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H-KERNELS BY WALKS IN SUBDIVISION DIGRAPH

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Abstract. Let H be a digraph possibly with loops and D a digraph without loops whose arcs are colored with the vertices of H (D is said to be an H-colored digraph). A directed walk W in D is said to be an H-walk if and only if the consecutive colors encountered on W form a directed walk in H. A subset N of the vertices of D is said to be an H-kernel by walks if (1) for every pair of different vertices in N there is no H-walk between them (N is H-independent by walks) and (2) for each vertex u in $V(D)$-N there exists an H-walk from u to N in D (N is H-absorbent by walks).

Suppose that D is a digraph possibly infinite. In this paper we will work with the subdivision digraph $S_H(D)$ of D, where $S_H(D)$ is an H-colored digraph defined as follows: $V(S_H(D)) = V(D) \cup A(D)$ and $A(S_H(D)) = \{(u,a) : a = (u,v) \in A(D)\} \cup \{(a,v) : a = (u,v) \in A(D)\}$, where (u, a, v) is an H-walk in $S_H(D)$ for every $a = (u,v)$ in $A(D)$. We will show sufficient conditions on D and on $S_H(D)$ which guarantee the existence or uniqueness of H-kernels by walks in $S_H(D)$.

1. Introduction

For general concepts we refer the reader to [2] and [3]. An arc of the form (x,x) is a *loop*. We will say that two digraphs D_1 and D_2 are equal, denoted by $D_1 = D_2$, if $A(D_1) = A(D_2)$ and $V(D_1) = V(D_2)$. A directed *walk* in a digraph D is a sequence $(v_1, v_2, \ldots, v_n)$ of vertices of D such that $(v_i,v_{i+1}) \in A(D)$ for each i in $\{1,\ldots,n-1\}$. If $v_i \neq v_j$ for all i and j such that $\{i, j\} \subseteq \{1,\ldots,n\}$ and $i \neq j$, it is called a directed *path*. A directed *cycle* is a directed walk $(v_1, v_2,\ldots,v_n, v_1)$ such that $v_i \neq v_j$

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for all i and j such that $\{i,\ j\} \subseteq \{1,\ldots,n\}$ and $i \neq j$. If D is an infinite digraph, an infinite outward path is an infinite sequence $(v_1,\ v_2,\ldots)$ of distinct vertices of D such that $(v_i,v_{i+1}) \in A(D)$ for each $i \in \mathbb{N}$. In this paper we are going to write walk, path, cycle, instead of directed walk, directed path, directed cycle, respectively. The union of walks will be denoted with $\cup$. Let $W = (v_0,\ v_1,\ldots,v_n)$ be a walk and $\{v_i,v_j\} \subseteq V(W)$, with $i<j$. Then the v_iv_j-walk $(v_i,v_{i+1},\ldots,v_{j-1},v_j)$ contained in W will be denoted by (v_i,W,v_j). Let S_1 and S_2 be two subsets of $V(D)$, a uv-walk in D will be called an S_1S_2-walk whenever $u \in S_1$ and $v \in S_2$. A digraph D is *transitive* whenever $\{(u,v),\ (v,w)\} \subseteq A(D)$ implies $(u,w) \in A(D)$. A digraph D is *asymmetric* whenever $(u,v) \in A(D)$ implies $(v,u) \notin A(D)$.

A digraph D is said to be m-colored if the arcs of D are colored with m colors. Let D be an m-colored digraph. A path is called monochromatic if all of its arcs are colored alike. For an arc (u,v) of D we will denote its color by $c_D(u,v)$. Let v be a vertex in $V(D)$ and $C = \{c_1,\ldots,c_m\}$ the set of colors used to color $A(D)$, the set $\{b \in C : c_D(v,u) = b$ for some u in $V(D)\}$ will be denoted by $\xi^+_D(v)$.

Let D be a digraph. A subset K of $V(D)$ is said to be a *kernel* if it is both independent (a vertex in K has no successor in K) and absorbing (a vertex not in K has a successor in K). The notion of kernel was introduced in [11] by von Neumann and Morgenstern in the context of Game Theory as a solution for cooperative n-player games. Kernels have been widely studied. For comprehensive surveys see, for example, [4], [6], [8] and [9]. Chvátal proved in [5] that recognizing digraphs that have a kernel is an NP complete problem, so finding sufficient conditions for a digraph to have a kernel or finding large families of digraphs with kernels have been a very prosperous line of investigation explored by many authors. The existence of kernels in digraphs formed by some operations from another digraphs have been studied by several authors. For example, in [14] Jerzy Topp defined the subdivision digraph $S(D)$ of a digraph D (D is possibly infinite) and he proved that $S(D)$ has at least one kernel for every digraph D. Moreover, Topp investigated some unitary operations on digraphs and some necessary or sufficient conditions for the existence or uniqueness of kernels in digraphs formed by these operations from another digraph.

Let D be an m-colored digraph. A subset N of $V(D)$ is said to be a *kernel by monochromatic paths* (mp-kernel) if it satisfies the following two conditions (1) No two vertices of N are connected by a monochromatic path (N is mp-independent) and (2) For every vertex x of D not in N there is a monochromatic path from x to a vertex in N (N is mp-absorbent). The existence of mp-kernels in m-colored digraphs was studied primarily by Sands, Sauer and Woodrow in [13]. In [7] Hortensia Galeana-Sánchez and Rocío Rojas-Monroy defined the subdivision digraph $S(D)$ of an m-colored digraph D and they proved that if D has no monochromatic infinite outward path, then $S(D)$ has an mp-kernel.

Let H be a digraph possibly with loops and D an H-colored digraph without loops. Notice that an arc in D is an H-path, that is to say, a singleton vertex is a walk in H, and the concatenation of two H-walks is not always an H-walk. In [1] Arpin and Linek introduced the concept of H-kernel by walks. The concept of H-kernel by walks generalizes the concept of kernel and mp-kernel since

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an H-kernel by walks is a kernel when $A(H) = \emptyset$ and it is an mp-kernel when $A(H) = \{(u,u) : u \in V(H)\}$.

Let H be and D two digraphs. In this paper we define the subdivision digraph $S_H(D)$, with respect to H, of D and we will prove the following: Let H be a digraph, D a digraph without infinite outward path and $S_H(D)$ a subdivision of D. Suppose that $|V(S_H(D))| \geq 2k+3$ and $|\xi^+_{S_H(D)}(v)| \leq k$ for every v in $V(S_H(D))$ and for some positive integer $k \geq 1$. Then $S_H(D)$ has an H-kernel by walks. We also will consider some sufficient conditions on D for the uniqueness of H-kernels by walks in $S_H(D)$. Then we are going to introduce a generalization of the subdivision digraph of D, with respect to H, denoted by $S'_H(D)$.

We need to introduce some notation in order to present our proofs more compactly.

Let H be a digraph and D an H-colored digraph. Consider $\{u, v\}$, S_1 and S_2 three subsets of $V(D)$. We will write: $u \xrightarrow[D]{H} v$ if there exists a uv-H-walk in D; $S_1 \xrightarrow[D]{H} S_2$ if there exists a S_1S_2-H-walk in D; $u \overset{H}{\underset{D}{\not\rightarrow}} v$ is the denial of $u \xrightarrow[D]{H} v$; $S_1 \overset{H}{\underset{D}{\not\rightarrow}} S_2$ is the denial of $S_1 \xrightarrow[D]{H} S_2$.

We will work with the following digraph in order to find an H-kernel by walks in $S_H(D)$.

Definition 1.1. *Let H be and D two digraphs and $S_H(D)$ a subdivision of D. D_A is the digraph such that $V(D_A) = A(D)$ and $A(D_A) = \{(a,b) : a \xrightarrow[S_H(D)]{H} b\}$.*

It follows from the definition of subdivision digraph that (1) $S_H(D)$ is obtained from D by changing the arc $a = (u,v)$ in D for an H-path (u, a, v) and (2) It is possible to have many subdivisions of D depending of $|A(H)|$.

We will use the following result in this paper.

Theorem 1.2 ([12])**.** *Let D be a digraph possibly infinite. Suppose that D is a transitive digraph such that every infinite outward path has at least one symmetric arc. Then D has a kernel.*

2. Preliminary results

We will start with some lemmas and one proposition which are going to be very useful in order to prove the main results.

Lemma 2.1. *Let H be and D two digraphs, $S_H(D)$ a subdivision of D and $\{a,b,c\} \subseteq V(S_H(D))$ such that $b \in A(D)$. If W_1 is an ab-H-walk in $S_H(D)$ and W_2 is a bc-H-walk in $S_H(D)$, then $W_1 \cup W_2$ is an H-walk in $S_H(D)$.*

Proof. Suppose that $W_1 = (a = v_0, \ldots, v_k = b)$ and $W_2 = (b = u_0, \ldots, u_n = c)$ for some subset $\{k,n\}$ of $\mathbb{N}$. If $a = b$ or $b = c$, it follows that $W_1 \cup W_2$ is an ac-H-walk in $S_H(D)$. Therefore, suppose that $b \notin \{a,c\}$. Since W_1 and W_2 are two H-walks in $S_H(D)$, it remains to prove that $(c_{S_H(D)}(v_{k-1}, v_k{=}b), c_{S_H(D)}(u_0{=}b, u_1)) \in A(H)$. Since $b \in A(D)$, it follows from the definition of $S_H(D)$ that $d^+_{S_H(D)}(b) =$

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$1 = d^{-}_{S_H(D)}(b)$ and, therefore, $(v_{k-1}, v_k = b, u_1)$ is an H-walk in $S_H(D)$ (by definition), which implies that $(c_{S_H(D)}(v_{k-1}, v_k{=}b), c_{S_H(D)}(u_0{=}b, u_1)) \in A(H)$. □

Lemma 2.2. *Let H be and D two digraphs and $S_H(D)$ a subdivision of D. If D has no infinite outward path, then $S_H(D)$ has no infinite outward path.*

Proof. Proceeding by contradiction, suppose that $(v_n)_{n\in\mathbb{N}}$ is an infinite outward path in $S_H(D)$ and $v_0 \in V(D)$. Thus, it follows from the definition of $S_H(D)$ that $v_{2i} \in V(D)$ and $v_{2i+1} \in A(D)$ for each i in $\mathbb{N}$, which implies that $(v_{2i})_{i\in\mathbb{N}}$ is an infinite outward path in D, a contradiction with the hypothesis of Lemma 2.2. Therefore, $S_H(D)$ does not have infinite outward path. □

Lemma 2.3 (flower)**.** *Let H be a digraph, D an H-colored digraph, with $|V(D)| \geq 2k+3$ and $|\xi^{+}_{D}(v)| \leq k$ for every v in $V(D)$, for some positive integer $k \geq 1$. Suppose that $\{a_1,\ldots,a_{2k+2}\}$ is a subset of $V(D)$ such that:*

(1) *$a_i \neq a_j$ for every subset $\{i,j\}$ of $\{1,\ldots,2k+2\}$ with $i \neq j$.*

(2) *There exists an a_ia_{i+1}-H-walk, say W_i, in D for each i in $\{1,\ldots, 2k+1\}$.*

(3) $\displaystyle\bigcap_{i=1}^{2k+1} [V(W_i) \setminus \{a_i, a_{i+1}\}] \neq \emptyset$.

Then there exists a subset $\{i,j\}$ of $\{2,\ldots, 2k+1\}$, with $i \leq j-1$, such that there is an a_ja_i-H-walk in D.

Proof. Let y be a vertex in $\displaystyle\bigcap_{i=1}^{2k+1} [V(W_i) \setminus \{a_i, a_{i+1}\}]$. For every i in $\{1, \ldots, 2k+1\}$ consider the arc $b_i = (y,x_{W_i})$ in W_i. We have two cases on $c_D(b_i)$ for every i in $\{1,\ldots, 2k+1\}$.

Case 1. There exists a subset $\{r,s\}$ of $\{1,\ldots, 2k+1\}$ such that $s \leq r-2$ and $c_D(b_r) = c_D(b_s)$.

In this case we have that $(a_r,W_r,y) \cup (y,W_s,a_{s+1})$ is an H-walk in D.

Case 2. $c_D(b_r) \neq c_D(b_s)$ for every subset $\{r,s\}$ of $\{1,\ldots, 2k+1\}$ such that $s \leq r-2$.

In this case, in particular, $b_{2k+1} \notin \{b_{2k-1},\ldots, b_1\}$, $b_{2k-1} \notin \{b_{2k-3},\ldots, b_1\},\ldots, b_5 \notin \{b_3, b_1\}$ and $b_3 \neq b_1$. Thus, $\{b_1,b_3,\ldots, b_{2k+1}\} \subseteq \xi^{+}_{D}(y)$, which implies that $|\xi^{+}_{D}(y)| \geq k+1$, a contradiction with $|\xi^{+}_{D}(y)| \leq k$.

Therefore, the only one possible case is 1. □

Lemma 2.4. *Let H be and D two digraphs and $S_H(D)$ a subdivision of D, with $|V(S_H(D))| \geq 2k+3$ and $|\xi^{+}_{S_H(D)}(v)| \leq k$ for every v in $V(S_H(D))$, for some positive integer $k \geq 1$. Suppose that $(a_n)_{n\in\mathbb{N}}$ is a sequence of different vertices of D_A such that:*

(1) *there exists an a_ia_{i+1}-H-walk, say W_i, in $S_H(D)$ for every i in $\mathbb{N}$.*

(2) $a_{i+1} \overset{H}{\underset{S_H(D)}{\not\longrightarrow}} a_i$ *for every i in $\mathbb{N}$.*

Then, there exist a path P in $S_H(D)$ of length at least 1, λ in $\mathbb{N}\setminus\{1\}$, a vertex x in W_λ and a sequence $(b_m)_{m\in\mathbb{N}}$ of different vertices of D_A such that:

1. *$x \in A(D)$.*

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2. *P is an a_1x-path in $S_H(D)$.*
3. *P is contained in $\bigcup_{n\in\mathbb{N}} W_n$.*
4. $V(P)\cap V((x,W_\lambda,a_{\lambda+1})\cup W_{\lambda+1})=\{x\}$.
5. $V(P)\cap V(W_{\lambda+j})=\emptyset$ *for every j in $\mathbb{N}\setminus\{1\}$.*
6. $b_1=x$ *and* $b_i=a_{\lambda+i}$ *for every i in $\mathbb{N}\setminus\{1\}$.*
7. $W_1^{(2)}=(x,W_\lambda,a_{\lambda+1})\cup W_{\lambda+1}$ *is a b_1b_2-H-walk in $S_H(D)$.*
8. $W_i^{(2)}=W_{\lambda+i}$ *is a b_ib_{i+1}-H-walk in $S_H(D)$ for every i in $\mathbb{N}\setminus\{1\}$.*
9. $b_{i+1}\ \underset{S_H(D)}{\overset{H}{\not\rightarrow}}\ b_i$ *for every i in $\mathbb{N}$.*

Proof. **Remark 1.** Notice that, since $\bigcup_{n\in\mathbb{N}} W_n$ is an H-walk (by Lemma 2.1) and $a_{i+1}\ \underset{S_H(D)}{\overset{H}{\not\rightarrow}}\ a_i$ for every i in $\mathbb{N}$, we get that $a_i\notin V(W_j)$ for every i in $\mathbb{N}$ and for every j in $\mathbb{N}$ such that $j\geq i+1$.

Let y be the first vertex in W_1 that appears in $V(W_h)$ for some h in $\mathbb{N}\setminus\{1\}$ (there exists y because $a_2\in V(W_1)\cap V(W_2)$). Notice that if $y=a_f$ for some f in $\mathbb{N}$, then $y\neq a_g$ for every g in $\mathbb{N}\setminus\{f\}$.

Consider the set $I=\{j\in\mathbb{N} : y\in V(W_j)\}$. **Claim 1.** I has a maximum element.

Proceeding by contradiction, suppose that I has no maximum element. Then, we can choose a subset $\{r_1,\ r_2,\ldots,\ r_{2k+1}\}$ of I such that $r_1<r_2<\cdots<r_{2k+1}$ and $y\notin\{a_{r_i},\ a_{r_i+1}\}$ for every i in $\{1,\ldots,\ 2k+1\}$.

Claim 1.1. There exists an $a_{r_i+1}a_{r_{i+1}+1}$-H-walk in $S_H(D)$ that contains y for every i in $\{1,\ldots,\ 2k+1\}$.

Let i be an index in $\{1,\ldots,\ 2k+1\}$. Then $\bigcup_{h=r_i+1}^{r_{i+1}} W_h$ is an $a_{r_i+1}a_{r_{i+1}+1}$-H-walk in $S_H(D)$ that contains y (by Lemma 2.1).

Let $d_1=a_{r_1}$ and $d_t=a_{r_{t-1}+1}$ for every t in $\{2,\ldots,\ 2k+2\}$. Since $\{d_1,\ldots,\ d_{2k+2}\}$ is a subset of $V(S_H(D))$ such that:

[1] $d_i\neq d_j$ for every subset $\{i,j\}$ of $\{1,\ldots,\ 2k+2\}$ with $i\neq j$.

[2] There exists a d_id_{i+1}-H-walk, say C_i, in $S_H(D)$ for each i in $\{1,\ldots,\ 2k+1\}$ (by Claim 1.1).

[3] $y\in\bigcap_{i=1}^{2k+1}[V(C_i)\setminus\{d_i,d_{i+1}\}]$.

Then, it follows from Lemma's flower that there exists a subset $\{\alpha,\beta\}$ of $\{2,\ldots,\ 2k+1\}$, with $\alpha\leq\beta-1$, such that there is a $d_\beta d_\alpha$-H-walk in $S_H(D)$, say W' (notice that W' is an $a_{r_{\beta-1}+1}a_{r_{\alpha-1}+1}$-$H$-walk in $S_H(D)$). Since $\alpha-1<\beta-1$, we have that $r_{\alpha-1}<r_{\beta-1}$. Thus, $C=\bigcup_{h=r_{\alpha-1}+1}^{r_{\beta-1}} W_h$ is an $a_{r_{\alpha-1}+1}a_{r_{\beta-1}+1}$-$H$-walk in $S_H(D)$, which implies that $W'\cup(a_{r_{\alpha-1}+1},C,a_{r_{\beta-1}})$ is an $a_{r_{\beta-1}+1}a_{r_{\beta-1}}$-$H$-walk in $S_H(D)$, a contradiction with $a_{i+1}\ \underset{S_H(D)}{\overset{H}{\not\rightarrow}}\ a_i$ for every i in $\mathbb{N}$.

Therefore, I has a maximum element, say λ. Let x be a vertex in $V(W_\lambda)$ such that

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$$x = \begin{cases} y & \text{if} \quad y \in A(D) \\ \\ \textit{the succesor} & \text{if} \quad y \notin A(D) \\ \textit{of } y \textit{ in } W_\lambda & \end{cases}$$

Notice that $x \in A(D)$ (by choice of x).

Claim 2. $x \notin V(W_{\lambda+n})$ for every n in $\mathbb{N}\backslash\{1\}$.

Proceeding by contradiction, suppose that $x \in V(W_{\lambda+m})$ for some $m \in \mathbb{N}\backslash\{1\}$. Then $(a_{\lambda+2},[\bigcup_{h=\lambda+2}^{\lambda+m} W_h],x) \cup (x, W_\lambda, a_{\lambda+1})$ is an $a_{\lambda+2}a_{\lambda+1}$-$H$-walk in $S_H(D)$ (by Lemma 2.1), a contradiction.

Claim 3. $a_{\lambda+2} \underset{S_H(D)}{\overset{H}{\not\longrightarrow}} x$.

Proceeding by contradiction, suppose that $a_{\lambda+2} \underset{S_H(D)}{\overset{H}{\longrightarrow}} x$. Since $(x, W_\lambda, a_{\lambda+1})$ is an $xa_{\lambda+1}$-H-walk in $S_H(D)$, it follows from Lemma 2.1 that $a_{\lambda+2} \underset{S_H(D)}{\overset{H}{\longrightarrow}} a_{\lambda+1}$, which is not possible.

Let P' be an a_1y-path contained in (a_1, W_1, y). If $y \in A(D)$, then set $P = P'$; if $y \notin A(D)$, then set $P = (P' \cup (y, x))$ (notice that in this case it follows from the choice of y and the choice of x that $x \notin V((a_1, W_1, y))$, which implies that P is a path). It follows from the definition of P that P is contained in $\bigcup_{n\in\mathbb{N}} W_n$. Since $a_1 \neq y$ (because $a_1 \notin W_i$ for every i in $\mathbb{N}\backslash\{1\}$) we get that the length of P is at least 1.

Claim 4. $V(P) \cap V((x, W_\lambda, a_{\lambda+1}) \cup W_{\lambda+1}) = \{x\}$.

If $y \in A(D)$, then it follows from the choice of y and by the choice of λ that $V((a_1, W_1, y)) \cap V((y, W_\lambda, a_{\lambda+1}) = \{y\}$, which implies that $V(P) \cap V((x, W_\lambda, a_{\lambda+1}) \cup W_{\lambda+1}) = \{y = x\}$.

If $y \notin A(D)$, then it follows from the choice of y that $V((a_1, W_1, y)) \cap V((y, W_\lambda, a_{\lambda+1})) = \{y\}$, which implies that $V(P) \cap V((x, W_\lambda, a_{\lambda+1}) \cup W_{\lambda+1}) = \{x\}$ (because by choice of λ, by choice of y and by choice of P' we have that $V(P') \cap V(W_{\lambda+1}) = \emptyset$).

Claim 5. $V(P) \cap V(W_{\lambda+j}) = \emptyset$ for every j in $\mathbb{N}\backslash\{1\}$.

Since $V((a_1, W_1, y)) \cap V(W_{\lambda+j}) = \emptyset$ for every j in $\mathbb{N}$ (by choice of y and by choice of λ), it follows that $V(P) \cap V(W_{\lambda+j}) = \emptyset$ for every j in $\mathbb{N}\backslash\{1\}$.

set $b_1 = x$ and set $b_i = a_{\lambda+i}$ for each i in $\mathbb{N}\backslash\{1\}$. It follows from the choice of x, Claim 2 and the hypothesis on the sequence $(a_n)_{n\in\mathbb{N}}$ that $(b_m)_{m\in\mathbb{N}}$ is a sequence of different vertices of D_A.

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Claim 6. $(x, W_\lambda, a_{\lambda+1}) \cup W_{\lambda+1}$ is a b_1b_2-H-walk in $S_H(D)$.

It follows from Lemma 2.1 that $(x, W_\lambda, a_{\lambda+1}) \cup W_{\lambda+1}$ is a b_1b_2-H-walk in $S_H(D)$.

Claim 7. $b_i \xrightarrow[S_H(D)]{H} b_{i+1}$ and $b_{i+1} \underset{S_H(D)}{\overset{H}{\not\longrightarrow}} b_i$ for each i in $\mathbb{N}$.

If $i \in \mathbb{N}\backslash\{1\}$, it follows from the definition of b_i and by hypothesis (1) and (2) of Lemma 2.4 that $b_i \xrightarrow[S_H(D)]{H} b_{i+1}$ and $b_{i+1} \underset{S_H(D)}{\overset{H}{\not\longrightarrow}} b_i$. If $i = 1$ we get from Claim 6 that $b_1 \xrightarrow[S_H(D)]{H} b_2$. On the other hand, $b_2 \underset{S_H(D)}{\overset{H}{\not\longrightarrow}} b_1$, otherwise if W is a b_2b_1-H-walk in $S_H(D)$ we get that $W \cup (x, W_\lambda, a_{\lambda+1})$ is an $a_{\lambda+2}a_{\lambda+1}$-$H$-walk in $S_H(D)$, a contradiction.

In order to conclude with the proof of Lemma 2.4, set $W_1^{(2)} = (x, W_\lambda, a_{\lambda+1}) \cup W_{\lambda+1}$ and set $W_i^{(2)} = W_{\lambda+i}$ for every i in $\mathbb{N}$. □

Corollary 2.5. *Let H be and D two digraphs and $S_H(D)$ a subdivision of D, with $|V(S_H(D))| \geq 2k+3$ and $|\xi^+_{S_H(D)}(v)| \leq k$ for every v in $V(S_H(D))$, for some positive integer $k \geq 1$. Suppose that $(a_n)_{n\in\mathbb{N}}$ is a sequence of different vertices of D_A such that:*

(1) $a_i \xrightarrow[S_H(D)]{H} a_{i+1}$ *for every i in $\mathbb{N}$.*

(2) $a_{i+1} \underset{S_H(D)}{\overset{H}{\not\longrightarrow}} a_i$ *for every i in $\mathbb{N}$.*

Then, $S_H(D)$ has an infinite outward path.

Proof. Set $a_n^{(1)} = a_n$ for every n in $\mathbb{N}$. Let $W_i^{(1)}$ be an a_ia_{i+1}-H-walk in $S_H(D)$ for every i in $\mathbb{N}$. It follows from Lemma 2.4 that there exist a path P_1 in $S_H(D)$ of length at least 1, λ_1 in $\mathbb{N}\backslash\{1\}$, a vertex x_1 in $W^{(1)}_{\lambda_1}$ and a sequence $(a_n^{(2)})_{n\in\mathbb{N}}$ of different vertices of D_A such that:

$1^{(1)}$. $x_1 \in A(D)$.

$2^{(1)}$. P_1 is an $a_1^{(1)}x_1$-path in $S_H(D)$.

$3^{(1)}$. P_1 is contained in $\bigcup_{n\in\mathbb{N}} W_n^{(1)}$.

$4^{(1)}$. $V(P_1) \cap V((x_1, W^{(1)}_{\lambda_1}, a^{(1)}_{\lambda_1+1}) \cup W^{(1)}_{\lambda_1+1}) = \{x_1\}$.

$5^{(1)}$. $V(P_1) \cap V(W^{(1)}_{\lambda_1+j}) = \emptyset$ for every j in $\mathbb{N}\backslash\{1\}$.

$6^{(1)}$. $a_1^{(2)} = x_1$ and $a_i^{(2)} = a^{(1)}_{\lambda_1+i}$ for every i in $\mathbb{N}\backslash\{1\}$.

$7^{(1)}$. $W_1^{(2)} = (x_1, W^{(1)}_{\lambda_1}, a^{(1)}_{\lambda_1+1}) \cup W_{\lambda_1+1}$ is an $a_1^{(2)}a_2^{(2)}$-H-walk in $S_H(D)$.

$8^{(1)}$. $W_i^{(2)} = W^{(1)}_{\lambda_1+i}$ is an $a_i^{(2)}a_{i+1}^{(2)}$-H-walk in $S_H(D)$ for every i in $\mathbb{N}\backslash\{1\}$.

$9^{(1)}$. $a_{i+1}^{(2)} \underset{S_H(D)}{\overset{H}{\not\longrightarrow}} a_i^{(2)}$ for every i in $\mathbb{N}$.

By $7^{(1)}$, $8^{(1)}$, $9^{(1)}$ and by Lemma 2.4 we get that for the sequence $(a_n^{(2)})_{n\in\mathbb{N}}$ there exist a path P_2 in $S_H(D)$ of length at least 1, λ_2 in $\mathbb{N}\backslash\{1\}$, a vertex x_2 in $W^{(2)}_{\lambda_2}$ and a sequence $(a_n^{(3)})_{n\in\mathbb{N}}$ of different vertices of D_A such that:

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$1^{(1)}$. $x_2 \in A(D)$.
$2^{(2)}$. P_2 is an $a_1^{(2)}x_2$-path in $S_H(D)$.
$3^{(2)}$. P_2 is contained in $\bigcup_{n\in\mathbb{N}} W_n^{(2)}$.
$4^{(2)}$. $V(P_2) \cap V((x_2, W_{\lambda_2}^{(2)}, a_{\lambda_2+1}^{(2)}) \cup W_{\lambda_2+1}^{(2)}) = \{x_2\}$.
$5^{(2)}$. $V(P_2) \cap V(W_{\lambda_2+j}^{(2)}) = \emptyset$ for every j in $\mathbb{N}\backslash\{1\}$.
$6^{(2)}$. $a_1^{(3)} = x_2$ and $a_i^{(3)} = a_{\lambda_2+i}^{(2)}$ for every i in $\mathbb{N}\backslash\{1\}$.
$7^{(2)}$. $W_1^{(3)} = (x_2, W_{\lambda_2}^{(2)}, a_{\lambda_2+1}^{(2)}) \cup W_{\lambda_2+1}^{(2)}$ is an $a_1^{(3)}a_2^{(3)}$-H-walk in $S_H(D)$.
$8^{(2)}$. $W_i^{(3)} = W_{\lambda_2+i}^{(2)}$ is an $a_i^{(3)}a_{i+1}^{(3)}$-H-walk in $S_H(D)$ for every i in $\mathbb{N}\backslash\{1\}$.
$9^{(2)}$. $a_{i+1}^{(3)} \underset{S_H(D)}{\overset{H}{\not\rightarrow}} a_i^{(3)}$ for every i in $\mathbb{N}$.

So, by Lemma 2.4, we have that for each m in $\mathbb{N}$, given a sequence of different vertices of D_A $(a_n^{(m)})_{n\in\mathbb{N}}$ such that:

[1] $a_i^{(m)} \underset{S_H(D)}{\overset{H}{\longrightarrow}} a_{i+1}^{(m)}$ for every i in $\mathbb{N}$,
[2] $a_{i+1}^{(m)} \underset{S_H(D)}{\overset{H}{\not\rightarrow}} a_i^{(m)}$ for every i in $\mathbb{N}$,

where if $W_i^{(m)}$ is an $a_i^{(m)}a_{i+1}^{(m)}$-H-walk in $S_H(D)$ for every i in $\mathbb{N}$, then there exist a path P_m in $S_H(D)$ of length at least 1, λ_m in $\mathbb{N}\backslash\{1\}$, a vertex x_m in $W_{\lambda_m}^{(m)}$ and a sequence $(a_n^{(m+1)})_{n\in\mathbb{N}}$ of different vertices of D_A such that:

$1^{(m)}$. $x_m \in A(D)$.
$2^{(m)}$. P_m is an $a_1^{(m)}x_m$-path in $S_H(D)$.
$3^{(m)}$. P_m is contained in $\bigcup_{n\in\mathbb{N}} W_n^{(m)}$.
$4^{(m)}$. $V(P_m) \cap V((x_m, W_{\lambda_m}^{(m)}, a_{\lambda_m+1}^{(m)}) \cup W_{\lambda_m+1}^{(m)}) = \{x_m\}$.
$5^{(m)}$. $V(P_m) \cap V(W_{\lambda_m+j}^{(m)}) = \emptyset$ for every j in $\mathbb{N}\backslash\{1\}$.
$6^{(m)}$. $a_1^{(m+1)} = x_m$ and $a_i^{(m+1)} = a_{\lambda_m+i}^{(m)}$ for every i in $\mathbb{N}\backslash\{1\}$.
$7^{(m)}$. $W_1^{(m+1)} = (x_m, W_{\lambda_m}^{(m)}, a_{\lambda_m+1}^{(m)}) \cup W_{\lambda_m+1}^{(m)}$ is an $a_1^{(m+1)}a_2^{(m+1)}$-H-walk in $S_H(D)$.
$8^{(m)}$. $W_i^{(m+1)} = W_{\lambda_m+i}^{(m)}$ is an $a_i^{(m+1)}a_{i+1}^{(m+1)}$-$H$-walk in $S_H(D)$ for every i in $\mathbb{N}\backslash\{1\}$.
$9^{(m)}$. $a_{i+1}^{(m+1)} \underset{S_H(D)}{\overset{H}{\not\rightarrow}} a_i^{(m+1)}$ for every i in $\mathbb{N}$.

Claim 1. $\bigcup_{i\in\mathbb{N}} P_i$ is an infinite outward path in $S_H(D)$.

Let i be an index in $\mathbb{N}$. Since P_i is an $a_1^{(i)}x_i$-path in $S_H(D)$, P_{i+1} is an $a_1^{(i+1)}x_{i+1}$-path in $S_H(D)$ and $a_1^{(i+1)} = x_i$, we get that $P_i \cup P_{i+1}$ is a walk in $S_H(D)$, which implies that $\bigcup_{i\in\mathbb{N}} P_i$ is a walk in $S_H(D)$.

Claim 1.1. $V(P_i) \cap V(P_{i+1}) = \{x_i\}$ for every i in $\mathbb{N}$.

Let i be an index in $\mathbb{N}$. Since P_{i+1} is contained in $\bigcup_{n\in\mathbb{N}} W_n^{(i+1)}$ (by property in $3^{(i)}$), $V(P_i) \cap V(W_k^{(i+1)}) = \emptyset$ for every k in $\mathbb{N}\backslash\{1\}$ (by property in $5^{(i)}$), $V(P_i) \cap V(W_1^{(i+1)}) = \{x_i\}$ (by property in $4^{(i)}$

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and the definition of $W_1^{(i+1)}$) and $P_i \cup P_{i+1}$ is a walk in $S_H(D)$, then we get that $V(P_i) \cap V(P_{i+1}) = \{x_i\}$.

Claim 1.2. $V(P_i) \cap V(P_j) = \emptyset$ for every i in $\mathbb{N}$ and for every j in $\mathbb{N}$ such that $i+1 < j$.

It follows from properties $7^{(\alpha)}$ and $8^{(\alpha)}$ for every α in $\{i, \ldots, j\}$ that $\bigcup_{n\in\mathbb{N}} W_n^{(j)}$ is contained in $\bigcup_{n\in\mathbb{N}} W_n^{(i+2)}$, which implies that P_j is contained in $\bigcup_{n\in\mathbb{N}} W_n^{(i+2)}$ (by property in $3^{(j)}$).

Since $V(P_i) \cap V(W_k^{(i+1)}) = \emptyset$ for every k in $\mathbb{N}\setminus\{1\}$ (by property in $5^{(i)}$), $W_1^{(i+2)} = (x_{i+1}, W_{\lambda_{i+1}}^{(i+1)}, a_{\lambda_{i+1}+1}^{(i+1)}) \cup W_{\lambda_{i+1}+1}^{(i+1)}$, $W_k^{(i+2)} = W_{\lambda_{i+1}+k}^{(i+1)}$ for every k in $\mathbb{N}\setminus\{1\}$ and $\lambda_{i+1} \in \mathbb{N} \setminus \{1\}$, we get that $V(P_i) \cap V(P_j) = \emptyset$.

Therefore, it follows from Claims 1.1 and 1.2 that $\bigcup_{i\in\mathbb{N}} P_i$ is an infinite outward path in $S_H(D)$. □

Proposition 2.6. *Let H be a digraph, D a digraph without infinite outward path and $S_H(D)$ a subdivision of D. Suppose that $|V(S_H(D))| \geq 2k+3$ and $|\xi^+_{S_H(D)}(v)| \leq k$ for every v in $V(S_H(D))$ and for some positive integer $k \geq 1$. Then D_A has a kernel.*

Proof. Consider the following claims.

Claim 1. D_A has no asymmetric infinite outward path.

Proceeding by contradiction, suppose that $(a_n)_{n\in\mathbb{N}}$ is an asymmetric infinite outward path in D_A. Then, it follows from the definition of D_A that $a_i \xrightarrow[S_H(D)]{H} a_{i+1}$ for every i in $\mathbb{N}$ and since $(a_n)_{n\in\mathbb{N}}$ is an asymmetric infinite outward path in D_A it follows that $a_{i+1} \underset{S_H(D)}{\overset{H}{\not\longrightarrow}} a_i$ for every i in $\mathbb{N}$. Therefore, we get from Corollary 2.5 that $S_H(D)$ has an infinite outward path, a contradiction with Lemma 2.2. Therefore, D_A has no asymmetric infinite outward path.

Claim 2. D_A is a transitive digraph.

Let $\{u, v, w\}$ be a subset of $V(D_A)$ such that $\{(u,v),(v,w)\} \subseteq A(D_A)$. We will prove that $(u,w) \in A(D_A)$. From the definition of D_A we have that $u \xrightarrow[S_H(D)]{H} v$ and $v \xrightarrow[S_H(D)]{H} w$. So, it follows from Lemma 2.1 that $u \xrightarrow[S_H(D)]{H} w$. Therefore, $(u,w) \in A(D_A)$.

Since every infinite outward path in D_A has a symmetric arc and D_A is a transitive digraph, it follows from Theorem 1.2 that D_A has a kernel. □

3. Main results

Theorem 3.1. *Let H be a digraph, D a digraph without infinite outward path and $S_H(D)$ a subdivision of D. Suppose that $|V(S_H(D))| \geq 2k+3$ and $|\xi^+_{S_H(D)}(v)| \leq k$ for every v in $V(S_H(D))$ and for some positive integer $k \geq 1$. Then $S_H(D)$ has an H-kernel by walks.*

Proof. It follows from Proposition 2.6 that D_A has a kernel N_1.

Consider the following sets:

$B = \{w \in V(D) : d^+_{S_H(D)}(w) = 0\}$.

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$N_2 = \{a \in N_1 : a \xrightarrow[S_H(D)]{H} w$ for some w in $B\}$.

The following claims on $N = (B \cup [N_1 \backslash N_2])$ will be useful.

Claim 1. N is an H-independent set by walks in $S_H(D)$.

From the definition of B we have that it is an H-independent set by walks in $S_H(D)$ and $B \overset{H}{\underset{S_H(D)}{\not\longrightarrow}} [N_1 \backslash N_2]$. On the other hand, we have that $[N_1 \backslash N_2] \overset{H}{\underset{S_H(D)}{\not\longrightarrow}} B$ (by the definition of N_2). Therefore, it remains to prove that $N_1 \backslash N_2$ is an H-independent set by walks in $S_H(D)$. Proceeding by contradiction, suppose that there exists a subset $\{u,v\}$ of $N_1 \backslash N_2$ such that $u \xrightarrow[S_H(D)]{H} v$. Since $\{u,v\} \subseteq N_1 \subseteq A(D)$, it follows from the definition of D_A that $(u,v) \in A(D_A)$, contradicting that N_1 is an independent set in D_A. Therefore, $N_1 \backslash N_2$ is an H-independent set by walks in $S_H(D)$.

Claim 2. N is an H-absorbent set by walks in $S_H(D)$.

Let u be a vertex in $V(S_H(D)) \backslash N$.

We will prove that $u \xrightarrow[S_H(D)]{H} w$ for some w in N.

Consider three cases on u.

Case 1. $u \in N_2$.

In this case, it follows from the definition of N_2 that there exists a vertex w in B such that $u \xrightarrow[S_H(D)]{H} w$.

Case 2. $u \in A(D)$ and $u \notin N_2$.

Since N_1 is a kernel of D_A, it follows that there exists a vertex a in N_1 such that $(u,a) \in A(D_A)$. Which implies that $u \xrightarrow[S_H(D)]{H} a$. Suppose that $a \in N_2$. So, there exists a vertex x in B such that $a \xrightarrow[S_H(D)]{H} x$. Thus, $u \xrightarrow[S_H(D)]{H} x$ (by Lemma 2.1).

Case 3. $u \in V(D)$.

Since $u \notin B$, it follows that $d^+_{S_H(D)}(u) \neq 0$. Let z be a vertex in $V(S_H(D))$ such that $(u,z) \in A(S_H(D))$. Notice that $z \in A(D)$ (by definition of $S_H(D)$).

Suppose that $z \notin N_1 \backslash N_2$. Then, it follows from cases 1 and 2 that there exists a vertex x in N such that $z \xrightarrow[S_H(D)]{H} x$. Therefore, $u \xrightarrow[S_H(D)]{H} x$ (by Lemma 1). Thus, N is an H-absorbent set by walks in $S_H(D)$.

So, it follows from Claims 1 and 2 that N is an H-kernel by walks in $S_H(D)$. □

Theorem 3.2. *Let H be a digraph, D a digraph without cycles and without infinite outward path and $S_H(D)$ a subdivision of D. Suppose that $|V(S_H(D))| \geq 2k+3$ and $|\xi^+_{S_H(D)}(v)| \leq k$ for every v in $V(S_H(D))$ and for some positive integer $k \geq 1$. Then, $S_H(D)$ has a unique H-kernel by walks.*

Proof. Consider the following remark.

Remark. Since D has no cycles, then $S_H(D)$ has no cycles. Which implies that $S_H(D)$ has no closed walks.

From Theorem 3.1 we have that $S_H(D)$ has an H-kernel by walks. Let N be and M two H-kernels by walks in $S_H(D)$. We will prove that $N = M$.

Let u be a vertex in N. We are going to prove that $u \in M$. Proceeding by contradiction, suppose that $u \notin M$. Then, since N and M are H-kernel by walks in $S_H(D)$, it follows that there exists u_1 in $M \backslash N$ such that $u \xrightarrow[S_H(D)]{H} u_1$, there exists u_2 in $N \backslash M$ such that $u_1 \xrightarrow[S_H(D)]{H} u_2$, there exists u_3 in $M \backslash N$ such that $u_2 \xrightarrow[S_H(D)]{H} u_3$, there exists u_4 in $N \backslash M$ such that $u_3 \xrightarrow[S_H(D)]{H} u_4$, and so on. Let P_i be a $u_{i-1}u_i$-H-walk in $S_H(D)$ for each i in $\mathbb{N}$, where $u_0 = u$ when $i = 1$.

Consider the following claim.

Claim. For every subset $\{i, j\}$ of $\mathbb{N}$ we have that

$$V(P_i) \cap V(P_j) = \begin{cases} \{u_i\} & \text{if} \quad j = i+1 \\ \\ \emptyset & \quad\;\; j \neq i+1 \end{cases}$$

Let $\{i, j\}$ be a subset of $\mathbb{N}$. If $j = i + 1$ and there exists a vertex x in $V(P_i) \cap V(P_{i+1})$ such that $x \neq u_i$. Then, $(x, P_i, u_i) \cup (u_i, P_{i+1}, x)$ is a closed walk in $S_H(D)$, a contradiction (by remark). Therefore, $[V(P_i) \cap V(P_{i+1})] = \{u_i\}$. If $j \neq i + 1$, with $i + 1 < j$, and there exists a vertex x in $V(P_i)$ $\cap\, V(P_j)$. Then, $(x, P_i, u_i) \cup [\bigcup_{n=i+1}^{j-1} P_n] \cup (u_{j-1}, P_j, x)$ is a closed walk in $S_H(D)$, a contradiction (by remark). Therefore, $[V(P_i) \cap V(P_j)] = \emptyset$.

Thus, it follows from claim that $\bigcup_{n \in \mathbb{N}} P_n$ is an infinite outward path in $S_H(D)$, a contradiction with Lemma 2.2. Therefore, $u \in M$.

Similarly we can prove that $M \subseteq N$. So, $N = M$. □

4. A generalization of the subdivision digraph of D, with respect to H, $S'_H(D)$.

Definition 4.1. *Let H be and D two digraphs, $S_H(D)$ a subdivision of D and β_a a ua-H-path for some $a = (u,v)$ in $A(D)$ such that:*

1. *The initial arc of β_a has the same color than the arc (u,a) in $S_H(D)$.*
2. *$\beta_a \cup (a,v)$ is an H-walk.*
3. *$V(\beta_a) \cap V(S_H(D)) = \{u,a\}$.*
4. *$(V(\beta_a) \cap V(\beta_b)) \backslash \{u\} = \emptyset$ for $b \neq a$.*

The generalization of the subdivision digraph of D, with respect to H, $S'_H(D)$ is the H-colored digraph defined as follows:

$V(S'_H(D)) = V(S_H(D)) \cup [\bigcup_{a \in A(D)} V(\beta_a)]$ *and*

$A(S'_H(D)) = [A(S_H(D)) \backslash \{(w,b) : b = (w,z) \in A(D)\}] \cup [\bigcup_{b \in A(D)} A(\beta_b)]$

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In other words, $S'_H(D)$ is obtained from $S_H(D)$ by changing the arc (u,a) in $S_H(D)$ for the H-path β_a for every $a = (u,v)$ in $A(D)$.

Remark 4.2. *It follows from the definition of $S'_H(D)$ that $d^+_{S'_H(D)}(x) = 1 = d^-_{S'_H(D)}(x)$ for every $a = (u,v)$ in $A(D)$ and for every x in $V(\beta_a)\setminus\{u,v\}$.*

The following proposition establishes a link between an H-kernel by walks of $S_H(D)$ and an H-kernel by walks of $S'_H(D)$.

Proposition 4.3. *Let H be and D two digraphs, $S_H(D)$ a subdivision of D, $S'_H(D)$ a generalization of $S_H(D)$ and N a subset of $V(S_H(D))$. Then, N is an H-kernel by walks of $S_H(D)$ if and only if N is an H-kernel by walks of $S'_H(D)$.*

Proof. Consider the following claims which will be very useful.

Claim 1. Let $\{u,v\}$ be a subset of $V(S_H(D))$, P a uv-H-walk in $S'_H(D)$, $z \in V(P) \cap (V(\beta_c)\setminus\{r\})$ for some $c = (r,s)$ in $A(D)$. Then, $\beta_c = (r,P,c)$.

We will prove that $V(\beta_c) \subseteq V(P)$. Let w be a vertex in $V(\beta_c)$. Proceeding by contradiction, suppose that $w \notin V(P)$. Assume that $\beta_c = (r = v_0, v_1, \ldots, v_\alpha = z, \ldots, v_\lambda = c)$. If $w \in V((z,\beta_c,c))$, then consider $i_0 = \min\{j : j \in \{\alpha, \ldots, \lambda\}$ and $v_j \notin V(P)\}$ (there exists i_0 because $w \notin V(P)$). It follows from the choice of v_{i_0} that $v_{i_0-1} \in V(P)$. Moreover, $v_{i_0-1} \in V(P)\setminus V(S_H(D))$. On the other hand, there exists h in $V(P)$ such that $(v_{i_0-1},h) \in A(P)$. So, $d^+_{S'_H(D)}(v_{i_0-1}) \geq 2$, a contradiction with Remark 4.2. Suppose that $w \in V((r,\beta_c,z))$. Consider $i_0 = \max\{j : j \in \{0,\ldots, \alpha\}$ and $v_j \notin V(P)\}$ (there exists i_0 because $w \notin V(P)$). Notice that $v_{i_0+1} \in V(P)\setminus V(S_H(D))$. Then, there exists h in $V(P)$ such that $(h, v_{i_0+1}) \in A(P)$. So, $d^-_{S'_H(D)}(v_{i_0+1}) \geq 2$, a contradiction with Remark 4.2.

Therefore, $V(\beta_c) \subseteq V(P)$. Thus, it follows from the definition of $S'_H(D)$ that $\beta_c = (r,P,c)$.

Claim 2. Let $\{u,v\}$ be a subset of $A(D)$. $u \xrightarrow[S_H(D)]{H} v$ if and only if $u \xrightarrow[S'_H(D)]{H} v$.

[Sufficiency] Let $P = (u = v_0, v_1, \ldots, v_n = v)$ be a uv-H-walk in $S_H(D)$, for some n in $\mathbb{N}$. Notice that, it follows from the definition of $S_H(D)$ that $v_r \in A(D)$ for each even positive integer r, with $r \in \{0\ldots, n\}$. Which implies that $n = 2m$ for some m in $\mathbb{N}$. Consider $\beta_{v_{2i}}$ for each i in $\{1,\ldots, m\}$. Thus, from the definition of $S'_H(D)$ we have that $(v_0,v_1) \cup \beta_{v_2} \cup (v_2,v_3) \cup \beta_{v_4} \cup \cdots \cup (v_{2m-2},v_{2m-1}) \cup \beta_{v_{2m}}$ is a uv-H-walk in $S'_H(D)$.

[Necessity] Let $P = (u = v_0, v_1, \ldots, v_n = v)$ be a uv-H-walk in $S'_H(D)$, for some n in $\mathbb{N}$ and $I = \{t : v_t \in A(D), 2 \leq t \leq n\}$. Suppose that $I = \{\alpha_1, \ldots, \alpha_m\}$ and $v_{\alpha_i} = (x_i,y_i)$ for every i in $\{1,\ldots, m\}$. It follows from Claim 1 that $\beta_{v_{\alpha_i}} = (x_i,P,v_{\alpha_i})$ for each i in $\{1,\ldots, m\}$. Moreover, it follows from the definition of $S_H(D)$ that $y_i \in V(P)$ for each i in $\{1,\ldots, m-1\}$. Therefore, $(v_0, P, x_1) \cup (x_1,v_{\alpha_1},y_1) \cup (y_1, P, x_2) \cup (x_2,v_{\alpha_2},y_2) \cup \cdots \cup (x_{m-1},v_{\alpha_{m-1}},y_{m-1}) \cup (y_{m-1}, P, x_m) \cup (x_m, v_n)$ is a uv-H-walk in $S_H(D)$ (by definition of $S_H(D)$).

Claim 3. Let $\{u,v\}$ be a subset of $V(S_H(D))$. $u \xrightarrow[S_H(D)]{H} v$ if and only if $u \xrightarrow[S'_H(D)]{H} v$.

[Sufficiency] Let $P = (u = v_0, v_1, \ldots, v_n = v)$ be a uv-H-walk in $S_H(D)$, for some n in $\mathbb{N}$.

Case 1. $\{u,v\} \subseteq A(D)$.

It follows from Claim 2 that $u \xrightarrow[S'_H(D)]{H} v$.

Case 2. $u \in A(D)$ and $v \in V(D)$.

If $n = 1$, then it follows from the definition of $S'_H(D)$ that P is also a uv-H-walk in $S'_H(D)$. Suppose that $n \geq 3$, which implies, in this case, that $v_{n-1} \in A(D)\backslash\{u\}$ and $v_{n-1} = (v_{n-2},v_n)$. So, it follows from Claim 2 that there exists a uv_{n-1}-H-walk in $S'_H(D)$, say $W = (u = u_0, u_1,\ldots, u_m = v_{n-1})$, for some m in $\mathbb{N}$ (because (u, P, v_{n-1}) is an H-walk in $S_H(D)$). On the other hand, it follows from Claim 1 that $\beta_{v_{n-1}} = (u_i = v_{n-2}, u_{i+1},\ldots, u_{m-1}, u_m = v_{n-1})$ for some i in $\{1,\ldots, m-1\}$. Therefore $W \cup (u_m = v_{n-1},v_n = v)$ is a uv-H-walk in $S'_H(D)$ (because $\beta_{v_{n-1}} \cup (u_m = v_{n-1},v_n = v)$ is an H-walk by definition of $S'_H(D)$).

Case 3. $\{u,v\} \subseteq V(D)$.

In this case we have that $n \geq 2$, which implies that $v_1 \in A(S_H(D))$. So, from Case 2 we have that there exists a v_1v-H-walk in $S'_H(D)$, say $W = (v_1 = w_0, w_1,\ldots, w_m = v)$, for some m in $\mathbb{N}$ (because (v_1, P, v) is an H-walk in $S_H(D)$). On the other hand, from the definition of $S'_H(D)$ we have that $(w_0,w_1) = (v_1,v_2)$, the color of the final arc of β_{v_1} and the color of the arc (v_1,v_2) form a walk in H. Therefore, $\beta_{v_1} \cup W$ is a uv-H-walk in $S'_H(D)$.

Case 4. $u \in V(D)$ and $v \in A(D)$.

If $n = 1$, then it follows from the definition of $S'_H(D)$ that β_v is a uv-H-walk in $S'_H(D)$. Suppose that $n \geq 3$. In this case, we have that $v_1 \in A(D)\backslash\{v\}$. Therefore, from Claim 2 we have that there exists a v_1v_n-H-walk in $S'_H(D)$, say $W = (v_1 = z_0, z_1,\ldots, z_m = v_n)$, for some m in $\mathbb{N}$ (because (v_1, P, v_n) is an H-walk in $S_H(D)$). On the other hand, we have from the definition of $S'_H(D)$ that $(z_0, z_1) = (v_1,v_2)$, the color of the final arc of β_{v_1} and the color of the arc (v_1,v_2) form a walk in H. Therefore, $\beta_{v_1} \cup W$ is a uv-H-walk in $S'_H(D)$.

[**Necessity**] Let $P = (u = v_0, v_1,\ldots, v_n = v)$ be a uv-H-walk in $S'_H(D)$, for some n in $\mathbb{N}$.

Case 1. $\{u,v\} \subseteq A(D)$.

It follows from Claim 2 that $u \xrightarrow[S_H(D)]{H} v$.

Case 2. $u \in A(D)$ and $v \in V(D)$.

If $n = 1$, then it follows from the definition of $S'_H(D)$ that P is also a uv-H-walk in $S_H(D)$. Suppose that $n \geq 2$, then it follows from the definition of $S'_H(D)$ that $v_{n-1} \in A(D)\backslash\{u\}$. So, it follows from Claim 2 that there exists a uv_{n-1}-H-walk in $S_H(D)$, say $W = (u = u_0, u_1,\ldots, u_m = v_{n-1})$, for some m in $\mathbb{N}$ (because (u, P, v_{n-1}) is an H-walk in $S'_H(D)$). On the other hand, from the definition of $S'_H(D)$ we have that $(v_{n-1},v_n) \in A(S_H(D))$, $v_{n-1} = (u_{m-1},v_n)$ and $(u_{m-1},u_m = v_{n-1}, v_n)$ forms an H-walk. Therefore, $W \cup (u_m = v_{n-1},v_n = v)$ is a uv-H-walk in $S_H(D)$.

Case 3. $u \in V(D)$ and $v \in A(D)$.

If $[V(P)\backslash\{v\}] \cap A(D) = \emptyset$, then it follows from the definition of $S'_H(D)$ that $(u,v) \in A(S_H(D))$. Suppose that $[V(P)\backslash\{v\}] \cap A(D) \neq \emptyset$. It follows from the definition of $S'_H(D)$ that $v_1 \in A(D)$ and it

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follows from Claim 2 that there exists a v_1v-H-walk in $S_H(D)$, say $W = (v_1 = u_0, u_1,\ldots, u_m = v)$, for some m in $\mathbb{N}$ (because (v_1, P, v) is an H-walk in $S'_H(D)$). On the other hand, from the definition of both $S_H(D)$ and $S'_H(D)$ we have that $v_1 = (u,u_1)$ and (u,v_1, u_1) forms an H-walk. Therefore, $(u,v_1) \cup W$ is a uv-H-walk in $S_H(D)$.

Case 4. $\{u,v\} \subseteq V(D)$.

In this case we have that $n \geq 2$, which implies that $v_{n-1} \in A(D)$ (by definition of $S'_H(D)$). So, from Case 3 we have that there exists a uv_{n-1}-H-walk in $S_H(D)$, say $W = (u = w_0, w_1,\ldots, w_m = v_{n-1})$, for some m in $\mathbb{N}$ (because (u, P, v_{n-1}) is an H-walk in $S'_H(D)$). On the other hand, from the definition of both $S_H(D)$ and $S'_H(D)$ we have that $v_{n-1} = (w_{m-1},v_n)$ and (w_{m-1},v_{n-1}, v_n) forms an H-walk. Therefore, $W \cup (v_{n-1},v_n)$ is a uv-H-walk in $S_H(D)$.

Therefore, for a subset N of $V(S_H(D))$, it follows from Claim 3 that N is an H-kernel by walks of $S_H(D)$ if and only if N is an H-kernel by walks of $S'_H(D)$. □

The following results are a direct consequence of Theorem 3.1, Theorem 3.2 and Proposition 4.3.

Theorem 4.4. *Let H be and D two digraphs, $S_H(D)$ a subdivision of D without infinite outward path and $S'_H(D)$ a generalization of $S_H(D)$. Suppose that $|V(S_H(D))| \geq 2k+3$ and $|\xi^+_{S_H(D)}(v)| \leq k$ for every v in $V(S_H(D))$ and for some positive integer $k \geq 1$. Then, $S'_H(D)$ has an H-kernel by walks.*

Proof. It follows from Theorem 3.1 that $S_H(D)$ has an H-kernel by walks, say N. Therefore, N is an H-kernel by walks of $S'_H(D)$ (by Proposition 4.3). □

Theorem 4.5. *Let H be a digraph, D a digraph without cycles, $S_H(D)$ a subdivision of D without infinite outward path and $S'_H(D)$ a generalization of $S_H(D)$. Suppose that $|V(S_H(D))| \geq 2k+3$ and $|\xi^+_{S_H(D)}(v)| \leq k$ for every v in $V(S_H(D))$ and for some positive integer $k \geq 1$. Then, $S'_H(D)$ has a unique H-kernel by walks.*

Proof. It follows from Theorem 3.1 that $S_H(D)$ has an unique H-kernel by walks, say N, which is an H-kernel by walks of $S'_H(D)$ (by Proposition 4.3). Therefore, it follows from Proposition 4.3 and the fact that N is unique in $S_H(D)$ that N is unique in $S'_H(D)$. □

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