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ON SCHEMES ORIGINATED FROM FERRERO PAIRS

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Abstract. The Frobenius complement of a given Frobenius group acts on its kernel. The scheme which is arisen from the orbitals of this action is called Ferrero pair scheme. In this paper, we show that the fibers of a Ferrero pair scheme consist of exactly one singleton fiber and every two fibers with more than one point have the same cardinality. Moreover, it is shown that the restriction of a Ferrero pair scheme on each fiber is isomorphic to a regular scheme. Finally, we prove that for any prime p, there exists a Ferrero pair p-scheme, and if $p > 2$, then the Ferrero pair p-schemes of the same rank are all isomorphic.

1. Introduction

Assume that N is a finite group and H is a fixed point free automorphism group on N, and so H is a subgroup of $\mathrm{Aut}(N)$ acting semiregularly on N. The pair (N, H) is called *Ferrero pair*. The orbitals of the action of H on N, denoted by $\mathrm{Inv}(H, N)$, generate a scheme called the *Ferrero pair scheme*.

In [1], a new design from the Ferrero pair is constructed and it is shown that if f is a design isomorphism between two Ferrero pairs (N_1, H_1) and (N_2, H_2), then $fH_1f^{-1} = H_2$. We prove an analog of this statement for Ferrero pair schemes. We also show that the restriction of a Ferrero pair scheme on each fiber is isomorphic to a regular scheme and a Ferrero pair scheme has thin thin residue. In other words, thin residue of a Ferrero pair scheme is a closed subset of its thin radical. In addition, we derive some necessary conditions under which the isomorphism between two Ferrero pair schemes induces an isomorphism between their groups. Moreover, the Ferrero pair scheme $\mathrm{Inv}(H, N)$ is a direct sum of two regular schemes if and only if $H \ltimes N$ is a 2-transitive Frobenius group. It is

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already known that p-schemes of the same rank are not isomorphic. Finally, we are going to show that if p is an odd prime, then the Ferrero pair p-schemes of the same rank are all isomorphic.

Let $\mathrm{Inv}(H,N)$ be a Ferrero pair scheme. Then the semidirect product $H \ltimes N$ is a Frobenius group with the complement $H \ltimes \langle 1_N \rangle$ and the kernel $\langle \mathrm{id}_N \rangle \ltimes N$. It is well-known that the complement of a Frobenius group acts fixed point free on its kernel. Therefore, there is a one-to-one correspondence between the set of all Frobenius groups and the set of all Ferrero pairs.

Let us fix the notations which we will use throughout the paper. We assume that V is a nonempty finite set, $\mathcal{R}$ is a set of binary relations on V, and $R \in \mathcal{R}$. We write

- $\mathcal{R}^*$ to denote the set of all unions of elements of $\mathcal{R}$;
- $\Delta(V)$ to denote the set of all pairs (v,v) with $v \in V$;
- R^t to denote the set of all pairs (u,v) with $(v,u) \in R$;
- $R(u)$ to denote the set of all elements $v \in V$ with $(u,v) \in R$;
- $\mathcal{R}_{X,Y}$ to denote the set of all nonempty relations $R \cap (X \times Y)$ with $R \in \mathcal{R}$ and $X, Y \subseteq V$. In particular, $\mathcal{R}_X = \{R \in \mathcal{R} : R \subset X \times X\}$.

Now we recall the definitions and concepts related to permutation groups and schemes which will be used in this paper. For more details, see [3, 5, 10].

Let $G \leq \mathrm{Sym}(V)$ be a permutation group and let V^m $(m > 1)$ denote the Cartesian product of m copies of V. Then G acts on V in a natural way, namely,

$$(v_1, \ldots, v_m)^g = (v_1^g, \ldots, v_m^g) \quad \text{for all } v_i \in V \text{ and } g \in G.$$

The above action partitions V^m into mutually disjoint classes, and each class is called an *m-orbit* of G on V. The set of all m-orbits of G is denoted by $\mathrm{Orb}_m(G)$. Moreover, we set $\mathrm{Orb}_1(G) = \mathrm{Orb}(G)$. The 2-orbits of G on V are called the *orbitals* of G.

Let $\mathcal{R}$ be a set of binary relations on V. The set of all permutations of V that preserve each relation of $\mathcal{R}$ forms a group called the *automorphism group* of $\mathcal{R}$, and it is denoted by $\mathrm{Aut}(\mathcal{R})$. For any permutation group G on V, the group $\mathrm{Aut}(\mathrm{Orb}_m(G))$, denoted by $G^{(m)}$, is called the *m-closure* of G. It is easy to see that $G \leq G^{(m)}$. We say that G is *m-closed* if $G^{(m)} = G$. A *base* of a permutation group G is a set of points whose pointwise stabilizer is trivial. The minimum cardinality of a base of the group G is called the *base number* of G and it is denoted by $b(G)$. It is not difficult to check that G is $(b(G)+1)$-closed.

A *coherent configuration* or a *scheme* $\mathcal{C} = (V, \mathcal{R})$ consists of a finite set V and a partition $\mathcal{R}$ of V^2 such that $\mathcal{R}$ is closed with respect to transposition, $\mathcal{R}^*$ contains the diagonal $\Delta(V)$ and the number $c_{R,S}^T = |\{v \in V : \ (u,v) \in R, (v,w) \in S\}|$ does not depend on the choice of $(u,w) \in T$ for all $R, S, T \in \mathcal{R}$. We refer to V and the elements of $\mathcal{R}$ as the set of points and the *basis relations*, respectively. The numbers $|V|$ and $|\mathcal{R}|$ are called the *order* and the *rank* of $\mathcal{C}$, respectively. Any set $X \subseteq V$ with $\Delta(X) \in \mathcal{R}$ is called a *fiber* of $\mathcal{C}$ and the set of all fibers of $\mathcal{C}$ and the set of all fibers of $\mathcal{C}$ with more than one point are denoted by $\mathrm{Fib}(\mathcal{C})$ and $\overline{\mathrm{Fib}}(\mathcal{C})$, respectively. Any fiber with exactly one point is called a *singleton fiber* and clearly, the set of points is the disjoint union of fibers. If X is a union of fibers, then the *restriction* of $\mathcal{C}$ to X is defined as the scheme $\mathcal{C}_X = (X, \mathcal{R}_X)$.

A nonempty subset U of $\mathcal{R}$ is called *closed* if the set $\{T \in \mathcal{R} : \ c^T_{R^t,S} \neq 0, \ R, S \in U\}$ is contained in U. An element R of $\mathcal{R}$ is called *thin* if the set $\{T \in \mathcal{R} : \ c^T_{R^t,R} \neq 0\}$ contains exactly one element. The set $O_\vartheta(\mathcal{C})$ of all thin relations of $\mathcal{C}$ is called the *thin radical* of $\mathcal{C}$. One can see that $O_\vartheta(\mathcal{C})$ is closed and contains $\Delta(X)$ for each $X \in \mathrm{Fib}(\mathcal{C})$. Let $O^\vartheta(\mathcal{C})$ be the smallest closed subset of $\mathcal{R}$ that contains RR^t for any $R \in \mathcal{R}$. Then $O^\vartheta(\mathcal{C})$ is called the *thin residue* of $\mathcal{C}$. For arbitrary prime number p, the scheme $\mathcal{C}$ is called a *p-scheme* if the cardinality of each basis relation of $\mathcal{C}$ is a power of p. The algebraic properties of p-schemes were studied in [8].

In the following example, we see that how a class of schemes are obtained from a permutation group.

Example 1.1. *Let $G \leq \mathrm{Sym}(V)$ be a permutation group. Then G acts on V^2 in a natural way. It is well-known that $(V, \mathrm{Orb}_2(G))$ is a scheme and it is denoted by $\mathrm{Inv}(G, V)$. Also, we have $\mathrm{Fib}(\mathrm{Inv}(G, V)) = \mathrm{Orb}(G)$.*

A scheme obtained from a permutation group is called *Schurian*. The Schurian schemes of a regular permutation group and $\mathrm{Inv}(\mathrm{id}_V, V)$ are called *regular* and *trivial* schemes, respectively.

Let $\mathcal{C} = (V, \mathcal{R})$ be a scheme and $U = \{u_1, \ldots, u_m\}$ be a subset of V. Let $\mathcal{R}_{(U)}$ be the set of the basis relations of the smallest scheme on V which contains $\mathcal{R}$ and $R_i = \{(u_i, u_i)\}$ for $i = 1, \ldots, m$. Then the scheme $\mathcal{C}_{(U)} = (V, \mathcal{R}_{(U)})$ is called an *m-point extension* of $\mathcal{C}$. Obviously, this extension is nontrivial if and only if $\{u_i\} \notin \mathrm{Fib}(\mathcal{C})$ for at least one i. The *base number* $b(\mathcal{C})$ of a scheme $\mathcal{C}$ is the minimal integer m such that the m-point extension of $\mathcal{C}$ is trivial. The base number of a trivial scheme is 0 as well as the base number of a scheme with rank 2 is $\deg(C) - 1$.

Let $\mathcal{C} = (V, \mathcal{R})$ be a scheme. A point $v \in V$ is called *regular* if $|R^t(v)| \leq 1$ for all $R \in \mathcal{R}$. If the set of all regular points of $\mathcal{C}$ is nonempty, then $\mathcal{C}$ is called *1-regular*. One can see that the set of all regular points of any scheme is a union of fibers. It is shown in [4, Theorem 9.3] that any 1-regular scheme is Schurian. For each given regular point $v \in V$, the scheme $\mathcal{C}_{(v)}$ is trivial. So the base number of any 1-regular scheme is at most 1.

Let $\mathcal{C}_1 = (V_1, \mathcal{R}_1)$ and $\mathcal{C}_2 = (V_2, \mathcal{R}_2)$ be two schemes. The *direct sum* of $\mathcal{C}_1$ and $\mathcal{C}_2$ is the scheme $\mathcal{C}_1 \boxplus \mathcal{C}_2 = (V, \mathcal{R})$, where V is a disjoint union of the sets V_1 and V_2 and $\mathcal{R}$ is the union of the sets $\mathcal{R}_1$, $\mathcal{R}_2$ and the set of all relations $X_1 \times X_2$ and $X_2 \times X_1$ with $X_i \in \mathrm{Fib}(\mathcal{C}_i)$ $(i = 1, 2)$.

Two given schemes $\mathcal{C}_1 = (V_1, \mathcal{R}_1)$ and $\mathcal{C}_2 = (V_2, \mathcal{R}_2)$ are called *similar* if

$$c^T_{R,S} = c^{T^\varphi}_{R^\varphi, S^\varphi}, \quad \text{for all } R, S, T \in \mathcal{C}_1,$$

where $R \overset{\varphi}{\longmapsto} R^\varphi$ is a bijection from $\mathcal{R}_1$ to $\mathcal{R}_2$. Two schemes $\mathcal{C}_1 = (V_1, \mathcal{R}_1)$ and $\mathcal{C}_2 = (V_2, \mathcal{R}_2)$ are said to be *isomorphic* if there exists a bijection $f : V_1 \to V_2$ such that $\mathcal{R}_1^f = \mathcal{R}_2$, where $\mathcal{R}_1^f = \{R^f \in \mathcal{R}_2 : \ R \in \mathcal{R}_1\}$ and $R^f = \{(u^f, v^f) : \ (u, v) \in R\}$. The bijection f is called an isomorphism from $\mathcal{C}_1$ to $\mathcal{C}_2$. The set of all isomorphisms from $\mathcal{C}_1$ to $\mathcal{C}_2$ is denoted by $\mathrm{Iso}(\mathcal{C}_1, \mathcal{C}_2)$. The set $\mathrm{Iso}(\mathcal{C}) = \mathrm{Iso}(\mathcal{C}, \mathcal{C})$ is obviously a permutation group on V. Also, every $f \in \mathrm{Iso}(\mathcal{C}_1, \mathcal{C}_2)$ induces a bijection from $\mathrm{Fib}(\mathcal{C}_1)$ onto $\mathrm{Fib}(\mathcal{C}_2)$ such that $\mathrm{Fib}(\mathcal{C}_1)^f = \mathrm{Fib}(\mathcal{C}_2)$. The group $\mathrm{Aut}(\mathcal{R})$ is called the *automorphism group* of the scheme $\mathcal{C}$ and is denoted by $\mathrm{Aut}(\mathcal{C})$. Obviously, $\mathrm{Aut}(\mathcal{C})$ is a normal subgroup of $\mathrm{Iso}(\mathcal{C})$.

2. Ferrero pair schemes

Let $\mathcal{C} = (V, \mathcal{R})$ be a scheme satisfying the following conditions:

($*$) The set $\mathrm{Fib}(\mathcal{C})$ contains exactly one singleton subset $Z = \{z\}$ of V and all of the nonsingleton fibers have the same cardinality. Also, for each $X, Y \in \overline{\mathrm{Fib}}(\mathcal{C})$, $X \times Z$ and $Z \times Y$ are basis relations of $\mathcal{C}$. Moreover, if $|R(z)| = |R^t(z)| = 0$, then R is a thin relation, where $R \in \mathcal{R}$.

Lemma 2.1. *Let $\mathcal{C} = (V, \mathcal{R})$ be a scheme satisfying ($*$). Then $\mathcal{C}$ is a Schurian scheme.*

Proof. It follows from ($*$) that $|R^t(v)| \leq 1$ for all $v \in V \setminus Z$ and $R \in \mathcal{R}$. Therefore, $\mathcal{C}$ is a 1-regular scheme and so $b(\mathcal{C}) \leq 1$. By applying [4, Theorem 9.3], $\mathcal{C}$ is a Schurian scheme and consequently, there exists a group $G \leq \mathrm{Sym}(V)$ such that $\mathcal{C} = \mathrm{Inv}(G, V)$. This completes the proof. □

Theorem 2.2. *Let $\mathcal{C} = \mathrm{Inv}(H, N)$ be a Ferrero pair scheme and let $\mathcal{R}$ be the set of all orbitals of H on N. Then the following statements hold:*

(1) $(|H|, |N|) = 1$ *and* $|H|$ *divides* $|N| - 1$.

(2) $\mathrm{Fib}(\mathcal{C})$ *contains exactly one singleton fiber and all of its nonsingleton fibers have the same cardinality. Moreover,* $|R| = 1$ *or* $|R| = |H|$ *for all* $R \in \mathcal{R}$.

(3) $\mathcal{C}$ *satisfies* ($*$).

(4) $\mathcal{C}_X$ *is a thin scheme and* $|\mathcal{R}_{X,Y}| = |H|$ *for all* $X, Y \in \overline{\mathrm{Fib}}(\mathcal{C})$.

(5) $O^{\vartheta}(\mathcal{C}) \subset O_{\vartheta}(\mathcal{C})$.

Proof. It is well-known that if $H \ltimes N$ is a Frobenius group, then $(|H|, |N|) = 1$ and $|H|$ divides $|N| - 1$ (see [6, Lemma 16.6]).

Since $\mathrm{Fib}(\mathcal{C}) = \mathrm{Orb}(H)$, it follows that $Z = \{1_N\}$ is the only fiber of cardinality 1 and other fibers have cardinality $|H|$. Obviously, H acts fixed point free on its orbitals of length greater than one, because otherwise there exist $R \in \mathcal{R}$ and $h \in H$ such that $(x, y) \in R$ and $(x, y)^h = (x, y)$, which contradicts to the fact that H acts fixed point free on $N^* = N \setminus \{1_N\}$. Thus, we obtain the equality $|R| = |H|$ for $R \neq \Delta(Z)$, which proves statement (2).

To prove statement (3), suppose that $R \in \mathcal{R}$ and let $1_N \in R(x)$ for some $x \in N^*$. Then

$$R = \{(x^h, 1_N) : \ h \in H\} = X \times Z,$$

where $X \in \overline{\mathrm{Fib}}(\mathcal{C})$ and $x \in X$. Similarly, if $1_N \in R^t(x)$, then $R = Z \times Y$ for a unique $Y \in \overline{\mathrm{Fib}}(\mathcal{C})$. Now suppose that $R \in \mathcal{R}$ and $R \subseteq X \times Y$ for some $X, Y \in \overline{\mathrm{Fib}}(\mathcal{C})$. If $|R(u)| > 1$ for some $u \in X$, then there exist $x, y \in Y$ and $h \in H$ such that $\{x, y\} \subseteq R(u)$ and $(u, x)^h = (u, y)$. So the element $h \in H$ has the fixed point u, which is a contradiction. Hence $|R(u)| = 1$. Similarly, $|R^t(u)| = 1$ for all $u \in X$, which shows that R is thin and so $\mathcal{C}$ satisfies ($*$).

Next, suppose that $X, Y \in \overline{\mathrm{Fib}}(\mathcal{C})$ and $R \subseteq X \times Y$. Then, the statement (2) implies that

$$|H| = |Y| = \sum_{R \in \mathcal{R}_{X,Y}} |R(u)| = \sum_{R \in \mathcal{R}_{X,Y}} 1 = |\mathcal{R}_{X,Y}|$$

for all $u \in X$, which proves statement (4).

Finally, let $R \in \mathcal{R}_{X,Y}$ for some $X, Y \in \mathrm{Fib}(\mathcal{C})$. Clearly, RR^t equals to $|X|\Delta(X)$ if $X = Z$ and $\Delta(X)$ if $X \neq Z$. Furthermore, it is straightforward to verify that

$$O^{\vartheta}(\mathcal{C}) = \{\Delta(X) : \; X \in \mathrm{Fib}(\mathcal{C})\} \subset O_{\vartheta}(\mathcal{C}).$$

Hence the proof of the theorem is complete. □

One may deduce the following remark from Lemma 2.1 and Theorem 2.2. This remark implies that the Ferrero pair schemes of the same rank are all similar.

Remark 2.3. *Let $\mathcal{C} = \mathrm{Inv}(H, N)$ be a Ferrero pair scheme and consider the cyclic permutation $i_X \in \mathrm{Sym}(X)$ of length $|X| = |H|$. Let G be the following permutation group:*

$$G = \langle \prod_{X \in \overline{\mathrm{Fib}}(\mathcal{C})} i_X \rangle \leq \mathrm{Sym}(N).$$

In fact, G is a permutation group of order $|H|$ which is generated by a product of disjoint cycles of length $|H|$. Obviously, $\mathrm{Inv}(G, N)$ satisfies $(*)$. So for each Ferrero pair scheme $\mathcal{C}$, there exists a cyclic permutation group G such that $\mathrm{Inv}(G, N)$ is similar to $\mathcal{C}$. Now, it is straightforward to show that the base number of G is at most 1. Since the permutation group G is $(b(G)+1)$-closed, we conclude that G is 2-closed. In other words, $\mathrm{Aut}(\mathrm{Inv}(G, N)) = G$.

Theorem 2.4. *Let $\mathrm{Inv}(H, N)$ be a Ferrero pair scheme and K be a finite group acting on N with $\mathrm{Inv}(H, N) = \mathrm{Inv}(K, N)$. Then $\mathrm{Inv}(K, N)$ is a Ferrero pair scheme and $|H| = |K|$.*

Proof. Let $\mathcal{C} = \mathrm{Inv}(H, N)$. First we show that $\mathrm{Aut}(\mathcal{C})$ acts fixed point free on N. Suppose on the contrary that $x^g = x$ for some $g \in \mathrm{Aut}(\mathcal{C}) \setminus \{\mathrm{id}_N\}$ and $x \in N^* = N \setminus \{1_N\}$. Let y be an arbitrary element in N^*. Then there exists a unique orbital R of $\mathcal{C}$ which contains (x, y). According to the choice of g, we have $(x^g, y^g) = (x, y^g) \in R$, and so there exists $h \in H$ such that $(x, y)^h = (x, y^g)$. Since H acts fixed point free on N, so $h = 1_N$. Hence, $y^g = y$. Then g is a trivial element of $\mathrm{Aut}(\mathcal{C})$ which contradicts the choice of $g \in \mathrm{Aut}(\mathcal{C}) \setminus \{\mathrm{id}_N\}$. By the hypothesis, one can see that

$$K \leq K^{(2)} = \mathrm{Aut}(\mathrm{Inv}(K, N)) = \mathrm{Aut}(\mathcal{C}),$$

which implies that K acts fixed point free on N. Hence $\mathcal{C}' = \mathrm{Inv}(K, N)$ is a Ferrero pair scheme. Clearly, $\mathrm{Fib}(\mathcal{C})$ and $\mathrm{Fib}(\mathcal{C}')$ have the same cardinality, say d. Since H and K act fixed point free on N, it follows that $1 + (d-1)|H| = |N| = 1 + (d-1)|K|$ and therefore, $|H| = |K|$. □

Corollary 2.5. *Suppose that H and K are finite groups satisfying the hypotheses of Theorem 2.4. If $\mathrm{Aut}(\mathrm{Inv}(H, N)) = H$, then $K \cong H$.*

Proof. Since $\mathrm{Aut}(\mathrm{Inv}(H, N)) = H$, therefore, $K \leq \mathrm{Aut}(\mathrm{Inv}(K, N)) = \mathrm{Aut}(\mathrm{Inv}(H, N)) = H$. To complete the proof, we note that $|H| = |K|$, by Theorem 2.4. □

Theorem 2.6. *Let $\mathrm{Inv}(H, N)$ be a Ferrero pair scheme and K be a group acting fixed point free on N. Suppose that H and K satisfy one of the following conditions:*

(1) *H is a p-group for $p > 2$;*

(2) *H is a square free order group;*

(3) *$H \ltimes N$ and $K \ltimes N$ are isomorphic.*

If $\operatorname{Inv}(H,N) = \operatorname{Inv}(K,N)$*, then* $K \cong H$*.*

Proof. Since $\operatorname{Inv}(H,N)$ is a Ferrero pair scheme, it follows that $H \ltimes N$ is a Frobenius group with complement H and kernel N. By considering [7, Theorem 18.1], we observe that for each odd prime p, the Sylow p-subgroups of H are cyclic. Also, if H is square free order acting on N, then by [9, Theorem 6.1.11], H is cyclic. Hence if (1) or (2) holds, H is 2-closed and so $\operatorname{Aut}(\operatorname{Inv}(H,N)) = H$. Moreover, by Corollary 2.5, $K \cong H$.

If (3) holds, then we know that $H \ltimes N \cong K \ltimes N$ if and only if there exists $g \in \operatorname{Aut}(N)$ such that $K = H^g$ and the proof of the theorem is complete. □

Remark 2.7. *Let $\mathcal{C}$ be a Ferrero pair scheme satisfying (1) or (2) of Theorem 2.6. Then $\mathcal{C}$ is isomorphic to the scheme* $\operatorname{Inv}(G,N)$ *introduced in Remark 2.3. Let p be a prime number. Then by* [8, Corollary 1.2]*,* $\operatorname{Inv}(H,N)$ *is a p-scheme if and only if H is a p-group. In particular, for $p > 2$, the Ferrero pair scheme* $\operatorname{Inv}(H,N)$ *is a p-scheme if and only if H is a cyclic p-group.*

The following example shows that the condition $p > 2$ is necessary in Theorem 2.6.

Example 2.8. *We know that there are two nonisomorphic Frobenius groups of order* 72*. Let G be a Frobenius group of order* 72*. Then G is a semidirect product of the elementary abelian group $N = C_3 \times C_3$ of order* 9 *and a group H of order* 8 *acting fixed point free on $N \setminus \{1_N\}$. The group H is cyclic or quaternion. Now consider the following permutation subgroups of S_9:*

$$H = \langle (2\ 4\ 5\ 8\ 3\ 7\ 9\ 6) \rangle, \qquad K = \langle (2\ 4\ 3\ 7)(5\ 6\ 9\ 8), (2\ 9\ 3\ 5)(4\ 6\ 7\ 8) \rangle,$$

$$N = \langle (1\ 2\ 3)(4\ 5\ 6)(7\ 8\ 9), (1\ 9\ 5)(2\ 7\ 6)(3\ 8\ 4) \rangle.$$

Then H and K act on N by conjugation and it is straightforward to show that $H \cong C_8$ and $K \cong Q_8$, where Q_8 is the quaternion group of order 8*. Now it is easy to see that* $\operatorname{Inv}(C_8,N) = \operatorname{Inv}(Q_8,N)$*, whereas $C_8 \not\cong Q_8$.*

Theorem 2.9. *For each prime number p, there exists a Ferrero pair p-scheme. Also, if $p > 2$, then the Ferrero pair p-schemes of the same rank are all isomorphic.*

Proof. Let p be a prime number and H be a cyclic group of order p^n for some positive integer n. From [2, Corollary 3.3], we can construct a Frobenius group with Frobenius complement H. Now by Remark 2.7, the Ferrero pair scheme corresponding to this Frobenius group is a p-scheme.

Next, let p be an odd prime and $\mathcal{C}_1 = \operatorname{Inv}(H_1,N_1)$ and $\mathcal{C}_2 = \operatorname{Inv}(H_2,N_2)$ be two Ferrero pair p-schemes with the same rank. Suppose that $X_1 \in \overline{\operatorname{Fib}}(\mathcal{C}_1)$ and $X_2 \in \overline{\operatorname{Fib}}(\mathcal{C}_2)$. Then by Theorem 2.2, $|\mathcal{R}_{X_i}| = |H_i|$ and the restriction of $\mathcal{C}_i$ to X_i is a thin scheme for $i = 1,2$. Hence $|H_1| = |H_2|$, since otherwise $\mathcal{C}_1$ and $\mathcal{C}_2$ have different ranks which contradicts the hypothesis of the theorem. Since H_1 and H_2 are the complements of the Frobenius groups $H_1 \ltimes N_1$ and $H_2 \ltimes N_2$, respectively, it follows by [7, Theorem 18.1] that they are cyclic groups. Therefore, H_1 and H_2 are isomorphic to each other and this isomorphism induces an isomorphism from $\mathcal{C}_1$ to $\mathcal{C}_2$. Consequently, $\mathcal{C}_1$ is isomorphic to $\mathcal{C}_2$. □

Lemma 2.10. *Let $\mathcal{C} = \mathrm{Inv}(H, N)$ be a Ferrero pair scheme and $\mathcal{C}_1$ and $\mathcal{C}_2$ be regular schemes of order 1 and $|H|$, respectively. Then $\mathcal{C} = \mathcal{C}_1 \boxplus \mathcal{C}_2$ if and only if $H \ltimes N$ is a 2-transitive Frobenius group.*

Proof. Let $G \leq \mathrm{Sym}(V)$ be a Frobenius group with complement H and kernel K. Then G is a transitive permutation group in which any two-point stabilizer is trivial. On the other hand, $H = G_x$ for some $x \in V$. Now assume that $G = H \ltimes N$ is a 2-transitive Frobenius group. Since H acts transitively on $V \setminus \{x\}$, it follows that $|H| = |V| - 1$. Then equality $|N| = |G : H| = |G : G_x| = |V|$ shows that $|H| = |N| - 1$. This implies that $\mathcal{C} = \mathrm{Inv}(H, N)$ has two fibers, say $\mathrm{Fib}(\mathcal{C}) = \{1_N, X\}$. Then Theorem 2.1 implies that $\mathcal{C}_X$ is similar to a regular scheme of order $|X| = |H|$. Also, the other basis relations of $\mathcal{C}$ are $R_0 = \{(1_N, 1_N)\}$, $R_1 = 1_N \times X = \{(1_N, x) : x \in X\}$, and R_1^t. Hence $\mathcal{C}$ is a direct sum of two regular schemes of orders 1 and $|H|$.

Conversely, if $\mathcal{C}$ is a direct sum of a regular scheme of order 1 and a regular scheme of order $|H|$, then $\mathrm{Fib}(\mathcal{C})$ consists of exactly two fibers such that one of them is singleton, and the other one has cardinality $|H|$. Thus, we obtain the equality $|N| = 1 + |H|$ and this implies that $|G| = |H||N| = |H| + |H|^2$. Now, we assume that g is an element of $G \setminus H$. Then H and HgH are disjoint double cosets of G. Since H is a Frobenius complement, we get $H \cap H^g = \{1_H\}$. Therefore, we obtain

$$|H| + |HgH| = |H| + \frac{|H||H^g|}{|H \cap H^g|} = |H| + |H|^2 = |G|,$$

which yields $G = H \cup HgH$. From [7, Proposition 3.7], G is 2-transitive if and only if $G = H \cup HgH$ for every $g \in G \setminus H$. This completes the proof. □

Theorem 2.11. *Let $\mathcal{C} = \mathrm{Inv}(H, N)$ and $\mathcal{C}' = \mathrm{Inv}(K, N)$ be two Ferrero pair schemes and let $f \in \mathrm{Iso}(\mathcal{C}, \mathcal{C}')$. Then $K^f = fKf^{-1} = H$.*

Proof. Let $x \in N$. Then $X = x^H$ is an orbit containing x. Since $X \in \mathrm{Fib}(\mathcal{C})$, it follows that $X^f \in \mathrm{Fib}(\mathcal{C}') = \mathrm{Orb}(K)$. Also, $(x^f)^K \cap X^f \neq \emptyset$ and so we have $X^f = (x^f)^K$. On the other hand, $(x^H)^f = (x^f)^K$, which means that the permutation actions of fKf^{-1} and H on N are equivalent. This completes the proof. □

Remark 2.12. *Let $\mathcal{C} = \mathrm{Inv}(H, N)$ be a Ferrero pair scheme and let $f \in \mathrm{Iso}(\mathcal{C})$. Then for each $x \in N \setminus \{1_N\}$, there exists a unique $\lambda_x \in H$ depending on x such that $(x^h)^f = (x^f)^{\lambda_x}$. Also, the set $\mathrm{Iso}(\mathcal{C})$ is obviously a permutation group on N and so $f^{-1} \in \mathrm{Iso}(\mathcal{C})$. Now by Theorem 2.11, $f^{-1}hf \in H$. Hence*

$$(x^f)^{\lambda_x} = (x^h)^f = (x^f)^{f^{-1}hf}.$$

Finally, since H acts fixed point free on N, it follows that $\lambda_x = f^{-1}hf$. Consequently, λ_x is independent of x.

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