A GENERALIZATION OF HALL’S THEOREM FOR $k$-UNIFORM $k$-PARTITE HYPERGRAPHS

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Abstract. In this paper we prove a generalized version of Hall’s theorem in graphs, for hypergraphs. More precisely, let $\mathcal{H}$ be a $k$-uniform $k$-partite hypergraph with some ordering on parts as $V_1, V_2, \ldots, V_k$ such that the subhypergraph generated on $\bigcup_{i=1}^{k-1} V_i$ has a unique perfect matching. In this case, we give a necessary and sufficient condition for having a matching of size $t = |V_i|$ in $\mathcal{H}$. Some relevant results and counterexamples are given as well.

1. Introduction

We refer to [7] and [6] for elementary backgrounds in graph theory and hypergraph theory, respectively.

Let $G$ be a simple finite graph with vertex set $V(G)$ and edge set $E(G)$. A matching in $G$, is a set $M$ of pairwise disjoint edges of $G$. A matching $M$ is said to be a perfect matching, if every $x \in V(G)$, lies in one of elements of $M$. A matching $M$ in $G$, is maximum whenever for every matching $M'$, $|M'| \leq |M|$.

For every set of vertices $A$, $N(A)$ which is called the neighborhood of $A$, is the set of vertices which are adjacent with at least one element of $A$. The following theorem is known as Hall’s theorem in bipartite graphs.

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Theorem 1.1. ([7, Theorem 5.2]) Let $G$ be a bipartite graph with bipartition $(X, Y)$. Then $G$ contains a matching that saturates every vertex in $X$, if and only if

$$|N(S)| \geq |S| \quad \text{for all } S \subseteq X.$$ 

A vertex cover in $G$, is a subset $C$ of $V(G)$ such that for every edge $e$ of $G$, $e$ intersects $C$. A vertex cover $C$ is called a minimum vertex cover, if for every vertex cover $C'$, $|C| \leq |C'|$. The following theorem is known as König’s theorem in graph theory.

Theorem 1.2. ([7, Theorem 5.3]) In a bipartite graph, the number of edges in a maximum matching is equal to the number of vertices in a minimum vertex cover.

Let $V$ be a finite nonempty set. A hypergraph $H$ on $V$ is the pair $(V, E)$, where $E$ is a collection of nonempty subsets of $V$. Also here we assume that $\bigcup_{e \in H} e = V$. Each subset is said to be a hyperedge and each element of $V$ is called a vertex. We denote the set of vertices and hyperedges of $H$, by $V(H)$ and $E(H)$, respectively. Two vertices $x$ and $y$ of a hypergraph are said to be adjacent whenever they lie in a hyperedge.

A matching in the hypergraph $H$ is a set $M$ of pairwise disjoint hyperedges of $H$. A perfect matching is a matching such that every $x \in V(H)$ lies in one of its elements. A matching $M$ in $H$ is called a maximum matching whenever for every matching $M'$, we have $|M'| \leq |M|$.

In a hypergraph $H$, a subset $C$ of $V(H)$ is called a vertex cover if every hyperedge of $H$ intersects $C$. A vertex cover $C$ is said to be minimum if for every vertex cover $C'$, $|C| \leq |C'|$. We denote the number of hyperedges in a maximum matching of the hypergraph $H$, by $\alpha'(H)$ and the number of vertices in a minimum vertex cover of $H$, by $\beta(H)$.

A hypergraph $H$ is said to be simple or a clutter if none of its two distinct hyperedges contains another. A hypergraph is called $t$-uniform (or $t$-graph), if all its hyperedges have the same size $t$. A $t$-uniform ($t \geq 2$) hypergraph $H$ is said to be $r$-partite ($r \geq t$), whenever $V(H)$ can be partitioned into $r$ subsets such that for every two vertices $x, y$ in one part, $x$ and $y$ are not adjacent. If $r = 2, 3$, the hypergraph is said to be bipartite and tripartite, respectively.

Many researches have been done about matchings and existence of perfect matchings in hypergraphs (see for instance [1], [9], [12]). Also some attempts have been produced in generalization of Hall’s theorem and König’s theorem to hypergraphs (see [2], [3], [4], [5], [10], [11]).

Definition 1.3. Let $H$ be a $k$-uniform hypergraph with $k \geq 2$. A subset $\epsilon \subseteq V(H)$ of size $k - 1$ is called a submaximal hyperedge if there is a hyperedge containing $\epsilon$. For a submaximal hyperedge $\epsilon$, define the neighborhood of $\epsilon$ as the set $N(\epsilon) := \{v \in V(H) \mid \epsilon \cup \{v\} \in E(H)\}$.

For a set $A$, consisting of submaximal hyperedges of $H$, $\{v \in V(H) \mid \exists \epsilon \in A, v \in N(\epsilon)\}$ is denoted by $N(A)$.

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Definition 1.4. Let \( \mathcal{H} \) be a hypergraph and \( \emptyset \neq V' \subseteq V(\mathcal{H}) \). The subhypergraph generated on \( V' \) is
\[
<V'> := (V', \{e \cap V' | e \in E(\mathcal{H}), e \cap V' \neq \emptyset\}).
\]

If \( k \geq 3 \) and \( \mathcal{H} \) be a \( k \)-uniform \( k \)-partite hypergraph with parts \( V_1, V_2, \ldots, V_k \), it is clear that the subhypergraph generated on the union of every \( k-1 \) distinct parts is a \( (k-1) \)-uniform \( (k-1) \)-partite hypergraph.

Let \( \mathcal{A} = (A_1, \ldots, A_n) \) be a family of subsets of a set \( E \). A subset \( \{x_1, \ldots, x_n\} \) of \( E \), where \( x_i \neq x_j \), is said to be a transversal (or SDR) for \( \mathcal{A} \), if for every \( i \) (\( 1 \leq i \leq n \)), \( x_i \in A_i \). A partial transversal (partial SDR) of length \( l \) (\( 1 \leq l \leq n-1 \)) for \( \mathcal{A} \), is a transversal for a subfamily of \( \mathcal{A} \) with \( l \) sets.[8]

The following theorem is known as Hall’s theorem in combinatorics.

Theorem 1.5. ([1], Theorem 4.1] The family \( \mathcal{A} = (A_1, \ldots, A_n) \) of subsets of a set \( E \) has a transversal if and only if
\[
|\bigcup_{i \in I'} A_i| \geq |I'|, \quad \forall I' \subseteq \{1, \ldots, n\}.
\]

Corollary 1.6. ([1], Corollary 4.3] The family \( \mathcal{A} = (A_1, \ldots, A_n) \) of subsets of a set \( E \) has a partial transversal of length \( l(>0) \) if and only if
\[
|\bigcup_{i \in I'} A_i| \geq |I'| - n + l, \quad \forall I' \subseteq \{1, \ldots, n\}.
\]

2. The main results

Now we are ready to present our first theorem.

Theorem 2.1. Let \( \mathcal{H} \) be a \( k \)-uniform \( k \)-partite hypergraph with some ordering on parts, as \( V_1, V_2, \ldots, V_k \), such that the subhypergraph generated on \( \bigcup_{i=1}^{k-1} V_i \) has a unique perfect matching \( M \). Then \( \mathcal{H} \) has a matching of size \( t = |V_1| \), if and only if for every subset \( A \) of \( M \), \( |N(A)| \geq |A| \).

Proof. Let \( t = |V_1| \) and let the elements of \( M \) are \( e_1, \ldots, e_t \). Assume \( \mathcal{H} \) has a matching of size \( t \) with elements \( e_1, \ldots, e_t \). Upon uniqueness of \( M \), \( M = \{e_1 - V_k, \ldots, e_t - V_k\} \). Therefore
\[
(N(e_1), \ldots, N(e_t)) = (N(e_1 - V_k), \ldots, N(e_t - V_k)).
\]
Then the family \( (N(e_1), \ldots, N(e_t)) \) has an SDR. Thus by Theorem 1.1,
\[
|\bigcup_{i \in I} N(e_i)| \geq |I|, \quad \forall I \subseteq \{1, \ldots, t\},
\]
and therefore for every subset \( A \) of \( M \), \( |N(A)| \geq |A| \).

Conversely, let for every subset \( A \) of \( M \), we have \( |N(A)| \geq |A| \). Now, \( (N(e_1), \ldots, N(e_t)) \) is a family such that
\[
|\bigcup_{i \in I} N(e_i)| \geq |I|, \quad \forall I \subseteq \{1, \ldots, t\}.
\]
Therefore by Theorem 1.3, the mentioned family has an SDR. That is, there are distinct elements $x_1, \ldots, x_t$ of $V_k$ such that $x_j \in N(v_j)$. Now, for every $1 \leq j \leq t$, $v_j \cup \{x_j\}$ is a hyperedge of $H$ and these hyperedges are pairwise disjoint. Then they form a matching of size $t$ for $H$.

Corollary 2.2. Let $H$ be a $k$-uniform $k$-partite hypergraph with some ordering on parts as $V_1, V_2, \ldots, V_k$ where $|V_1| = |V_2| = \cdots = |V_k|$ such that the subhypergraph generated on $\bigcup_{i=1}^{k-1} V_i$ has a unique perfect matching $M$. Then $H$ has a perfect matching if and only if for every subset $A$ of $M$, $|N(A)| \geq |A|$.

Remark 2.3. Theorem 2.1 implies Theorem 1.1 (Hall’s theorem) in case $k = 2$.

Remark 2.4. In Theorem 2.1, if the hypothesis of uniqueness of perfect matching of subhypergraph generated on $\bigcup_{i=1}^{k-1} V_i$ is removed, only one side of theorem will remains correct. That is, from this fact that for every subset $A$ of $M$, $|N(A)| \geq |A|$, we conclude that $H$ has a matching of size $t = |V_1|$. The following example shows that the inverse case is not true in general.

Example 2.5. Assume the 3-uniform 3-partite hypergraph $H$ with the following presentation.

Indeed, $E(H) = \{\{x_1, y_1, z_1\}, \{x_1, y_2, z_2\}, \{x_2, y_1, z_2\}, \{x_2, y_2, z_2\}\}$, where the parts of $H$ are $V_1 = \{x_1, x_2\}, V_2 = \{y_1, y_2\}, V_3 = \{z_1, z_2\}$.

In this case, there is a perfect matching $M_1 = \{\{x_2, y_1\}, \{x_1, y_2\}\}$ for subhypergraph generated on $V_1 \cup V_2$. Although the hypergraph $H$ has a matching $M' = \{\{x_1, y_1, z_1\}, \{x_2, y_2, z_2\}\}$ of size 2, if $A = M_1$, we have $N(A) = \{z_2\}$. Therefore $|N(A)| \neq |A|$. Note that $M_1$ is not the unique perfect matching of subhypergraph generated on $V_1 \cup V_2$ because $M_2 = \{\{x_1, y_1\}, \{x_2, y_2\}\}$ is also yet.

Theorem 2.6. Let $H$ be a $k$-uniform $k$-partite hypergraph with some ordering on parts as $V_1, V_2, \ldots, V_k$ such that the subhypergraph generated on $\bigcup_{i=1}^{k-1} V_i$ has a perfect matching $M$. If for every subset $A$ of $M$, we have $|N(A)| \geq |A| - p$, where $p$ is a fixed integer and $1 \leq p \leq t - 1$, then $H$ has a matching of size $t - p$, where $t$ is the size of $V_1$.

Proof. Let the elements of $M$ are $v_1, \ldots, v_t$. $(N(v_1), \ldots, N(v_t))$ is a family such that the cardinality of the union of each $s$ terms is greater than or equal to $s - t + (t - p)$. Then by Corollary 1.6, the family $(N(v_1), \ldots, N(v_t))$ has a partial SDR of size $t - p$. That is, there are distinct elements $y_1, \ldots, y_{t-p}$.
of $V_k$ such that $y_j \in N(e_j)$. Now, for every $1 \leq j \leq t - p$, $e_j \cup \{y_j\}$ is a hyperedge of $H$ and these hyperedges are pairwise disjoint. Thus they form a matching of size $t - p$ for $H$.

Theorem 2.7. Let $H$ be a $k$-uniform $k$-partite hypergraph with some ordering on parts as $V_1, V_2, \ldots, V_k$, and let $t = |V_1|$. Then $H$ has a matching of size $t$ if and only if $\alpha' = \beta = t$, where $\alpha'$ and $\beta$ denotes the number of hyperedges in a maximum matching, and the number of vertices in a minimum vertex cover of $H$, respectively.

Proof. Let $H$ has a matching of size $t$. We show that $\alpha' = \beta = t$. Clearly $\beta \geq \alpha'$ because for covering each hyperedge of maximum matching, one vertex is needed. But since there is a matching of size $t$, then $\alpha' \geq t$. Now, $V_1$ is a minimal vertex cover of $H$ because each hyperedge has only one vertex in $V_1$ and each vertex of $V_1$ lies in a hyperedge. Therefore $t \geq \beta$, which implies that $\alpha' \geq \beta$. Then $\alpha' = \beta$.

The matching of size $t$ is the maximum matching because it covers all vertices of $V_1$.

Conversely, if $\alpha' = \beta = t$, it is clear that $H$ has a matching of size $t$.

The following example shows that removing the condition $t = |V_1|$ in Theorem 2.7 is not possible even if the subhypergraph generated on union of every $k - 1$ parts, has a perfect matching.

Example 2.8. Assume the 3-uniform 3-partite hypergraph $H$ with the following presentation, where the parts of $H$ are $V_1 = \{1, 2\}$, $V_2 = \{3, 4\}$, $V_3 = \{5, 6\}$.

Indeed, $E(H) = \{\{1, 3, 5\}, \{2, 3, 6\}, \{2, 4, 5\}\}$.

In this hypergraph, we have the matching $\{\{1, 3, 5\}\}$ of size 1. But $\alpha' \neq \beta$ because $\alpha' = 1$ and $\beta = 2$. Note that each one of subhypergraph generated on $V_1 \cup V_2$, $V_2 \cup V_3$ and $V_1 \cup V_3$, have a perfect matching.

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