GUTMAN INDEX, EDGE-WIENER INDEX AND EDGE-CONNECTIVITY

JAYA PERCIVAL MAZORODZE, SIMON MUKWEMBI AND Tomáš VETRÍK

Abstract. We study the Gutman index Gut(G) and the edge-Wiener index We(G) of connected graphs G of given order n and edge-connectivity λ. We show that the bound Gut(G) ≤ \frac{2^{\lambda^2} + 1}{2^{\lambda + 1}} n^5 + O(n^4) is asymptotically tight for λ ≥ 8. We improve this result considerably for λ ≤ 7 by presenting asymptotically tight upper bounds on Gut(G) and We(G) for 2 ≤ λ ≤ 7.

1. Introduction

Let G be a connected graph with vertex set V(G) and edge set E(G). The number of vertices adjacent to v is the degree deg(v) of a vertex v ∈ V(G). The number of edges in a shortest path between two vertices u, v ∈ V(G) is the distance d(u, v) between u and v. The distance between v and any vertex furthest from v is the eccentricity ec(v) of v in G. The distance between any two furthest vertices in G is the diameter of G. The i-th neighbourhood N_i(v) of v in G is the set of vertices at distance i from v. N_1(v) = N(v) is the neighbourhood of v and N[v] = N(v) ∪ {v}. The edge-connectivity of G is the smallest number of edges whose removal disconnects G. The number of vertices in a shortest path between two edges e, f ∈ E(G) is the distance d(e, f) between e and f.
The Gutman index and the edge-Wiener index have been studied because of their extensive applications. The edge-Wiener index

\[ W_e(G) = \sum_{\{e,f\} \subseteq E(G)} d(e, f) \]

of a connected graph \( G \) was introduced by Iranmanesh et al. \([11]\) and Khalifeh et al. \([12]\). Azari and Iranmanesh \([3]\) studied the edge-Wiener index for the sum of graphs, relations between the edge-Wiener index and other indices were investigated in \([5]\), \([14]\) and \([18]\).

The Gutman index

\[ \text{Gut}(G) = \sum_{\{u,v\} \subseteq V(G)} \deg(u)\deg(v)d(u, v) \]

of a connected graph \( G \) has been introduced in \([10]\). The Gutman index of acyclic structures was considered by Gutman \([10]\). Unicyclic graphs were investigated by Feng \([8]\), bicyclic graphs by Feng and Liu \([9]\) and graphs of given vertex-connectivity in \([16]\). Andova et. al. \([2]\) gave bounds on the Gutman index for graphs with maximal and graphs with minimal Gutman index. and lower bounds on this index were studied also by Chen \([4]\). Knor, Potočnik and Škrekovski \([13]\) studied relations between the Gutman index and the edge-Wiener index. Das, Su and Xiong \([7]\) investigated relations between the Gutman index and the degree distance. Bounds on the number of edges with respect to edge-connectivity and other invariants were given by Ali et al. \([1]\).

Mukwembi \([17]\) proved that for a connected graph \( G \) with \( n \) vertices,

\[ \text{Gut}(G) \leq \frac{2^4}{5^4} n^5 + O(n^4). \]

The Gutman index of connected graphs \( G \) of order \( n \) and minimum degree \( \delta \) was studied in \([15]\). It was proved that

\[ \text{Gut}(G) \leq \frac{2^4 \cdot 3}{5^5 (\delta + 1)} n^5 + O(n^4). \]

Since the edge-connectivity is an important graph invariant, in this paper we study the Gutman index of graphs of given order \( n \) and edge-connectivity \( \lambda \). For \( \lambda = 1 \), from \((1.1)\) we have \( \text{Gut}(G) \leq \frac{2^4}{5^5} n^5 + O(n^4) \) and the bound is asymptotically tight, since the extremal graph given in \([17]\) has edge-connectivity one.

For every graph \( G \), we have \( \lambda \leq \delta \), thus from \((1.2)\) we obtain the inequality

\[ \text{Gut}(G) \leq \frac{2^4 \cdot 3}{5^5 (\lambda + 1)} n^5 + O(n^4). \]

We show that this bound is asymptotically tight for \( \lambda \geq 8 \). The main challenge of this paper is to obtain asymptotically tight upper bounds on the Gutman index for graphs of given order and edge-connectivity \( \lambda \), where \( 2 \leq \lambda \leq 7 \). We prove that the bound \((1.3)\) can be improved considerably for \( 2 \leq \lambda \leq 7 \). We also obtain asymptotically tight upper bounds on the edge-Wiener index of graphs of given order and edge-connectivity \( \lambda \geq 2 \).

http://dx.doi.org/10.22108/toc.2020.124104.1749
2. Results

First, we consider the Gutman index of graphs with edge-connectivity at least 8.

**Theorem 2.1.** Let $G$ be any graph with $n$ vertices and edge-connectivity $\lambda$, where $\lambda \geq 8$ is a constant. Then

$$\text{Gut}(G) \leq \frac{2^4 \cdot 3}{5^5(\lambda + 1)} n^5 + O(n^4)$$

and the bound is asymptotically tight.

**Proof.** We obtain the bound from (1.2) by applying the inequality $\lambda \leq \delta$, thus We prove that the bound is asymptotically tight. Let us present the graph $G'$ having diameter $d = 3k + 1$ for $k \geq 1$. Let $G_0 = K_{[\frac{1}{3}[n+\lambda+1]]}$ and $G_{3k+1} = K_{[\frac{1}{3}[n-\lambda+1]]}$.

If $\lambda \equiv 2 \pmod{3}$, then $G_1 = G_2 = \cdots = G_{3k} = K_{\frac{\lambda + 1}{3}}$. If $\lambda \equiv 0 \pmod{3}$, then $G_{3i-1} = G_{3i-2} = G_{3i+1}$ and $G_{3i} = K_{\frac{\lambda + 1}{3}}$ for $i = 1, 2, \ldots, k$. If $\lambda \equiv 1 \pmod{3}$, then $G_{3i-2} = G_{3i-1} = K_{\frac{\lambda + 2}{3}}$ and $G_{3i} = K_{\frac{\lambda - 1}{3}}$ for $i = 1, 2, \ldots, k$.

The graph $G'$ consists of the graphs $G_0, G_1, G_2, \ldots, G_{3k+1}$, where each vertex of $G_i$ is adjacent to each vertex of $G_{i+1}$ for $i = 0, 1, 2, \ldots, 3k$. We have $|V(G_i)| + |V(G_{i+1})| + |V(G_{i+2})| = \lambda + 1$ for $i = 1, 2, \ldots, 3k-2$, so the degree of every vertex in $V(G_2) \cup V(G_3) \cup \cdots \cup V(G_{3k-1})$ is $\lambda$. Since $n$ is large, the degree of the other vertices is greater than $\lambda$. Note that $|V(G_i)||V(G_{i+1})| \geq \lambda$ for every $i = 0, 1, \ldots, 3k$, thus the edge-connectivity of $G'$ is $\lambda$.

We have $|V(G_1)| + |V(G_2)| + \cdots + |V(G_{3k})| = k(\lambda + 1)$. Since $|V(G_0)| + |V(G_{3k+1})| = n - k(\lambda + 1)$, we get $|V(G')| = n$. Let Gut$(u, v) = \text{deg}(u)\text{deg}(v)d(u, v)$, where $u, v \in V(G')$. Clearly,

$$\text{Gut}(G) = \sum_{\{u, v\} \subseteq V(G')} \text{Gut}(u, v).$$

For any $u \in V(G_0)$ and $v \in V(G_{3k+1})$, we obtain

$$\text{Gut}(u, v) = \left(\left\lfloor \frac{n - k(\lambda + 1)}{2} \right\rfloor + O(1)\right)\left(\left\lfloor \frac{n - k(\lambda + 1)}{2} \right\rfloor + O(1)\right)(3k + 1)$$

$$\quad = \frac{(n - k(\lambda + 1))^2(3k + 1)}{4} + O(nk).$$

Then for $k = \frac{n}{5(\lambda + 1)}$, we obtain

$$\sum_{u \in V(G_0), v \in V(G_{3k+1})} \text{Gut}(u, v) = \frac{(n - k(\lambda + 1))^4(3k + 1)}{16} + O(n^3k)$$

$$\quad = \frac{(\frac{n}{5})^4(\frac{3n}{5\lambda + 1} + 1)}{16} + O(n^3k)$$

$$\quad = \frac{2^4 \cdot 3}{5^5(\lambda + 1)} n^5 + O(n^4).$$

http://dx.doi.org/10.22108/toc.2020.124104.1749
For any \( u \in V(G_1) \cup V(G_{3k}) \) and \( v \in H = V(G_0) \cup V(G_1) \cup V(G_{3k}) \cup V(G_{3k+1}) \), we obtain \( \text{Gut}(u, v) = O(n^2k) \) since \( \deg(u) = O(n) \). Consequently, \( \sum_{u \in V(G_1) \cup V(G_{3k})}, v \in H \) \( \text{Gut}(u, v) = O(n^2k) \) since \( |V(G_1) \cup V(G_{3k})| = O(1) \) and \( |H| = O(n) \).

If \( \{u, v\} \subseteq V(G_0) \) or \( \{u, v\} \subseteq V(G_{3k+1}) \), then \( \text{Gut}(u, v) = O(n^2) \) since \( d(u, v) = 1 \). Thus

\[
\sum_{\{u, v\} \subseteq V(G_0)} \text{Gut}(u, v) = O(n^4) \quad \text{and} \quad \sum_{\{u, v\} \subseteq V(G_{3k+1})} \text{Gut}(u, v) = O(n^4).
\]

Finally, for \( u \in H' = V(G_2) \cup V(G_3) \cup \cdots \cup V(G_{3k-1}) \) and \( v \in V(G') \), we have \( \text{Gut}(u, v) = O(nk) \) since \( \deg(u) = O(1) \). Consequently, \( \sum_{u \in H', v \in V(G')} \text{Gut}(u, v) = O(n^2k^2) \) since \( |H'| = O(k) \) and \( V(G') = O(n) \). Hence \( \text{Gut}(G') = \frac{2^{d-3}}{3^{d-1}}n^5 + O(n^4) \).

\[\square\]

In Lemma 2.2 we study the degrees of vertices in graphs having small edge-connectivity.

**Lemma 2.2.** Let \( G \) be any graph with \( n \) vertices, edge-connectivity \( \lambda \) and diameter \( d \). Let \( v, v' \in V(G) \) such that \( d(v, v') \geq 3 \).

If \( \lambda = 2 \), then

\[
\deg(v) \leq n - \frac{3}{2}d + O(1) \quad \text{and} \quad \deg(v) + \deg(v') \leq n - \frac{3}{2}d + O(1).
\]

If \( \lambda = 3 \) or \( 4 \), then

\[
\deg(v) \leq n - 2d + O(1) \quad \text{and} \quad \deg(v) + \deg(v') \leq n - 2d + O(1).
\]

If \( \lambda = 5 \) or \( 6 \), then

\[
\deg(v) \leq n - \frac{5}{2}d + O(1) \quad \text{and} \quad \deg(v) + \deg(v') \leq n - \frac{5}{2}d + O(1).
\]

If \( \lambda = 7 \), then

\[
\deg(v) \leq n - 3d + O(1) \quad \text{and} \quad \deg(v) + \deg(v') \leq n - 3d + O(1).
\]

**Proof.** Let \( G \) be any graph having order \( n \), edge-connectivity \( \lambda \) and diameter \( d \). Let \( v_0 \) be a vertex of \( G \) having eccentricity \( d \). The \( i \)-th neighbourhood of \( v_0 \) is denoted by \( N_i \), where \( i = 0, 1, 2, \ldots, d \). Let \( v, v' \in V(G) \). Then \( v \in N_i \) and \( v' \in N_l \), where \( i, l \in \{0, 1, 2, \ldots, d\} \). Note that \( N(v) \subseteq N_{i-1} \cup N_{i} \cup N_{i+1} \) and \( N(v) \subseteq N_{i-1} \cup N_{i} \cup N_{i+1} \). Thus \( \deg(v) \leq |N_{i-1}| + |N_{i}| + |N_{i+1}| - 1 \) and \( \deg(v') \leq |N_{i-1}| + |N_{i}| + |N_{i+1}| - 1 \). Since \( d(v, v') \geq 3 \), we have \( N(v) \cap N(v') = \emptyset \). The edge-connectivity of \( G \) is \( \lambda \), thus \( |N_j||N_{j+1}| \geq \lambda \) for every \( j = 0, 1, 2, \ldots, d - 1 \).

If \( \lambda = 2 \), then \( |N_j||N_{j+1}| \geq 2 \). Thus \( |N_j| + |N_{j+1}| \geq 3 \). It follows that \( \sum_{j=0}^{i-2}|N_j| + \sum_{j=i+2}^{d}|N_j| \geq \frac{3}{2}(d - 2) - 1 \), and consequently

\[
n = \sum_{j=0}^{d}|N_j| \geq \deg(v) + \frac{3}{2}(d - 2) = \deg(v) + \frac{3}{2}d - O(1).
\]
Hence $\deg(v) \leq n - \frac{3}{2}d + O(1)$. Note that the inequalities hold also if $v \in N_i$ where $i \in \{0, 1, d - 1, d\}$. Similarly,

$$n \geq (\deg(v) + 1) + (\deg(v') + 1) + \frac{3}{2}(d - 5) - O(1)$$

$$= \deg(v) + \deg(v') + \frac{3}{2}d - O(1).$$

Thus $\deg(v) + \deg(v') \leq n - \frac{3}{2}d + O(1)$.

If $\lambda = 3$, then $|N_j||N_{j+1}| \geq 3$, and if $\lambda = 4$, then $|N_j||N_{j+1}| \geq 4$ for every $j = 0, 1, 2, \ldots, d - 1$. In both cases, it follows that $|N_j| + |N_{j+1}| \geq 4$. Then $\sum_{j=0}^{d-2} |N_j| + \sum_{j=i+2}^{d} |N_j| \geq 2d - O(1)$. We get

$$n = \sum_{j=0}^{d} |N_j| \geq \deg(v) + 2d - O(1), \quad \text{thus} \quad \deg(v) \leq n - 2d + O(1).$$

Similarly, $n \geq \deg(v) + \deg(v') + 2d - O(1)$, thus $\deg(v) + \deg(v') \leq n - \frac{3}{2}d + O(1)$.

If $\lambda = 5$, then $|N_j||N_{j+1}| \geq 5$, and if $\lambda = 6$, then $|N_j||N_{j+1}| \geq 6$ for every $j = 0, 1, 2, \ldots, d - 1$. In both cases, it follows that $|N_j| + |N_{j+1}| \geq 6$. Thus $\sum_{j=0}^{d-2} |N_j| + \sum_{j=i+2}^{d} |N_j| \geq \frac{5}{2}d - O(1)$ and $n \geq \deg(v) + \frac{5}{2}d - O(1)$. Similarly, $n \geq \deg(v) + \deg(v') + \frac{5}{2}d - O(1)$.

If $\lambda = 7$, then $|N_j||N_{j+1}| \geq 7$. Therefore $|N_j| + |N_{j+1}| \geq 7$ for any $j = 0, 1, 2, \ldots, d - 1$. Thus $\sum_{j=0}^{d-2} |N_j| + \sum_{j=i+2}^{d} |N_j| \geq 3d - O(1)$ and $n \geq \deg(v) + 3d - O(1)$. Similarly, $n \geq \deg(v) + \deg(v') + 3d - O(1)$. \hfill \Box

In the following theorem, we obtain an upper bound on the Gutman index for graphs $G$ of given order, edge-connectivity 2 and diameter.

**Theorem 2.3.** Let $G$ be any graph with $n$ vertices, edge-connectivity 2 and diameter $d$. Then

$$\text{Gut}(G) \leq \frac{d}{16} \left(n - \frac{3d}{2}\right)^4 + O(n^4),$$

and the bound is asymptotically tight.

**Proof.** We denote by $v_0 \in V(G)$ any vertex having eccentricity $d$. The $i$-th neighbourhood of $v_0$ is denoted by $N_i$, $i = 0, 1, 2, \ldots, d$. The edge-connectivity is 2, thus $|N_i||N_{i+1}| \geq 2$ for every $i = 0, 1, 2, \ldots, d - 1$. Therefore $|N_i| + |N_{i+1}| \geq 3$. For $i = 1, 2, \ldots, \lceil \frac{d}{2} \rceil$, each set $N_{2i-2} \cup N_{2i-1}$ contains at least three vertices $v_{i1}, v_{i2}, v_{i3}$. We define $P_i = \{v_{i1}, v_{i2}, v_{i3}\}$ and $P = \bigcup_{i=1}^{\lceil \frac{d}{2} \rceil} P_i$. So

$$|P| = 3 \left\lfloor \frac{d}{2} \right\rfloor.$$

The set $P$ has no vertices from $N_d$ if $d$ is even.

Let us partition the set $Z = \{ \{u,v\} : u,v \in V(G) \}$ into three sets $A$, $B$ and $C$, where

$$C = \{ \{u,v\} : u \in P \text{ and } v \in V(G) \},$$

$$A = \{ \{u,v\} \in Z - C : d(u,v) \geq 3 \},$$

$$B = \{ \{u,v\} \in Z - C : d(u,v) \leq 2 \}.$$
So \( Z = C \cup A \cup B \). Let \(|A| = a\) and \(|B| = b\). Then \( \binom{n}{2} = |C| + a + b \) and from \((2.1)\) we obtain
\[
(2.2) \quad a + b = \left( n - \frac{|P|}{2} \right) = \frac{1}{2} \left( n - 3 \left\lfloor \frac{d}{2} \right\rfloor \right) \left( n - 3 \left\lfloor \frac{d}{2} \right\rfloor - 1 \right).
\]

We have
\[
\text{Gut}(G) = \sum_{\{u,v\} \in A} \text{deg}(u)\text{deg}(v)d(u,v) + \sum_{\{u,v\} \in B} \text{deg}(u)\text{deg}(v)d(u,v) + \sum_{\{u,v\} \in C} \text{deg}(u)\text{deg}(v)d(u,v).
\]

Let us bound these three terms in the following claims.

**Claim 1.** \( \sum_{\{u,v\} \in C} \text{deg}(u)\text{deg}(v)d(u,v) \leq O(n^4) \).

Let \( P = V_1 \cup V_2 \cup \cdots \cup V_6 \) where
\[
V_1 = \{v_{11}, v_{31}, v_{51}, \ldots\}, \quad V_2 = \{v_{12}, v_{32}, v_{52}, \ldots\}, \quad V_3 = \{v_{13}, v_{33}, v_{53}, \ldots\},
\]
\[
V_4 = \{v_{21}, v_{41}, v_{61}, \ldots\}, \quad V_5 = \{v_{22}, v_{42}, v_{62}, \ldots\}, \quad V_6 = \{v_{23}, v_{43}, v_{63}, \ldots\}.
\]

Since \( d(u,v) \geq 3 \), we get \( N(u) \cap N(v) = \emptyset \) for each pair of distinct vertices \( u, v \) in the same set \( V_i \), where \( i = 1, 2, \ldots, 6 \). Therefore \( \sum_{u \in V_i} \text{deg}(u) < n \).

Let us define the score \( s(u) \) for each vertex \( u \in P \) as
\[
(2.3) \quad s(u) = \sum_{v \in V(G)} \text{deg}(u)\text{deg}(v)d(u,v) = \text{deg}(u) \sum_{v \in V(G)} \text{deg}(v)d(u,v).
\]

Then by Lemma 2.2,
\[
s(u) \leq \text{deg}(u) \sum_{v \in V(G)} \left( n - \frac{3}{2} d + O(1) \right) d(u,v)
\]
\[
= \text{deg}(u) \left( n - \frac{3}{2} d + O(1) \right) \sum_{v \in V(G)} d(u,v)
\]
\[
< \text{deg}(u) \left( n - \frac{3}{2} d + O(1) \right) nd
\]
and
\[
\sum_{u \in P} s(u) = \sum_{u \in V_1} s(u) + \sum_{u \in V_2} s(u) + \cdots + \sum_{u \in V_6} s(u)
\]
\[
< \sum_{u \in V_1} \text{deg}(u) \left( n - \frac{3}{2} d + O(1) \right) nd + \cdots + \sum_{u \in V_5} \text{deg}(u) \left( n - \frac{3}{2} d + O(1) \right) nd
\]
\[
= \left( \sum_{u \in V_1} \text{deg}(u) + \sum_{u \in V_2} \text{deg}(u) + \cdots + \sum_{u \in V_6} \text{deg}(u) \right) \left( n - \frac{3}{2} d + O(1) \right) nd
\]
\[
< 6n \left( n - \frac{3}{2} d + O(1) \right) nd \leq O(n^4).
\]

Since \( \sum_{\{u,v\} \in C} \text{deg}(u)\text{deg}(v)d(u,v) \leq \sum_{u \in P} s(u) \), the proof of Claim 1 is complete.
Claim 2. \( \sum_{\{u,v\} \in B} \deg(u)\deg(v)d(u,v) \leq O(n^4). \)

If \( \{u,v\} \in B \), then \( d(u,v) \leq 2 \) and \( b = O(n^2) \). Using these facts and Lemma 2.2, we get

\[
\sum_{\{u,v\} \in B} \deg(u)\deg(v)d(u,v) \leq \sum_{\{u,v\} \in B} 2\left(n - \frac{3}{2}d + O(1)\right)^2 = 2b\left(n - \frac{3}{2}d + O(1)\right)^2 \leq O(n^4).
\]

Claim 3. \( \sum_{\{u,v\} \in A} \deg(u)\deg(v)d(u,v) \leq \frac{d}{16}\left(n - \frac{3d}{2}\right)^4 + O(n^4). \)

Let \( \{u',v'\} \) be any pair in \( A \), where \( \deg(u') + \deg(v') \) is maximum. Let \( \deg(u') + \deg(v') = s \). We have \( \deg(u')\deg(v') \leq \frac{1}{4}(\deg(u') + \deg(v'))^2 \), thus

\[
\text{(2.4)} \quad \deg(u')\deg(v') \leq \frac{1}{4}s^2.
\]

By (2.2), we get

\[
\text{(2.5)} \quad a = \frac{1}{2}\left(n - 3\left\lceil \frac{d}{2} \right\rceil\right)\left(n - 3\left\lceil \frac{d}{2} \right\rceil - 1\right) - b.
\]

Clearly, all pairs \( \{u,v\}, u,v \in N[u'] - P \) and all \( \{u,v\}, u,v \in N[v'] - P \) are in \( B \). Since \( u' \in N_i \) for some \( i = 0, 1, \ldots, d \), we have \( N[u'] \subseteq N_i - N_i \cup N_i \cup N_{i+1} \). Since \( |N[u'] \cap P| \leq 6 \) and \( |N[v'] \cap P| \leq 6 \), we obtain

\[
b \geq \left(\frac{\deg(u') + 1}{2}\right) + \left(\frac{\deg(v') + 1}{2}\right)
= \frac{1}{2}\left((\deg(u'))^2 + (\deg(v'))^2\right) - 11(\deg(u') + \deg(v')) + 30
\geq \frac{1}{4}s^2 - 30.
\]

Then by (2.5),

\[
a \leq \frac{1}{2}\left(n - 3\left\lceil \frac{d}{2} \right\rceil\right)\left(n - 3\left\lceil \frac{d}{2} \right\rceil - 1\right) - \frac{1}{4}s^2 + 11s - 30,
\]

and by (2.4), we get

\[
\sum_{\{u,v\} \in A} \deg(u)\deg(v)d(u,v) \leq \sum_{\{u,v\} \in A} \frac{s^2d}{4} = \frac{s^2da}{4}
\leq \frac{s^2d}{4}\left(\frac{1}{2}\left(n - 3\left\lceil \frac{d}{2} \right\rceil\right)\left(n - 3\left\lceil \frac{d}{2} \right\rceil - 1\right) - \frac{1}{4}s^2 + 11s - 30\right)
= \frac{s^2d}{4}\left(\left(n - \frac{3d}{2}\right)^2 + O(n)\right) - \frac{1}{4}s^2 + O(n)
= \frac{s^2d}{4}\left(\left(n - \frac{3d}{2}\right)^2 - \frac{1}{4}s^2\right) + O(n^4)
\]

http://dx.doi.org/10.22108/toc.2020.124104.1749
From Lemma 2.2, we have $s \leq n - \frac{3d}{2} + O(1)$. Subject to this condition, $\frac{s^2d}{4} \left[ \frac{1}{2} (n - \frac{3d}{2})^2 - \frac{1}{4}s^2 \right]$ is maximized for $s = n - \frac{3d}{2} + O(1)$ and we obtain

$$
\sum_{\{u,v\} \in A} \deg(u)\deg(v)d(u,v) \leq \frac{d}{4} \left[ \left( n - \frac{3d}{2} \right)^2 + O(n) \right] \left[ \frac{1}{2} \left( n - \frac{3d}{2} \right)^2 - \frac{1}{4} \left( n - \frac{3d}{2} \right)^2 + O(n) \right] + O(n^4)
$$

$$
= \frac{d}{16} \left( n - \frac{3d}{2} \right)^4 + O(n^4),
$$

as claimed. By Claims 1, 2 and 3, we have

$$
\text{Gut}(G) = \sum_{\{u,v\} \in A} \deg(u)\deg(v)d(u,v) + \sum_{\{u,v\} \in B} \deg(u)\deg(v)d(u,v) + \sum_{\{u,v\} \in C} \deg(u)\deg(v)d(u,v)
$$

$$
\leq \frac{d}{16} \left( n - \frac{3d}{2} \right)^4 + O(n^4).
$$

We construct the graph $G_{n,d,2}$ to prove that our bound is asymptotically tight. For $1 \leq i \leq d - 1$, let $G_i = K_1$ if $i$ is odd, and $G_i = K_2$ if $i$ is even. $G_0 = K_{\left\lceil \frac{1}{2}(n - \left\lfloor \frac{1}{2}(d-1) \right\rfloor) \right\rceil}$ and $G_d = K_{\left\lfloor \frac{1}{2}(n - \left\lceil \frac{1}{2}(d-1) \right\rceil) \right\rfloor}$. The graph $G_{n,d,2}$ consists of the graphs $G_0, G_1, G_2, \ldots, G_d$, where every vertex of $G_{i-1}$ is adjacent to every vertex of $G_i$ for $i = 1, 2, \ldots, d$.

Since $|V(G_1) \cup V(G_2) \cup \cdots \cup V(G_{d-1})| = \left\lfloor \frac{3}{2}(d-1) \right\rfloor$ and $|V(K_{\left\lfloor \frac{1}{2}(n - \left\lfloor \frac{1}{2}(d-1) \right\rfloor) \right\rceil})| = n - \left\lceil \frac{3}{2}(d-1) \right\rceil$, the graph $G_{n,d,2}$ has $n$ vertices. The diameter of $G_{n,d,2}$ is $d$, the edge-connectivity is 2 and $\text{Gut}(G_{n,d,2}) = \frac{d}{16} \left( n - \frac{3d}{2} \right)^4 + O(n^4)$.

Now we give upper bounds on the Gutman index for graphs with $n$ vertices, diameter $d$ and edge-connectivity $\lambda$ for $3 \leq \lambda \leq 7$.

**Theorem 2.4.** Let $G$ be any graph with $n$ vertices, edge-connectivity $\lambda$ and diameter $d$.

(a) If $\lambda = 3$ or 4, then $\text{Gut}(G) \leq \frac{d}{16} (n - 2d)^4 + O(n^4)$.

(b) If $\lambda = 5$ or 6, then $\text{Gut}(G) \leq \frac{d}{16} (n - \frac{5d}{2})^4 + O(n^4)$.

(c) If $\lambda = 7$, then $\text{Gut}(G) \leq \frac{d}{16} (n - 3d)^4 + O(n^4)$.

The bounds are asymptotically tight.

Theorems 2.3 and 2.4 have similar proofs. We include only the main differences between the proofs of Theorem 2.3 and Theorem 2.4, part (b).

If $\lambda = 5$, then $|N_{i-1}||N_i| \geq 5$, and if $\lambda = 6$, then $|N_{i-1}||N_i| \geq 6$ for each $i = 1, 2, \ldots, d$. In both cases, $|N_{i-1}| + |N_i| \geq 6$. Let $v_1, v_2, v_3, v_4, v_5, v_6$ be any six vertices in $N_{2i-2} \cup N_{2i-1}$ and let $P_i = \{v_1, v_2, v_3, v_4, v_5, v_6\}$, where $i = 1, 2, \ldots, \left\lceil \frac{d}{2} \right\rceil$. We define $P = \bigcup_{i=1}^{\left\lceil \frac{d}{2} \right\rceil} P_i$. We have $|P| = 6 \left\lceil \frac{d}{2} \right\rceil$. 

http://dx.doi.org/10.22108/toc.2020.124104.1749
Let us partition $P$ into 12 sets

\[
V_1 = \{v_{11}, v_{31}, v_{51}, \ldots\}, \quad V_2 = \{v_{12}, v_{32}, v_{52}, \ldots\}, \\
V_3 = \{v_{13}, v_{33}, v_{53}, \ldots\}, \quad V_4 = \{v_{14}, v_{34}, v_{54}, \ldots\}, \\
V_5 = \{v_{15}, v_{35}, v_{55}, \ldots\}, \quad V_6 = \{v_{16}, v_{36}, v_{56}, \ldots\}, \\
V_7 = \{v_{21}, v_{41}, v_{61}, \ldots\}, \quad V_8 = \{v_{22}, v_{42}, v_{62}, \ldots\}, \\
V_9 = \{v_{23}, v_{43}, v_{63}, \ldots\}, \quad V_{10} = \{v_{24}, v_{44}, v_{64}, \ldots\}, \\
V_{11} = \{v_{25}, v_{45}, v_{65}, \ldots\}, \quad V_{12} = \{v_{26}, v_{46}, v_{66}, \ldots\}.
\]

So $P = V_1 \cup V_2 \cup \cdots \cup V_{12}$. The rest of the proof of part (b) can be easily obtained by following the proof of Theorem 2.3 and using Lemma 2.2.

Let us present the graphs $G_{n,d,\lambda}$ for $\lambda = 3, 4, 5, 6, 7$, to prove that the bounds presented in Theorem 2.4 are asymptotically tight. Let $G_{n,d,\lambda}$ be the graph which consists of the graphs $G_0, G_1, G_2, \ldots, G_d$ defined below, where each vertex of $G_{j-1}$ is adjacent to each vertex of $G_j$, $j = 1, 2, \ldots, d$.

For $\lambda = 3$, let $G_1 = K_1$, $G_2 = K_3$, $G_j = K_2$ for $j = 3, 4, \ldots, d - 1$, $G_0 = K_{\left\lceil \frac{1}{2} (n-2(d-1)) \right\rceil}$ and $G_d = K_{\left\lceil \frac{1}{2} (n-2(d-1)) \right\rceil}$.

For $\lambda = 4$, let $G_j = K_2$ if $j$ is odd, and $G_j = K_3$ if $j$ is even, where $3 \leq j \leq d - 1$. Let $G_1 = K_1, G_2 = K_5, G_0 = K_{\left\lceil \frac{1}{2} (n-3(d-1))-1 \right\rceil}$ and $G_d = K_{\left\lceil \frac{1}{2} (n-3(d-1))-1 \right\rceil}$.

For $\lambda = 5$, let $G_j = K_2$ if $j$ is odd, and $G_j = K_3$ if $j$ is even, where $1 \leq j \leq d - 1$. Let $G_0 = K_{\left\lceil \frac{1}{2} (n-3(d-1)) \right\rceil}$ and $G_d = K_{\left\lceil \frac{1}{2} (n-3(d-1)) \right\rceil}$.

For $\lambda = 6$, let $G_j = K_2$ if $j$ is odd, and $G_j = K_3$ where $3, 4, \ldots, d - 1$, $G_0 = K_{\left\lceil \frac{1}{2} (n-3d+1) \right\rceil}$ and $G_d = K_{\left\lceil \frac{1}{2} (n-3d+1) \right\rceil}$.

The graphs $G_{n,d,\lambda}$ have $n$ vertices, edge-connectivity $\lambda$, diameter $d$ and $\text{Gut}(G_{n,d,\lambda})$ is equal to the bound given in Theorem 2.4.

We use Theorems 2.3 and 2.4 to bound the Gutman index only in terms of order and edge-connectivity 2.

**Theorem 2.5.** Let $G$ be any graph with $n$ vertices and edge-connectivity $\lambda$.

(a) If $\lambda = 2$, then $\text{Gut}(G) \leq \frac{2^5}{35^5} n^5 + O(n^4)$.

(b) If $\lambda = 3$ or 4, then $\text{Gut}(G) \leq \frac{2^4}{35^5} n^5 + O(n^4)$.

(c) If $\lambda = 5$ or 6, then $\text{Gut}(G) \leq \frac{2^3}{35^5} n^5 + O(n^4)$.

(d) If $\lambda = 7$, then $\text{Gut}(G) \leq \frac{2^4}{35^5} n^5 + O(n^4)$.

The bounds are asymptotically tight.

http://dx.doi.org/10.22108/toc.2020.124104.1749
Proof. (a) By Theorem 2.3, Gut($G$) $\leq \frac{d}{16} (n - \frac{3d}{2})^4 + O(n^4)$ for graphs $G$ with $n$ vertices, diameter $d$ and edge-connectivity 2. With respect to $d$,

$$\frac{d}{16} (n - \frac{3d}{2})^4$$

is maximized for $d = \frac{2n}{15}$, therefore Gut($G$) $\leq \frac{2^5}{3 \cdot 5^3} n^5 + O(n^4)$ for graphs $G$ of order $n$ and edge-connectivity 2.

The bound is asymptotically tight since the Gutman index of the graph $G_{n, \frac{2n}{15}, 2}$ is $\frac{2^5}{3 \cdot 5^3} n^5 + O(n^4)$ for an integer $d = \frac{2n}{15}$, where $G_{n,d,2}$ is the graph introduced in the proof of Theorem 2.3.

(b) Let $\lambda = 3$ or 4. Then by Theorem 2.4, we get Gut($G$) $\leq \frac{d}{16} (n - 2d)^4 + O(n^4)$ for graphs $G$ with $n$ vertices and diameter $d$. Since $\frac{d}{16} (n - 2d)^4$ is maximized for $d = \frac{n}{16}$, we obtain Gut($G$) $\leq \frac{2^3}{3 \cdot 5^3} n^5 + O(n^4)$ for graphs $G$ of order $n$.

Let $\frac{n}{16}$ be an integer. Then the graphs $G_{n, \frac{n}{16}, 3}$ described above for $\lambda = 3$ and 4 have the Gutman index $\frac{2^3}{3 \cdot 5^3} n^5 + O(n^4)$.

(c) Let $\lambda = 5$ or 6. Then $\frac{d}{16} (n - \frac{5}{2}d)^4$ is maximized for $d = \frac{2n}{25}$ and we obtain the bound Gut($G$) $\leq \frac{2^3}{5} n^5 + O(n^4)$.

If $\frac{2n}{25}$ is an integer, then the graphs $G_{n, \frac{2n}{25}, 3}$ described above for $\lambda = 5$ and 6 have the Gutman index $\frac{4}{5} n^5 + O(n^4)$.

(d) Let $\lambda = 7$. Then $\frac{d}{16} (n - 3d)^4$ is maximized for $d = \frac{n}{15}$ and we obtain the bound Gut($G$) $\leq \frac{2^4}{3 \cdot 5^3} n^5 + O(n^4)$.

If $\frac{n}{15}$ is an integer, then the graph $G_{n, \frac{n}{15}, 7}$ described above for $\lambda = 7$ has the Gutman index $\frac{2^4}{3 \cdot 5^3} n^5 + O(n^4)$.

Dankelmann et. al. [6] proved the following relation between the Gutman index and the edge-Wiener index.

**Lemma 2.6.** Let $G$ be any connected graph with $n$ vertices. Then

$$\left| W_e(G) - \frac{1}{4} \text{Gut}(G) \right| \leq \frac{n^4}{8}.$$

We use Lemma 2.6 to get bounds on the edge-Wiener index of graphs of given order and and edge-connectivity.

**Corollary 2.7.** Let $G$ be a graph of order $n$ and edge-connectivity $\lambda$.

(a) If $\lambda = 2$, then $W_e(G) \leq \frac{2^3}{3 \cdot 5^3} n^5 + O(n^4)$.

(b) If $\lambda = 3$ or 4, then $W_e(G) \leq \frac{2^3}{5^3} n^5 + O(n^4)$.

(c) If $\lambda = 5$ or 6, then $W_e(G) \leq \frac{2^3}{5^3} n^5 + O(n^4)$.

(d) If $\lambda = 7$, then $W_e(G) \leq \frac{2^4}{3 \cdot 5^3} n^5 + O(n^4)$.

(e) If $\lambda \geq 8$ is a constant, then $W_e(G) \leq \frac{2^2 \cdot 3}{5^3(\lambda+1)} n^5 + O(n^4)$.

The bounds are asymptotically tight.
Proof. From Theorem 2.5 and Lemma 2.6 we obtain the results (a), (b), (c) and (d). From Theorem 2.1 and Lemma 2.6 we get $W_e(G) \leq \frac{2^2 - 3}{5^5(\lambda + 1)} n^5 + O(n^4)$. The graphs which have the largest Gutman index in terms of order and edge-connectivity $\lambda \geq 2$ achieve also the bounds given in this corollary, thus the bounds on $W_e(G)$ are asymptotically tight. \hfill \Box

Acknowledgments
The work of T. Vetrík is based on the research supported by the National Research Foundation of South Africa (Grant Number 126894).

References


http://dx.doi.org/10.22108/toc.2020.124104.1749


Jaya Percival Mazorodze
Department of Mathematics, University of Zimbabwe, P. O. Box MP 167, Mount Pleasant, Harare, Zimbabwe
Email: mazorodzejaya@gmail.com

Simon Mukwembi
Department of Mathematics, University of Zimbabwe, P. O. Box MP 167, Mount Pleasant, Harare, Zimbabwe and
School of Mathematics, University of the Witwatersrand, Private Bag 3, Wits 2050, South Africa Email: SimonMukwembi@gmail.com

Tomáš Vetřík
Department of Mathematics and Applied Mathematics, University of the Free State, P. O. Box 339, Bloemfontein, 9300, South Africa
Email: vetrikt@ufs.ac.za

http://dx.doi.org/10.22108/toc.2020.124104.1749