths in terms of the number of vertices. From that time, different approaches and methods were attempted by several authors to reduce the upper bound for $\hat{r}(P_n)$ (see [3, 5] and the references therein). Currently, the best known upper bound is due to Dudek et al. [6] which proved that $\hat{r}(P_n) \leq 74n$, for sufficiently large n. All of these results proves the existence of a graph F with $O(n)$ number of edges which is Ramsey for P_n when n is sufficiently large. Beside the weakness of having no upper bound for the

DOI: http://dx.doi.org/10.22108/toc.2019.111317.1573

small numbers n, these existence methods give no insight about the structure of such a graph F and the way one can explicitly construct it.

Here, we provide an alternative algebraic method which uses bipartite Ramanujan graphs to provide some upper bounds for the size Ramsey numbers of graphs. As a matter of fact, d-regular Ramanujan graphs are the graphs with a linear number of edges (in terms of the number of vertices) which have some good expanding properties. So, they are good candidates for a graph which is Ramsey for some sparse graphs. Our results, in contrast to the previous ones, are hold for every n (it does not need to be sufficiently large) and the method is constructive.

2. Main results

In this section, we will use the bipartite Ramanujan graphs and their expanding properties to give an upper bound for the size-Ramsey number of a graph. In particular, we give a linear upper bound for the size Ramsey number of the path P_n and the cycle C_{2n}, for every positive integer n. First, we define bipartite Ramanujan graphs and explore some of their expanding properties.

Let G be a d-regular graph. The eigenvalues of G is defined as the eigenvalues of its adjacency matrix. It is evident that d is the largest eigenvalue of G and $-d$ is an eigenvalue of G if and only if G is bipartite. When G is bipartite, d and $-d$ are called the trivial eigenvalues of G. Now, we give the definition of a bipartite Ramanujan graph.

Definition 2.1. *Let G be a $d-$regular bipartite graph on n vertices with eigenvalues $d = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n = -d$. Define*

$$\lambda(G) = \max\{|\lambda_i| : 2 \leq i \leq n-2\}.$$

*G is said to be **bipartite Ramanujan** if $\lambda(G) \leq 2\sqrt{d-1}$.*

Trivial examples of Ramanujan graphs are complete graphs and regular complete bipartite graphs. The following lemma is a generalization of [1, Theorem 9.2.4] and show that bipartite Ramanujan graphs are good expanders. The proof is similar to the classic arguments. For the completeness, we give a proof here.

Lemma 2.2. *Let $G = (V, E)$ be a $d-$regular bipartite graph with two color classes X and Y such that $|X| = |Y| = n$ and suppose that $\lambda = \lambda(G)$ is defined as in Definition 2.1. Then, for every subset B of X, with $|B| = bn$, we have*

$$\sum_{v \in Y} (|N_B(v)| - bd)^2 \leq \lambda^2 b(1-b)n.$$

Proof. Let A be the adjacency matrix of G and define a vector $f : V \to \mathbb{R}$ by $f(v) = 1-b$ for $v \in B$, $f(v) = -b$ for $v \in X - B$ and $f(v) = 0$ for $v \in Y$. Clearly, $\sum_{v\in X} f(v) = \sum_{v\in Y} f(v) = 0$. So, f is orthogonal to the eigenvectors corresponding to the eigenvalues d and $-d$. Therefore,

$$\langle Af, Af\rangle \leq \lambda^2 \langle f, f\rangle.$$

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Also, on the one hand,

$$\langle f, f\rangle = bn(1-b)^2 + (1-b)n(-b)^2 = b(1-b)n,$$

and the other hand,

$$\langle Af, Af\rangle = \sum_{v\in Y}(|N_B(v)|(1-b) + (d-|N_B(v)|)(-b))^2 = \sum_{v\in Y}(|N_B(v)| - bd)^2.$$

So, the desired result follows. □

Also, the following recent result due to Marcus, Spielman, and Srivastava guarantees the existence of regular bipartite Ramanujan graphs of all orders and all degrees. Also, their method gives an explicit construction for such graphs.

Theorem 2.3. [10, 11] *For all integers n, d, $n \geq d \geq 3$, there is a $d-$regular $n\times n$ bipartite Ramanujan graphs.*

We also need the definition of the bipartite Ramsey number. Given two bipartite graphs G_1, G_2, the *bipartite Ramsey number* of G_1, G_2, denoted by $br(G_1, G_2)$, is the smallest integer n such that for any edge coloring of $K_{n,n}$ with two colors blue and red, there is either a blue copy of G_1, or a red copy of G_2. In other words, it is the smallest integer n such that $K_{n,n} \to (G_1, G_2)$. The existence of such a positive integer is guaranteed by a result of Erdős and Rado [7]. It is easy to see that for bipartite graphs G_1, G_2, we have $R(G_1, G_2) \leq 2br(G_1, G_2)$.

Now, we are ready to prove our main result.

Theorem 2.4. *Let G_1, G_2 be two bipartite graphs, N, N', d be positive integers and $0 < b < 1$ be a number. Also suppose that*

(i) $br(G_1, K_{N',N'}) \leq N$,
(ii) $br(G_2, K_{\lceil bN\rceil,\lceil bN\rceil}) \leq N'$, *and*
(iii) $d^2/(d-1) > 4(1/b-1)^2$.

Then, every $d-$regular $N \times N$ bipartite Ramanujan graph is Ramsey for G_1, G_2 and thus, $\hat{r}(G_1, G_2) \leq dN$.

Proof. Let F be a $d-$regular $N \times N$ bipartite Ramanujan graph (which exists due to Theorem 2.3) with bipartite sets X, Y. We are going to prove that $F \to (G_1, G_2)$. Consider an edge coloring of F by two colors red and blue. Extend this coloring to an edge coloring of the graph $K_{N,N}$ by coloring non-edges of F between X and Y by green. So, by (i), there is either a red copy of G_1, or a blue-green copy of $K_{N',N'}$ in $K_{N,N}$. In the former case, we are done (since F contains a red G_1). In the latter case, we have a graph $K_{N',N'}$ whose edges are colored by blue and green. So, by (ii), there is either a blue copy of G_2 or a green copy of $K_{\lceil bN\rceil,\lceil bN\rceil}$. Again, in the former case, we are done. Suppose that

DOI: http://dx.doi.org/10.22108/toc.2019.111317.1573

the latter case occurs. Thus, there are two sets $B \subseteq X$ and $C \subseteq Y$ with $|B| = |C| = b'N$, for some $b' \geq b$, such that $e_F(B, C) = 0$. Thus, by Lemma 2.2, we have

$$\begin{aligned}\lambda^2 b'(1-b')N &\geq \sum_{v \in Y}(|N_B(v)| - b'd)^2 = \sum_{v \in C}(b'd)^2 + \sum_{v \in Y \setminus C}(|N_B(v)| - b'd)^2 \\ &\geq b'N(b'd)^2 + \frac{1}{|Y \setminus C|}\left(\sum_{v \in Y \setminus C}(|N_B(v)| - b'd)\right)^2 \\ &= b'^3 d^2 N + \frac{1}{(1-b')N}\left(db'N - b'd(1-b')N\right)^2 \\ &= b'^3 d^2 N + \frac{1}{(1-b')} b'^4 d^2 N,\end{aligned}$$

where the second inequality holds due to Cauchy-Schwartz inequality. Now, since F is a bipartite Ramanujan graph, $\lambda^2 \leq 4(d-1)$. This implies that $4(d-1)(1-b')^2 \geq \lambda^2 (1-b')^2 \geq b'^2 d^2 (1-b') + b'^3 d^2 = b'^2 d^2$. Thus, $d^2/(d-1) \leq 4(1/b'-1)^2 \leq 4(1/b-1)^2$ which is in contradiction with (iii). This completes the proof. $\square$

Now, we show how one may apply Theorem 2.4 to prove an upper bound for the size Ramsey number of paths. First, using the following result from [5], we can give an upper bound for the bipartite Ramsey number of a path versus a complete bipartite graph.

Lemma 2.5. [5] *Let G be a graph on N vertices and assume that G does not contain the path P_n as a subgraph. Then $V(G)$ can be partitioned into three sets R, S, T such that $|R| \leq n-1$, $G[R]$ contains a Hamiltonian path, $e_G(S, T) = 0$ and S, T have almost equal sizes, i.e. $||S| - |T|| \leq 1$.*

Lemma 2.6. *Let m, n be positive integers. Then, we have $br(P_n, K_{m,m}) \leq 2m + \lceil (n-1)/2 \rceil - 1$.*

Proof. Let H be the complete bipartite graph with bipartition (X, Y), where $|X| = |Y| = N = 2m + \lceil (n-1)/2 \rceil - 1$. Also, suppose that the edges of H are colored by red and blue and there is no red P_n in H. We prove that H contains a blue copy of $K_{m,m}$. Let H_r be the subgraph of H induced by red edges. By applying Lemma 2.5 to the graph H_r, since H_r contains no P_n as a subgraph, there are subsets R, S, T having the properties mentioned there. Let $X_1 = S \cap X$, $X_2 = T \cap X$, $Y_1 = S \cap Y$ and $Y_2 = T \cap Y$ and without loss of generality assume that $|X_1| \geq |X_2|$ and $|Y_2| \geq |Y_1|$ (this can be done since $||S| - |T|| \leq 1$). Since $e_{H_r}(S, T) = 0$, all edges between X_1 and Y_2 are blue. So, it is enough to show that $|X_1|, |Y_2| \geq m$. Since $G[R]$ contains a Hamiltonian path, we have $|R \cap X| \leq \lceil (n-1)/2 \rceil$. Thus, $|X_1| \geq \lceil (N - \lceil (n-1)/2 \rceil)/2 \rceil = m$. Similarly, $|Y_2| \geq m$ and therefore, H contains a blue copy of $K_{m,m}$. This completes the proof. $\square$

Remark 2.7. *Note that the upper bound in Lemma 2.6 is tight, in the sense that there are infinitely many integers m, n for which equality holds in the upper bound. To see this, let $n \geq 3$ be an odd integer and let $r = (n-1)/2$. Also, suppose that m is an integer such that r divides $m-1$. Let $N = 2m+r-2$,*

DOI: http://dx.doi.org/10.22108/toc.2019.111317.1573

so r divides N. Considering the graph $F = K_{N,N}$ with bipartition (X, Y), we provide an edge coloring for $F = K_{N,N}$ by two colors such that there is neither a red P_n, nor a blue $K_{m,m}$. For this, partition X into N/r sets $X_1, \ldots, X_{N/r}$ all of equal sizes r. Analogously, take the sets $Y_1, \ldots, Y_{N/r}$ of equal sizes partitioning Y. Now, for every $1 \leq i \leq N/r$, color all edges between X_i and Y_i by red. Also, color remaining edges by blue. Every red P_n should be contained in $X_i \cup Y_i$, for some i. Nevertheless, $|X_i \cup Y_i| = 2r \leq n-1$, so there is no red copy of P_n. Suppose that F contains a blue copy of $K_{m,m}$ with bipartition (X', Y'), where $X' \subset X$ and $Y' \subset Y$. Then, clearly the set $I = \{i : X_i \cap X' \neq \emptyset\}$ has no intersection with the set $J = \{i : Y_i \cap Y' \neq \emptyset\}$. Therefore, either $|I| \leq N/2r$, or $|J| \leq N/2r$. On the other hand, since $|X'| = |Y'| = m$, we have $|I|, |J| \geq \lceil m/r \rceil = (m-1)/r + 1$. So, $(m-1)/r + 1 \leq N/2r$ which is a contradiction. Hence, $br(P_n, K_{m,m}) \geq 2m + r - 2$ and equality holds in Lemma 2.6.

As a result of Theorem 2.4 and Lemma 2.6, we can prove a linear upper bound for the size Ramsey number of P_n.

Corollary 2.8. *Let $n \geq 13$ be an integer and let G be a $78-$regular $6n \times 6n$ bipartite Ramanujan graph. Then, we have $G \to (P_n, P_n)$ and thus, $\hat{r}(P_n) \leq 468\,n$.*

Proof. Set $b = .184$, $d = 78$, $N = 6n$ and $N' = 2\lceil bN \rceil + \lceil (n-1)/2 \rceil - 1$. Thus, by Lemma 2.6, $br(P_n, K_{\lceil bN \rceil, \lceil bN \rceil}) \leq 2\lceil bN \rceil + \lceil (n-1)/2 \rceil - 1 = N'$. Also, again by Lemma 2.6,

$$br(P_n, K_{N',N'}) \leq 2N' + \lceil \frac{n-1}{2} \rceil - 1 \leq 4bN + \frac{3n}{2} + 1 \leq 6n = N.$$

Also, $d^2/(d-1) > 4(1/b - 1)^2$. Hence, the result follows from Theorem 2.4. □

For the next result, we need the following result from [9].

Lemma 2.9. *Let n, m_1, m_2 be positive integers such that n is even with $m_2 \geq n \geq (\lceil \log m_1 \rceil + \lceil \log m_2 \rceil + 1)$. Then, $br(C_n, K_{m_1,m_2}) \leq 18m_1 + 25m_2$.*

Now, we can prove an upper bound for the size Ramsey number of an even cycle.

Corollary 2.10. *Let $n \geq 7390$ be an even integer and $d = 13660415$. Also, let G be a $d-$regular $43^2 n \times 43^2 n$ bipartite Ramanujan graph. Then, we have $G \to (C_n, C_n)$ and thus, $\hat{r}(C_n) \leq 43^2 dn$.*

Proof. Let $b = 1/43^2$, $N = 43^2 n$, $N' = 43n$ and $d = 13660415$. Then, by Lemma 2.9, $br(C_n, K_{N',N'}) \leq 43N' \leq N$. Also, we have $br(C_n, K_{bN,bN}) \leq 43bN \leq N'$ and $d^2/(1-d) > 4(1/b - 1)^2$. Therefore, the result follows from Theorem 2.4. □

DOI: http://dx.doi.org/10.22108/toc.2019.111317.1573

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DOI: http://dx.doi.org/10.22108/toc.2019.111317.1573