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**Transactions on Combinatorics**

ISSN (print): 2251-8657, ISSN (on-line): 2251-8665

Vol. 2 No. 1 (2013), pp. 27-37.

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## NEW SKEW LAPLACIAN ENERGY OF SIMPLE DIGRAPHS

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Communicated by Ivan Gutman

**ABSTRACT.** For a simple digraph  $G$  of order  $n$  with vertex set  $\{v_1, v_2, \dots, v_n\}$ , let  $d_i^+$  and  $d_i^-$  denote the out-degree and in-degree of a vertex  $v_i$  in  $G$ , respectively. Let  $D^+(G) = \text{diag}(d_1^+, d_2^+, \dots, d_n^+)$  and  $D^-(G) = \text{diag}(d_1^-, d_2^-, \dots, d_n^-)$ . In this paper we introduce  $\widetilde{SL}(G) = \widetilde{D}(G) - S(G)$  to be a new kind of skew Laplacian matrix of  $G$ , where  $\widetilde{D}(G) = D^+(G) - D^-(G)$  and  $S(G)$  is the skew-adjacency matrix of  $G$ , and from which we define the skew Laplacian energy  $SLE(G)$  of  $G$  as the sum of the norms of all the eigenvalues of  $\widetilde{SL}(G)$ . Some lower and upper bounds of the new skew Laplacian energy are derived and the digraphs attaining these bounds are also determined.

### 1. Introduction

In chemistry, there is a close relation between the molecular orbital energy levels of  $\pi$ -electrons in conjugated hydrocarbons and the eigenvalues of the corresponding molecular graphs. On these grounds, in 1970s, Gutman [5] introduced the concept of the energy for a simple undirected graph  $G$ :

$$E(G) = \sum_{i=1}^n |\lambda_i|,$$

where  $\lambda_1, \lambda_2, \dots, \lambda_n$  are the eigenvalues of the adjacency matrix  $A(G)$  of  $G$ . Recall that  $A(G) = [a_{ij}]$  is the  $n \times n$  matrix, where  $a_{ij} = 1$  if  $v_i$  and  $v_j$  are adjacent, and  $a_{ij} = 0$  otherwise. Due to its applications in chemistry, this concept has attracted much attention and a series of related papers have been published. We refer to [7, 10] for details.

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MSC(2010): Primary: 05C50, 05C31, 05C20, 05C90.

Keywords: energy; Laplacian energy; skew energy; skew Laplacian energy; eigenvalues; simple digraph.

Received: 24 April 2013, Accepted: 30 April 2013.

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In spectral graph theory [4], the eigenvalues of several other matrices have been studied, of which the Laplacian matrix plays an important role. Therefore, based on the definition of graph energy, Gutman and Zhou [6] defined the Laplacian energy for a simple undirected graph  $G$  possessing  $n$  vertices and  $m$  edges, which is given as follows:

$$LE_g(G) = \sum_{i=1}^n \left| \mu_i - \frac{2m}{n} \right|,$$

where  $\mu_1, \mu_2, \dots, \mu_n$  are the eigenvalues of the Laplacian matrix  $L(G) = D(G) - A(G)$  of  $G$ . Recall that  $D(G) = \text{diag}(d_1, d_2, \dots, d_n)$  is the diagonal matrix of the degrees of vertices. Some bounds were derived from the definition in [6].

**Theorem 1.1.** [6] *Let  $G$  be a simple undirected graph possessing  $n$  vertices,  $m$  edges and  $p$  components. Assume that  $d_1, d_2, \dots, d_n$  is the degree sequence of  $G$ . Then*

- (i)  $2\sqrt{M} \leq LE_g(G) \leq \sqrt{2Mn}$  ;
- (ii)  $LE_g(G) \leq \frac{2m}{n}p + \sqrt{(n-p)[2M - p(\frac{2m}{n})^2]}$ ;
- (iii) *If  $G$  has no isolated vertices, then  $LE_g(G) \leq 2M$ ,*

where  $M = m + \frac{1}{2} \sum_{i=1}^n (d_i - \frac{2m}{n})^2$ .

Moreover, Kragujevac [9] considered another definition for the Laplacian energy using the second spectral moment, namely  $LE_k(G) = \sum_{i=1}^n \mu_i^2$ . And the author proved the following result.

**Theorem 1.2.** [9]

- (i) *For any undirected graph  $G$  on  $n$  vertices whose degrees are  $d_1, d_2, \dots, d_n$ ,  $LE_k(G) = \sum_{i=1}^n d_i(d_i + 1)$ ;*
- (ii) *For any connected undirected graph  $G$  on  $n \geq 2$  vertices,  $LE_k(G) \geq 6n - 8$ , where the equality holds if and only if  $G$  is a path on  $n$  vertices.*

Since there are situations when chemists use digraphs rather than undirected graphs, Adiga et al. [1] first introduced the skew energy of a simple digraph. Let  $G$  be a simple digraph with vertex set  $V(G) = \{v_1, v_2, \dots, v_n\}$ . The skew-adjacency matrix of  $G$  is the  $n \times n$  matrix  $S(G) = [s_{ij}]$ , where  $s_{ij} = 1$  if  $(v_i, v_j)$  is an arc of  $G$ ,  $s_{ij} = -1$  if  $(v_j, v_i)$  is an arc of  $G$ , and  $s_{ij} = 0$  otherwise. Then the skew energy of  $G$  is the sum of the norms of all eigenvalues of  $S(G)$ , that is,

$$E_s(G) = \sum_{i=1}^n |\lambda_i|,$$

where  $\lambda_1, \lambda_2, \dots, \lambda_n$  are the eigenvalues of the skew-adjacency matrix  $S(G)$ , which are all pure imaginary numbers or 0 since  $S(G)$  is skew symmetric.

Similar to  $LE_k(G)$ , Adiga and Smitha [2] defined the skew Laplacian energy for a simple digraph  $G$  as

$$SLE_k(G) = \sum_{i=1}^n \mu_i^2,$$

where  $\mu_1, \mu_2, \dots, \mu_n$  are the eigenvalues of the skew Laplacian matrix  $SL(G) = D(G) - S(G)$  of  $G$ . In analogy with Theorem 1.1, the following results were obtained.

**Theorem 1.3.** [2]

- (i) For any simple digraph  $G$  on  $n$  vertices whose degrees are  $d_1, d_2, \dots, d_n$ ,  $SLE_k(G) = \sum_{i=1}^n d_i(d_i - 1)$ ;
- (ii) For any connected simple digraph  $G$  on  $n \geq 2$  vertices,  $2n - 4 \leq SLE_k(G) \leq n(n - 1)(n - 2)$ , where the left equality holds if and only if  $G$  is the directed path on  $n$  vertices and the right equality holds if and only if  $G$  is the complete digraph on  $n$  vertices.

Note that Theorem 1.3 shows that the skew Laplacian energy of a simple digraph defined in this way is independent of its orientation, which does not reflect the adjacency of the digraph. Being aware of this, later Adiga and Khoshbakht [3] gave another definition  $SLE_g(G) = \sum_{i=1}^n |\mu_i - \frac{2m}{n}|$ , just like  $LE_g(G)$ , and established some analogous bounds.

**Theorem 1.4.** [3] Let  $G$  be a simple digraph possessing  $n$  vertices and  $m$  edges. Assume that  $d_1, d_2, \dots, d_n$  is the degree sequence of  $G$  and  $\mu_1, \mu_2, \dots, \mu_n$  are the eigenvalues of the skew Laplacian matrix  $SL(G) = D(G) - S(G)$ . Let  $\gamma_i = \mu_i - \frac{2m}{n}$  and  $|\gamma_1| \leq |\gamma_2| \leq \dots \leq |\gamma_n| = k$ . Then

- (i)  $2\sqrt{M} \leq SLE_g(G) \leq \sqrt{2M_1n}$  ;
  - (ii)  $SLE_g(G) \leq k + \sqrt{(n - 1)(2M_1 - k^2)}$ ;
  - (iii) If  $G$  has no isolated vertices, then  $SLE_g(G) \leq 2M_1$ ,
- where  $M = -m + \frac{1}{2} \sum_{i=1}^n (d_i - \frac{2m}{n})^2$  and  $M_1 = M + 2m = m + \frac{1}{2} \sum_{i=1}^n (d_i - \frac{2m}{n})^2$ .

In 2010, Kissani and Mizoguchi [11] introduced a different Laplacian energy for directed graphs, in which only the out-degrees of vertices are considered rather than both the out-degrees and in-degrees. Let  $G$  be a digraph on  $n$  vertices. Suppose that  $\mu_1, \mu_2, \dots, \mu_n$  are the eigenvalues of the matrix  $L^+(G) = D^+(G) - A^+(G)$ , where  $D^+(G) = \text{diag}(d_1^+, d_2^+, \dots, d_n^+)$  is the diagonal matrix of the out-degrees of vertices in  $G$ , and  $A^+(G) = [a_{ij}]$  is the  $n \times n$  matrix, where  $a_{ij} = 1$  if  $(v_i, v_j)$  is an arc of  $G$  and 0 otherwise. Then the Laplacian energy of  $G$  defined in [11] is

$$LE_m(G) = \sum_{i=1}^n \mu_i^2.$$

By calculation, it is not hard to see that  $LE_m(G) = \sum_{i=1}^n (d_i^+)^2$  for a simple digraph  $G$ , and  $LE_m(G) = \sum_{i=1}^n d_i^+(d_i^+ + 1)$  for a symmetric digraph  $G$ . Furthermore, in [11] the authors found some relations between undirected and directed graphs of  $LE_m$  and used the so-called minimization maximum out-degree (MMO) algorithm to determine the digraphs with minimum Laplacian energy. The shortage of this definition is that it does not make use of the in-adjacency information of a digraph.

In this paper, we will introduce a brand-new definition for the skew Laplacian energy of a simple digraph and obtain some lower and upper bounds about it.

## 2. A New Skew Laplacian Energy of Simple Digraphs

We start with some notation and terminology which will be used in the sequel of this paper.

Given a simple digraph  $G$  with vertex set  $V(G) = \{v_1, v_2, \dots, v_n\}$ , let  $d_i^+$  and  $d_i^-$  denote the out-degree and in-degree of a vertex  $v_i$  in  $G$ , respectively.  $A(G), S(G), D(G), D^+(G), A^+(G)$  are defined as above. Similar to  $D^+(G)$ , we define  $D^-(G) = \text{diag}(d_1^-, d_2^-, \dots, d_n^-)$ , and  $\tilde{D}(G) = D^+(G) - D^-(G) = \text{diag}(d_1^+ - d_1^-, d_2^+ - d_2^-, \dots, d_n^+ - d_n^-)$ . Obviously,  $D(G) = D^+(G) + D^-(G)$ . Moreover, similar to  $A^+(G)$ , let  $A^-(G)$  be the  $n \times n$  matrix, where  $a_{ij} = 1$  if  $(v_j, v_i)$  is an arc of  $G$  and 0 otherwise. Clearly,  $A^-(G) = (A^+(G))^T$ . It is easy to see that the adjacency matrix of the underlying undirected graph  $G_U$  of  $G$  satisfies  $A(G_U) = A^+(G) + A^-(G)$  and the skew-adjacency matrix of  $G$  satisfies  $S(G) = A^+(G) - A^-(G)$ . Note that the Laplacian matrix of the underlying undirected graph  $G_U$  of  $G$  can be written as

$$\begin{aligned} L(G_U) &= D(G_U) - A(G_U) \\ &= (D^+(G) + D^-(G)) - (A^+(G) + A^-(G)) \\ &= (D^+(G) - A^+(G)) + (D^-(G) - A^-(G)). \end{aligned}$$

Inspired by this, we define a new kind of skew Laplacian matrix  $\tilde{S}L(G)$  of  $G$  as

$$\begin{aligned} \tilde{S}L(G) &= (D^+(G) - A^+(G)) - (D^-(G) - A^-(G)) \\ &= (D^+(G) - D^-(G)) - (A^+(G) - A^-(G)) \\ &= \tilde{D}(G) - S(G). \end{aligned}$$

Let  $\mu_1, \mu_2, \dots, \mu_n$  be the eigenvalues of the skew Laplacian matrix  $\tilde{S}L(G) = \tilde{D}(G) - S(G)$ . Since  $\tilde{S}L(G)$  is not symmetric, it does not give real eigenvalues always. However, we have the following two propositions about the eigenvalues of  $\tilde{S}L(G)$ :

**Proposition 2.1.**  $\sum_{i=1}^n \mu_i = \sum_{i=1}^n (d_i^+ - d_i^-) = 0$ .

*Proof.* The relation is evident from  $\sum_{i=1}^n \mu_i = \text{trace}(\tilde{S}L(G))$ . □

**Proposition 2.2.**  $0$  is an eigenvalue of  $\tilde{S}L(G)$  with multiplicity at least  $p$ , the number of components of  $G$ .

*Proof.* Let  $\sigma_{\tilde{S}L}(G)$  denote the set of eigenvalues of the skew Laplacian matrix  $\tilde{S}L(G)$ . Assume that  $C_1, C_2, \dots, C_p$  are all the components of  $G$ . Clearly,  $\sigma_{\tilde{S}L}(G) = \bigcup_{i=1}^p \sigma_{\tilde{S}L}(C_i)$ . So it suffices to prove that  $0 \in \sigma_{\tilde{S}L}(C_i)$  for  $1 \leq i \leq p$ . Now we restrict our attention to the induced subgraph  $C_i$ . The sum of each row in  $\tilde{S}L(C_i)$  is 0, thus 0 is an eigenvalue of  $\tilde{S}L(C_i)$  with eigenvector  $[1, 1, \dots, 1]^T$ . □

Note that for the Laplacian matrix of an undirected graph, 0 is also an eigenvalue with eigenvector  $[1, 1, \dots, 1]^T$ .

Now we give the formal definition for a new kind of skew Laplacian energy.

**Definition 2.3.** Let  $G$  be a simple digraph on  $n$  vertices. Then the skew Laplacian energy of  $G$  is defined as

$$SLE(G) = \sum_{i=1}^n |\mu_i|,$$

where  $\mu_1, \mu_2, \dots, \mu_n$  are the eigenvalues of the skew Laplacian matrix  $\widetilde{SL}(G) = \widetilde{D}(G) - S(G)$  of  $G$ .

We illustrate the concept with computing the skew Laplacian energy of two digraphs.

**Example 1 :** Let  $P_4$  be a directed path on four vertices with the arc set  $\{(1, 2)(2, 3)(3, 4)\}$ . Then

$$\widetilde{SL}(P_4) = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}.$$

The eigenvalues of  $\widetilde{SL}(P_4)$  are  $i\sqrt{2}, -i\sqrt{2}, 0, 0$ , and hence the skew Laplacian energy of  $P_4$  is  $2\sqrt{2}$ .

**Example 2 :** Let  $C_4$  be a directed cycle on four vertices with the arc set  $\{(1, 2)(2, 3)(3, 4)(4, 1)\}$ .

Then

$$\widetilde{SL}(C_4) = \begin{pmatrix} 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \end{pmatrix}.$$

The eigenvalues of  $\widetilde{SL}(C_4)$  are  $2i, -2i, 0, 0$ , and hence the skew Laplacian energy of  $C_4$  is 4, which is the same as the skew energy of  $C_4$ . Actually, we have the following more general result:

**Theorem 2.4.** *If  $G$  is an Eulerian digraph, then  $SLE(G) = E_s(G)$ .*

*Proof.* Since  $G$  is Eulerian, the out-degree and the in-degree are equal for each vertex in  $G$ , and so  $\widetilde{D} = 0$ , which results in  $\widetilde{SL}(G) = -S(G)$ , and consequently  $SLE(G) = E_s(G)$ . □

### 3. Some Lower and Upper Bounds for the New $SLE(G)$

This section is devoted to obtaining some lower and upper bounds for the skew Laplacian energy  $SLE(G)$  and determining the digraphs attaining these bounds.

**Theorem 3.1.** *Let  $G$  be a simple digraph possessing  $n$  vertices,  $m$  edges and  $p$  components. Assume that  $d_i^+$  ( $d_i^-$ ) is the out-degree (in-degree) of a vertex  $v_i$  in  $G$ . Then*

$$2\sqrt{|M|} \leq SLE(G) \leq \sqrt{2M_1(n-p)},$$

where  $M = -m + \frac{1}{2} \sum_{i=1}^n (d_i^+ - d_i^-)^2$  and  $M_1 = M + 2m = m + \frac{1}{2} \sum_{i=1}^n (d_i^+ + d_i^-)^2$ . Moreover, these bounds are sharp.

*Proof.* The proof is very similar to those of Theorems 1 and 4, but the extremal digraphs are determined very differently. Let  $\mu_1, \mu_2, \dots, \mu_n$  be the eigenvalues of the skew Laplacian matrix  $\widetilde{SL}(G) = \widetilde{D}(G) - S(G)$ , where  $\widetilde{D} = \text{diag}(d_1^+ - d_1^-, d_2^+ - d_2^-, \dots, d_n^+ - d_n^-)$  and  $S(G) = [s_{ij}]$  is the skew-adjacency matrix of  $G$ . Then from Proposition 1, we have

$$\sum_{i=1}^n \mu_i = \sum_{i=1}^n (d_i^+ - d_i^-) = 0. \tag{1}$$

Note that  $\sum_{i < j} \mu_i \mu_j$  is equal to the sum of the determinants of all  $2 \times 2$  principle submatrices of  $\widetilde{SL}(G)$ , which implies

$$\begin{aligned} \sum_{i < j} \mu_i \mu_j &= \sum_{i < j} \det \begin{pmatrix} d_i^+ - d_i^- & -s_{ij} \\ -s_{ji} & d_j^+ - d_j^- \end{pmatrix} \\ &= \sum_{i < j} [(d_i^+ - d_i^-)(d_j^+ - d_j^-) - s_{ij}s_{ji}] \\ &= \sum_{i < j} [(d_i^+ - d_i^-)(d_j^+ - d_j^-) + s_{ij}^2] \\ &= \sum_{i < j} [(d_i^+ - d_i^-)(d_j^+ - d_j^-)] + m. \end{aligned}$$

So

$$\sum_{i \neq j} \mu_i \mu_j = 2 \sum_{i < j} \mu_i \mu_j = \sum_{i \neq j} [(d_i^+ - d_i^-)(d_j^+ - d_j^-)] + 2m. \tag{2}$$

Combing (1) and (2), we get

$$\begin{aligned} \sum_{i=1}^n \mu_i^2 &= \left(\sum_{i=1}^n \mu_i\right)^2 - \sum_{i \neq j} \mu_i \mu_j \\ &= \left[\sum_{i=1}^n (d_i^+ - d_i^-)\right]^2 - \left[\sum_{i \neq j} (d_i^+ - d_i^-)(d_j^+ - d_j^-) + 2m\right] \\ &= \sum_{i=1}^n (d_i^+ - d_i^-)^2 - 2m \\ &= 2M. \end{aligned} \tag{3}$$

Let  $\widetilde{SL}(G) = [\ell_{ij}]$ . By Schur's unitary triangularization theorem [8], there exists a unitary matrix  $U$  such that  $U^* \widetilde{SL}(G) U = T$ , where  $T = [t_{ij}]$  is an upper triangular matrix with diagonal entries  $t_{ii} = \mu_i$ ,  $i = 1, 2, \dots, n$ . Therefore

$$\sum_{i,j=1}^n |\ell_{ij}|^2 = \sum_{i,j=1}^n |t_{ij}|^2 \geq \sum_{i=1}^n |t_{ii}|^2 = \sum_{i=1}^n |\mu_i|^2, \tag{4}$$

that is,

$$\sum_{i=1}^n |\mu_i|^2 \leq \sum_{i,j=1}^n |\ell_{ij}|^2 = \sum_{i=1}^n (d_i^+ - d_i^-)^2 + 2m = 2M_1.$$

Without loss of generality, assume that  $|\mu_1| \geq |\mu_2| \geq \dots \geq |\mu_n|$ . From Proposition 2, we know that  $\mu_{n-i} = 0$  for  $i = 0, 1, \dots, p - 1$ . Applying Cauchy-Schwarz Inequality, it yields that

$$SLE(G) = \sum_{i=1}^n |\mu_i| = \sum_{i=1}^{n-p} |\mu_i| \leq \sqrt{(n-p) \sum_{i=1}^{n-p} |\mu_i|^2} = \sqrt{(n-p) \sum_{i=1}^n |\mu_i|^2} \leq \sqrt{2M_1(n-p)}. \tag{5}$$

Now we turn to the proof of the left-hand inequality.

Since  $\sum_{i=1}^n \mu_i = 0$ ,  $\sum_{i=1}^n \mu_i^2 + 2 \sum_{i<j} \mu_i \mu_j = 0$ . Using (3), we get  $2 \sum_{i<j} \mu_i \mu_j = -2M$ , which follows that

$$2|M| = 2 \left| \sum_{i<j} \mu_i \mu_j \right| \leq 2 \sum_{i<j} |\mu_i| |\mu_j|. \tag{6}$$

Using (3) again,

$$2|M| = \left| \sum_{i=1}^n \mu_i^2 \right| \leq \sum_{i=1}^n |\mu_i|^2. \tag{7}$$

From (6) and (7), we arrive at

$$SLE(G)^2 = \left( \sum_{i=1}^n |\mu_i| \right)^2 = \sum_{i=1}^n |\mu_i|^2 + 2 \sum_{i<j} |\mu_i| |\mu_j| \geq 4|M|.$$

Consequently,  $SLE(G) \geq 2\sqrt{|M|}$ .

We proceed with the discussion for the sharpness of these bounds.

**Claim 1:**  $SLE(G) = 2\sqrt{|M|}$  holds if and only if for each pair of  $\mu_{i_1} \mu_{j_1}$  and  $\mu_{i_2} \mu_{j_2}$  ( $i_1 \neq j_1, i_2 \neq j_2$ ), there exists a non-negative real number  $k$  such that  $\mu_{i_1} \mu_{j_1} = k \mu_{i_2} \mu_{j_2}$ ; and for each pair of  $\mu_{i_1}^2$  and  $\mu_{i_2}^2$ , there exists a non-negative real number  $\ell$  such that  $\mu_{i_1}^2 = \ell \mu_{i_2}^2$ .

*Proof of Claim 1.* It follows from (6) and (7) that the equality is attained if and only if  $\left| \sum_{i<j} \mu_i \mu_j \right| =$

$\sum_{i<j} |\mu_i| |\mu_j|$  and  $\left| \sum_{i=1}^n \mu_i^2 \right| = \sum_{i=1}^n |\mu_i|^2$ . In other words, the equality holds if and only if for each pair of  $\mu_{i_1} \mu_{j_1}$  and  $\mu_{i_2} \mu_{j_2}$  ( $i_1 \neq j_1, i_2 \neq j_2$ ), there exists a non-negative real number  $k$  such that  $\mu_{i_1} \mu_{j_1} = k \mu_{i_2} \mu_{j_2}$ ; and for each pair of  $\mu_{i_1}^2$  and  $\mu_{i_2}^2$ , there exists a non-negative real number  $\ell$  such that  $\mu_{i_1}^2 = \ell \mu_{i_2}^2$ , which proves Claim 1.

A question arises: do such graphs exist? The answer is yes. Let  $G_1$  be an orientation of  $K_{2n,2n}$ . Assume that  $\{X, Y\}$  is the bipartition of  $G_1$ . We divide  $X$  ( $Y$ ) into two disjoint sets  $X_1, X_2$  ( $Y_1, Y_2$ ) such that  $|X_1| = |X_2| = |Y_1| = |Y_2| = n$ . The arc set is  $\{(u_1, v_1) | u_1 \in X_1, v_1 \in Y_1\} \cup \{(u_2, v_2) | u_2 \in X_2, v_2 \in Y_2\} \cup \{(v_1, u_2) | u_2 \in X_2, v_1 \in Y_1\} \cup \{(v_2, u_1) | u_1 \in X_1, v_2 \in Y_2\}$ ; see Figure 1.

Note that  $d_i^+ = d_i^-$  for each vertex  $v_i$  in  $G_1$ . So we get  $2\sqrt{|M|} = 2\sqrt{m} = 4n$ , and the skew Laplacian matrix of  $G_1$  is

$$\widetilde{SL}(G_1) = -S(G_1) = \begin{pmatrix} 0 & 0 & -J & J \\ 0 & 0 & J & -J \\ J & -J & 0 & 0 \\ -J & J & 0 & 0 \end{pmatrix},$$

where  $J$  is the  $n \times n$  matrix in which each entry is 1.

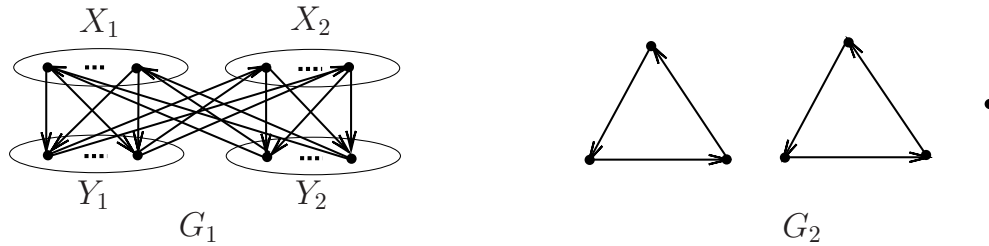


Figure 1 The graph for Claim 1.

Then the skew Laplacian characteristic polynomial  $P_{\widetilde{S}L}(G_1; x) = \det(xI - \widetilde{S}L(G_1)) = x^{4n-2}(x^2 + 4n^2)$ , and the eigenvalues of  $\widetilde{S}L(G_1)$  are  $2ni, -2ni, 0$  with multiplicity 1, 1,  $4n - 2$ , respectively. Hence the skew Laplacian energy of  $G_1$  is  $4n$ , which implies that the lower bound is sharp.

**Claim 2:**  $SLE(G) = \sqrt{2M_1(n - p)}$  holds if and only if (i)  $G$  is 0-regular or (ii) for each  $v_i \in V(G)$ ,  $d_i^+ = d_i^-$ , and the eigenvalues of  $\widetilde{S}L(G)$  are  $0, ai, -ai$  ( $a > 0$ ) with multiplicity  $p, \frac{n-p}{2}, \frac{n-p}{2}$ , respectively. *Proof of Claim 2.* It is evident from (4) and (5) that the equality holds if and only if  $T = [t_{ij}]$  is a diagonal matrix and  $|\mu_1| = |\mu_2| = \dots = |\mu_{n-p}|$ .

From Schur's unitary triangularization theorem [8], we know that  $T = [t_{ij}]$  is a diagonal matrix if and only if  $\widetilde{S}L(G)$  is a normal matrix. That is

$$\widetilde{S}L^*(G) \cdot \widetilde{S}L(G) = \widetilde{S}L(G) \cdot \widetilde{S}L^*(G).$$

Since  $\widetilde{S}L(G) = \widetilde{D}(G) - S(G)$ ,  $\widetilde{S}L^*(G) = \widetilde{D}(G) + S(G)$ , we have

$$(\widetilde{D}(G) + S(G)) \cdot (\widetilde{D}(G) - S(G)) = (\widetilde{D}(G) - S(G)) \cdot (\widetilde{D}(G) + S(G)).$$

By direct calculation,  $S(G) \cdot \widetilde{D}(G) = \widetilde{D}(G) \cdot S(G)$ . Comparing the element on the  $i$ th row and the  $j$ th column of the matrices on both sides, we arrive at

$$s_{ij}(d_j^+ - d_j^-) = (d_i^+ - d_i^-)s_{ij}.$$

If  $v_i$  and  $v_j$  are not adjacent (i.e.,  $s_{ij} = 0$ ), then it holds surely; if  $v_i$  and  $v_j$  are adjacent, then  $s_{ij} \neq 0$ , and consequently  $d_i^+ - d_i^- = d_j^+ - d_j^-$ .

Now we are concerned with each component  $C_k$  ( $1 \leq k \leq p$ ) of  $G$ . Since  $C_k$  is connected, any two vertices  $u, w$  in  $C_k$  are connected by a path  $P : u = v_0, v_1, \dots, v_t = w$ . Then  $d^+(v_i) - d^-(v_i) = d^+(v_{i+1}) - d^-(v_{i+1})$  for  $0 \leq i \leq t - 1$ , which implies that  $d^+(u) - d^-(u) = d^+(w) - d^-(w)$ . It is easy to see that  $\sum_{v \in V(C_k)} [d^+(v) - d^-(v)] = 0$ . Therefore  $d^+(v) - d^-(v) = 0$  for each vertex  $v$  in  $C_k$ . It follows that  $d_i^+ = d_i^-$  for  $v_i \in V(G)$ , i.e.,  $\widetilde{D}(G) = 0, \widetilde{S}L(G) = -S(G), \widetilde{S}L(C_k) = -S(C_k)$ .

From Proposition 2, we know that 0 is an eigenvalue of  $\widetilde{S}L(G)$  with multiplicity at least  $p$ , and 0 is also an eigenvalue of  $\widetilde{S}L(C_k)$  ( $k = 1, 2, \dots, p$ ) with multiplicity at least 1. We distinguish the following two cases.

**Case 1:**  $|\mu_1| = |\mu_2| = \dots = |\mu_{n-p}| = 0$ .

0 is the unique eigenvalue of  $\widetilde{S}L(G)$  with multiplicity  $n$ , and consequently  $G$  is a 0-regular graph.



**Case 2:**  $|\mu_1| = |\mu_2| = \dots = |\mu_{n-p}| = a > 0$ .

That is, 0 is an eigenvalue of  $\widetilde{SL}(G)$  with multiplicity exactly  $p$ , and the norms of all the other  $n - p$  eigenvalues of  $\widetilde{SL}(G)$  are equal to  $a$ . Since  $\widetilde{SL}(G) = -S(G)$  is a skew-symmetric matrix, its eigenvalues are 0 or pure imaginary numbers which appear in pairs. We conclude that the eigenvalues of  $\widetilde{SL}(G)$  are  $0, ai, -ai$  ( $a \geq 0$ ) with multiplicity  $p, \frac{n-p}{2}, \frac{n-p}{2}$ , respectively. This proves Claim 2.

Next, we give an example to verify the existence of such graphs. Let  $G_2 = \alpha K_3 \cup \beta K_1$ , where  $\alpha, \beta \in \mathcal{N}$  and  $3\alpha + \beta = n$ ; see Figure 1 for  $\alpha = 2, \beta = 1$ .  $K_3$  is oriented with the arc set  $\{(1, 2), (2, 3), (3, 1)\}$ . The eigenvalues of  $\widetilde{SL}(G_2)$  are  $\sqrt{3}i, -\sqrt{3}i, 0$  with multiplicity  $\alpha, \alpha, \alpha + \beta$ , respectively. Hence  $SLE(G_2) = 2\sqrt{3}\alpha$  and  $\sqrt{2M_1[n - (\alpha + \beta)]} = \sqrt{4m\alpha} = 2\sqrt{3}\alpha$ , which implies that the upper bound is also sharp.

Combining all above, we complete our proof. □

**Corollary 3.2.** *Let  $G$  be a simple digraph possessing  $p$  components  $C_1, C_2, \dots, C_p$ . If  $SLE(G) = \sqrt{2M_1(n - p)}$ , then each component  $C_i$  is Eulerian with odd number of vertices.*

*Proof.* If  $G$  is a 0-regular graph, each component of  $G$  is an isolated vertex, which obviously satisfies the conclusion. Otherwise, from Claim 2 we know that  $d_i^+ = d_i^-$  for each  $v_i \in V(C_k)$ , and hence  $C_k$  is Eulerian. Furthermore, the eigenvalues of  $\widetilde{SL}(G)$  are  $0, ai, -ai$  ( $a > 0$ ) with multiplicity  $p, \frac{n-p}{2}, \frac{n-p}{2}$ , respectively. It turns out that, for each component  $C_k$ , 0 is an eigenvalue of  $\widetilde{SL}(C_k)$  with multiplicity exactly one and all the other eigenvalues are  $ai, -ai$ , which appear in pairs. It follows that the number of vertices in  $C_k$  is odd. □

**Corollary 3.3.**  $SLE(G) \leq \sqrt{2M_1n}$ .

**Corollary 3.4.** *If  $G$  has no isolated vertices, then  $SLE(G) \leq 2M_1$ .*

*Proof.* If  $G$  has no isolated vertices, then  $n \leq 2m$ . Therefore,

$$SLE(G) \leq \sqrt{2M_1n} \leq 2\sqrt{M_1m} \leq 2M_1.$$

□

We may mention that the bounds in Theorem 3.1, Corollary 3.3 and Corollary 3.4 are in correspondence with those in Theorem 1.1 and Theorem 1.4.

#### 4. Concluding Remarks

Graph energy is one of the most active topics in chemical graph theory. There have appeared several different definitions for the (skew) Laplacian energy of undirected graphs and directed graphs. Here we would like to summarize them below:

1. *The Laplacian energy of undirected graphs*

For an undirected graph  $G$ , there are two kinds of Laplacian energies  $LE_k(G) = \sum_{i=1}^n \mu_i^2$  and  $LE_g(G) = \sum_{i=1}^n |\mu_i - \frac{2m}{n}|$ , where  $\mu_1, \mu_2, \dots, \mu_n$  are the eigenvalues of  $L(G) = D(G) - A(G)$ .

### 2. The Laplacian energy of directed graphs

For a directed graph  $G$ , there is one kind of Laplacian energy  $LE_m(G) = \sum_{i=1}^n \mu_i^2$ , where  $\mu_1, \mu_2, \dots, \mu_n$  are the eigenvalues of  $L^+(G) = D^+(G) - A^+(G)$ .

### 3. The skew Laplacian energy of simple directed graphs

For a simple directed graph  $G$ , there are two kinds of skew Laplacian energies  $SLE_k(G) = \sum_{i=1}^n \mu_i^2$  and  $SLE_g(G) = \sum_{i=1}^n |\mu_i - \frac{2m}{n}|$ , where  $\mu_1, \mu_2, \dots, \mu_n$  are the eigenvalues of  $SL(G) = D(G) - S(G)$ .

4. In this paper, we introduce a new kind of skew Laplacian matrix  $\widetilde{SL}(G) = \widetilde{D}(G) - S(G) = (D^+(G) - D^-(G)) - (A^+(G) - A^-(G))$ , which is inspired by the popularly used Laplacian matrix  $L(G) = D(G) - A(G) = (D^+(G) + D^-(G)) - (A^+(G) + A^-(G))$ . From the definition, we can see that the matrix  $\widetilde{SL}(G)$  fully reflects both the in-adjacency and the out-adjacency of a digraph  $G$ . Moreover, it has some good properties such as  $\text{trace}(\widetilde{SL}(G)) = 0$ , the sum of each row is 0, 0 is an eigenvalue and  $(1, 1, \dots, 1)^T$  is an eigenvector, and so on. These properties make  $\widetilde{SL}(G)$  to be regarded as a skew Laplacian matrix more reasonable. From the new skew Laplacian matrix, we define a new skew Laplacian energy as  $SLE(G) = \sum_{i=1}^n |\mu_i|$ , where  $\mu_1, \mu_2, \dots, \mu_n$  are the eigenvalues of  $\widetilde{SL}(G) = \widetilde{D}(G) - S(G)$ . That new skew Laplacian energy is well defined can also be seen from the following bounds, which should be compared with those in Theorem 1.1:

$$(i) \ 2\sqrt{|M|} \leq SLE(G) \leq \sqrt{2M_1n};$$

$$(ii) \ SLE(G) \leq \sqrt{2M_1(n-p)};$$

$$(iii) \ \text{If } G \text{ has no isolated vertices, then } SLE(G) \leq 2M_1,$$

where  $M = -m + \frac{1}{2} \sum_{i=1}^n (d_i^+ - d_i^-)^2$  and  $M_1 = M + 2m = m + \frac{1}{2} \sum_{i=1}^n (d_i^+ - d_i^-)^2$ .

## Acknowledgments

This work was supported by NSFC. The authors wish to thank the referee for helpful comments and suggestions.

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