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ON THE COMPLEXITY OF THE COLORFUL DIRECTED PATHS IN VERTEX COLORING OF DIGRAPHS

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Abstract. The colorful paths and rainbow paths have been considered by several authors. A colorful directed path in a digraph G is a directed path with $\chi(G)$ vertices whose colors are different. A v-colorful directed path is such a directed path, starting from v. We prove that for a given 3-regular triangle-free digraph G determining whether there is a proper $\chi(G)$-coloring of G such that for every $v \in V(G)$, there exists a v-colorful directed path is **NP**-complete.

1. Introduction

Graph coloring is a well-studied area of graph theory. For a graph G, a *proper k-coloring* of G is a function $c : V(G) \longrightarrow \{1, \cdots, k\}$ such that $c(u) \neq c(v)$ for every two adjacent vertices $u, v \in V(G)$. The chromatic number of G denoted by $\chi(G)$, is the smallest k for which G has a proper k-coloring. For a given coloring of a graph G, we say path P of G is a *rainbow path* if all vertices of P have different colors. A *v-rainbow path* is a rainbow path starting from the vertex v. A *v-colorful path* is a rainbow path starting from the vertex v with $\chi(G)$ vertices. Let G be a graph. We recall that a path in G is said to represent all $\chi(G)$ colors if all the colors $1, \dots, \chi(G)$ appear on this path. A *colorful directed path* in a digraph G is a directed path with $\chi(G)$ vertices whose colors are different. A *v-colorful directed path* is such a directed path, starting from v. The colorful paths and rainbow paths have been considered by several authors, for instance see [1, 2, 3, 4, 6, 7]. In 2007, Lin posed the following problem [7].

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Problem 1.1. [7] *Let G be a connected graph. Does there always exist a proper vertex coloring of G with $\chi(G)$ colors such that every vertex of G is on a path with $\chi(G)$ vertices which represents all $\chi(G)$ colors?*

Afterwards, Akbari et al. proposed the following stronger conjecture [1].

Conjecture 1.2. [1] *Let $G \neq C_7$ be a connected graph. Then there exists a proper $\chi(G)$-coloring of G such that for every $v \in V(G)$, there exists a v-colorful path.*

In [2] this was proved with $\lfloor \frac{\chi(G)}{2} \rfloor$ vertices instead of $\chi(G)$ vertices. Afterwards, Alishahi et al. strengthen this to $\chi(G)-1$ vertices [3]. Also in [2] it was proved that, there exists a proper $(\Delta(G)+1)$-coloring of G with a v-colorful path for every $v \in V(G)$. Furthermore, in [2] it was proved that this result is true if one replaces $(\Delta(G)+1)$ colors with $2\chi(G)$ colors.

A proper vertex coloring of a digraph D is defined, simply a vertex coloring of its underlying graph G, and its chromatic number $\chi(D)$ is defined to be the chromatic number $\chi(G)$ of G. The chromatic number of a digraph provides interesting information about its subdigraphs. The following well-known result, due to Gallai, gives a relationship between the length of the longest path and the chromatic number (for example see [9]).

Theorem 1.3. [Gallai Theorem] *Every digraph G has a directed path with at least $\chi(G)$ vertices.*

In 2001, Li generalized the Gallai Theorem by specifying the starting vertex of the directed path [6].

Theorem 1.4. [6] *If G is a digraph in which v is a vertex that can reach all other vertices, then G has a directed path starting at v with at least $\chi(G)$ vertices.*

Li gave the following conjecture for the digraph [6].

Conjecture 1.5. [6] *For any proper $\chi(G)$-coloring of a digraph G and any vertex $v \in V(G)$ that can reach all other vertices, there is a directed path starting at v whose vertices use all $\chi(G)$ colors.*

Chang et al. gave a counterexample to the above conjecture [4]. In this note, we are interested in the following problem.

Problem: *Colorful Directed Paths*

Input: A connected digraph G.

Question: Is there a proper $\chi(G)$-coloring of G such that for every $v \in V(G)$, there is a v-colorful directed path?

Our main result is that *Colorful Directed Paths* is **NP**-complete for 3-regular triangle-free digraphs. In contrast, we show that *Colorful Directed Paths* can be solved in polynomial time for 2-regular digraphs.

In [8] it was proved that, it is **NP**-complete to decide whether G is colorable with $\chi(G)$ colors in such a way that for a given vertex $v \in V(G)$ there is a path starting at v representing all $\chi(G)$ colors.

Next, by a similar argument, we prove that the following problem is **NP**-complete for disconnected graphs.

Problem: *Colorful Paths*
INPUT: A graph G
QUESTION: Is there a proper $\chi(G)$-coloring of G such that for every $v \in V(G)$, there exists a v-colorful path?

We follow [5, 9] for terminology and notation not defined here, and we consider finite simple graphs and digraphs. We denote the vertex set and the edge set of G by $V(G)$ and $E(G)$, respectively. We denote the maximum degree and the minimum degree of G by $\Delta(G)$ and $\delta(G)$, respectively. The union of simple graphs G and H is the graph $G \cup H$ with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. If G and H are disjoint, we refer to their union as a disjoint union, and generally denote it by $G + H$. By starting with a disjoint union of two graphs G and H and adding edges joining every vertex of G to every vertex of H, one obtains the join of G and H, denoted $G \vee H$. Also, for every $v \in V(G)$, $d(v)$ denotes the degree of v. For a natural number r, a graph G is called an *r-regular graph* if $d(v) = r$, for each $v \in V(G)$.

2. NP-completeness

Theorem 2.1. Colorful Directed Paths *is* **NP***-complete for* 3*-regular triangle-free digraphs and it can be solved in polynomial time for* 2*-regular digraphs.*

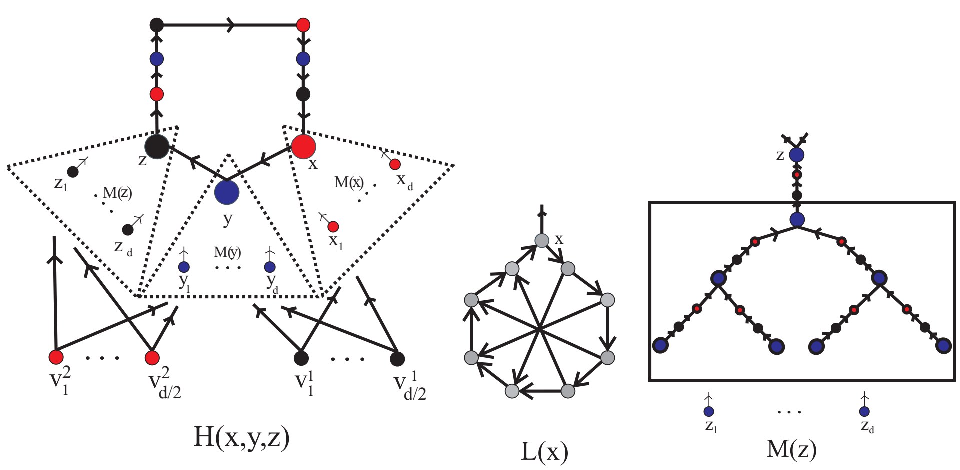

FIGURE 1. The auxiliary digraphs $H(x, y, z)$, $L(x)$ and $M(z)$.

Proof. First, we show that *Colorful Directed Paths* can be solved in polynomial time for 2-regular digraphs. Let G be a connected 2-regular digraph. We have the following straightforward characterization. If G is a connected 2-regular digraph, then there exists a proper $\chi(G)$-coloring of G such that for every $v \in V(G)$, there exists a v-colorful directed path, if and only if, for every vertex $v \in V(G)$,

$d^+(v) = 1$ and $|V(G)| = 2k$ or $3k$. Next, we prove that *Colorful Directed Paths* is **NP**-complete for 3-regular triangle-free digraphs. Clearly, the problem is in **NP**. We reduce 3-*Sat* to our problem. Let Φ be a 3-Sat formula with clauses $C = \{c_1, \cdots, c_k\}$ and variables $X = \{x_1, \cdots, x_n\}$. Also let $d = 10(k + n)$. We use the auxiliary digraphs T, $M(z)$, $H(x, y, z)$, $A(x_j)$, $B(c_j)$ and $L(x)$, which are shown in Figure 1 and Figure 2. We construct a digraph $G(\Phi)$ as the digraph arising from the following construction:

Algorithm 1 : Construction of $G(\Phi)$.

1: We start $H(x, y, z)$ as the digraph $G(\Phi)$.
2: For each variable x_j, put a copy of $A(x_j)$ also put two directed edges $x_j v^1_{2j-1}$, $\neg x_j v^1_{2j}$ from x_j and $\neg x_j$ to v^1_{2j-1} and v^1_{2j}, respectively.
3: For each clause c_j, put a copy of $B(c_j)$ also put five directed edges $v^2_j s^1_j$, $v^1_{4j+2n} s^1_j$, $c^1_j v^1_{4j-1+2n}$, $c^2_j v^1_{4j-2+2n}$ and $c^3_j v^1_{4j-3+2n}$.
4: For each clause $c_j = l_1 \vee l_2 \vee l_3$, for every i, $1 \leq i \leq 3$, add the directed edge $c^i_j a^j_{l_i}$ from c^i_j to $a^j_{l_i}$.
5: For each vertex v, if $d(v) = 1$, put two auxiliary graphs $L(v_x)$, $L(v_{x'})$ and also put two directed edges vv_x and $vv_{x'}$, from v to v_x and $v_{x'}$.
6: For each vertex v, if $d(v) = 2$, put the auxiliary graphs $L(v_x)$ and the directed edges vv_x from v to v_x.

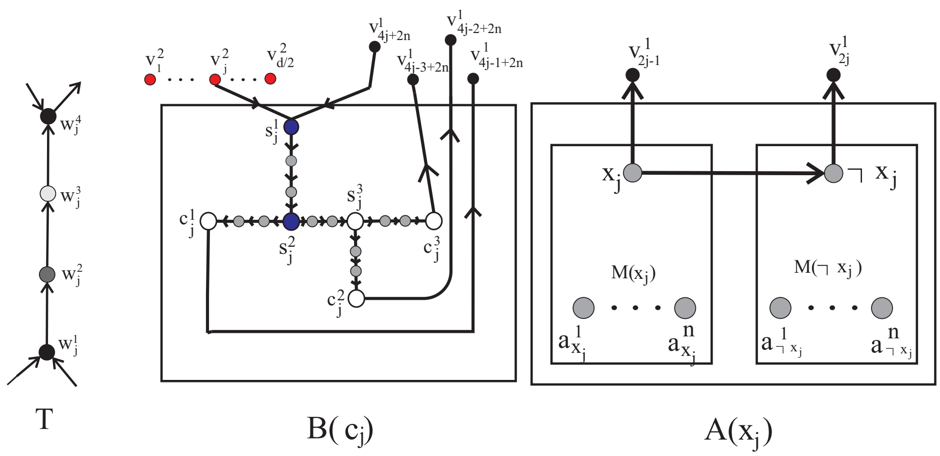

FIGURE 2. The auxiliary digraphs $A(x_j)$, $B(c_j)$ and T.

Next, we discuss basic properties of the digraph $G(\Phi)$. The digraph $G(\Phi)$ is 3-regular and triangle-free. Let f be a proper 3-coloring, such that for every vertex v, there exists a v-colorful directed path. We have:

$$\{f(\neg x_j), f(x_j)\} = \{Red, Blue\}, \qquad f(x_j) = f(a^1_{x_j}) = \cdots = f(a^n_{x_j}),$$
$$f(\neg x_j) = f(a^1_{\neg x_j}) = \cdots = f(a^n_{\neg x_j}), \qquad f(s^1_j) = f(s^2_j) = Blue,$$

Moreover, for every copy of T we have $f(w_j^1) = f(w_j^4)$ and for every copy of $M(z)$ we have $f(z) = f(z_1) = \cdots = f(z_d)$. Also for every copy of $H(x, y, z)$ we have $f(z) = f(z_1) = \cdots = f(z_d)$, $f(x) = f(x_1) = \cdots = f(x_d)$, $f(y) = f(y_1) = \cdots = f(y_d)$, $f(x) = f(v_1^2) = \cdots = f(v_{d/2}^2)$ and $f(z) = f(v_1^1) = \cdots = f(v_{d/2}^1)$. First, suppose that Φ is satisfiable with the satisfying assignment Γ. Now we present the proper 3-coloring f for $G(\Phi)$, such that for every $v \in V(G(\Phi))$, there exists a v-colorful directed path. Let $f(x) = Red$, $f(y) = Blue$ and $f(z) = Black$. Now, for every vertex v, $v \in V(H(x, y, z))$, the color of v, is determined uniquely. For each variable x_i, if $x_i = True$, then let $f(x_i) = Red$ and $f(\neg x_i) = Blue$. Otherwise let $f(x_i) = Blue$ and $f(\neg x_i) = Red$. For every $c_j = l_1 \vee l_2 \vee l_3$, color the vertices of $B(c_j)$ according to the Figure 3. Now, for every vertex v, $v \in V(A(x_j))$, the color of v, is determined uniquely. Finally, color the vertices of every copy of $L(x)$. It is easy to see that for every $v \in V(G(\Phi))$, there exists a v-colorful directed path.

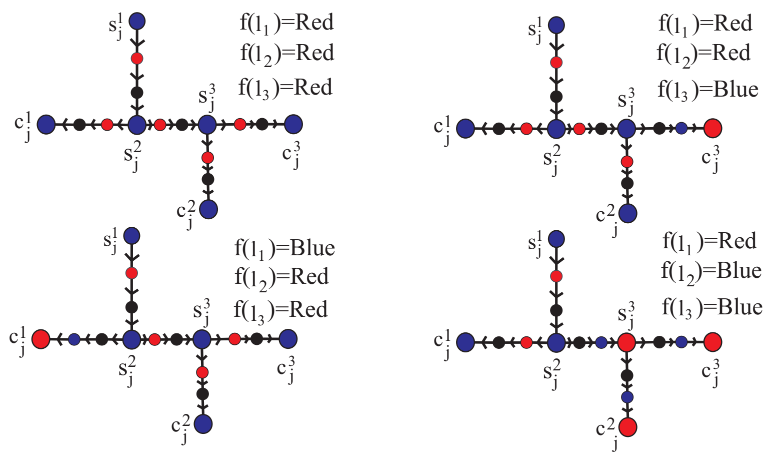

Figure 3. Four cases.

Next, suppose that $G(\Phi)$ has the proper 3-coloring f, such that for every $v \in V(G(\Phi))$, there exists a v-colorful directed path. With no loss of generality suppose that $f(x) = Red$, $f(y) = Blue$ and $f(z) = Black$. For each variable x_i, let $x_i = True$ in Γ if and only if $f(x_i) = Red$. Let $c_j = l_1 \vee l_2 \vee l_3$ be an arbitrary clause. We have $f(s_j^1) = f(s_j^2) = Blue$, therefore $Blue \in \{f(c_j^1), f(s_j^3)\}$, so $Blue \in \{f(c_j^1), f(c_j^2), f(c_j^3)\}$. Consequently $Red \in \{f(a_{l_1}^j), f(a_{l_2}^j), f(a_{l_3}^j)\}$, so $Red \in \{f(l_1), f(l_2), f(l_3)\}$. Therefore; Γ is a satisfying assignment for Φ.

□

Theorem 2.2. *For every $r \geq 3$, the following problem is* **NP***-complete: "given an r-regular triangle-free digraph G with $\chi(G) = 3$, does there exist a proper 3-coloring of G such that for every $v \in V(G)$, there exists a v-colorful directed path?"*

Proof. The proof is similar to the proof of Theorem 2.1. Consider two disjoint copies of $G(\Phi)$ ($G(\Phi)$ is introduced in the proof of the previous theorem), then for every vertex v, put a directed edge from v to the corresponding vertex in the second copy of $G(\Phi)$. By repeating this procedure, we find an r-regular triangle-free digraph G' with $2^{r-3}|V(G(\Phi))|$ vertices. Clearly, there exists a proper 3-coloring of G' such that for every $v \in V(G')$, there exists a v-colorful directed path, if and only if, there exists a proper 3-coloring of $G(\Phi)$ such that for every $v \in V(G(\Phi))$, there exists a v-colorful directed path. □

Theorem 2.3. Colorful Paths *is* **NP**-*complete.*

Proof. Clearly, Colorful Paths is in **NP**. We reduce Hamilton Path to this problem (for a given graph G, does G have a Hamilton path? [5]). Consider a graph G with $|V(G)| = n$, as an instance of Hamilton Path. We construct a new graph G' with the property that, there exists a proper $\chi(G')$-coloring of G' such that for every $v \in V(G')$, there exists a v-colorful path, if and only if G has a Hamilton path. Let $G' = (G \vee K_1) + K_{n+1}$. If G has a Hamilton path then $G \vee K_1$ has a Hamilton cycle, so there exists a proper $\chi(G')$-coloring of G' such that for every $v \in V(G')$, there exists a v-colorful path. Next, suppose that there exists a proper $\chi(G')$-coloring f of G' such that for every $v \in V(G')$, there exists a v-colorful path. Now consider a u-colorful path $uv_1v_2 \ldots v_n$ for G', clearly $v_1v_2 \ldots v_n$ is a Hamilton path for G. So Colorful Paths is **NP**-complete. □

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References

[1] S. Akbari, F. Khaghanpoor and S. Moazzeni, Colorful paths in vertex coloring of graphs, *Preprint*.

[2] S. Akbari, V. Liaghat and A. Nikzad, Colorful paths in vertex coloring of graphs, *Electron. J. Combin.*, **18** no. 1 (2011) Paper 17, 9 pp.

[3] M. Alishahi, A. Taherkhani and C. Thomassen, Rainbow paths with prescribed ends, *Electron. J. Combin.*, **18** no. 1 (2011) Paper 86, 5 pp.

[4] G. J. Chang, L. D. Tong, J. H. Yan and H. G. Yeh, A note on the Gallai-Roy-Vitaver theorem, *Discrete Math.*, **256** no. 1–2 (2002) 441–444.

[5] M. R. Garey and D. S. Johnson, *Computers and intractability: A guide to the theory of NP-completeness*, A Series of Books in the Mathematical Sciences, W. H. Freeman and Co., San Francisco, Calif., 1979.

[6] H. Li, A generalization of the Gallai-Roy theorem, *Graphs Combin.*, **17** no. 4 (2001) 681–685.

[7] C. Lin, Simple proofs of results on paths representing all colors in proper vertex-colorings, *Graphs Combin.*, **23** no. 2 (2007) 201–203.

[8] P. M. Pardalos and A. Migdalas, A note on the complexity of longest path problems related to graph coloring, *Appl. Math. Lett.*, **17** no. 1 (2004) 13–15.

[9] D. B. West, *Introduction to graph theory*, Prentice Hall Inc., Upper Saddle River, NJ, 1996.

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