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MODULAR CHROMATIC NUMBER OF $C_m \square P_n$

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ABSTRACT. A modular k -coloring, $k \geq 2$, of a graph G is a coloring of the vertices of G with the elements in \mathbb{Z}_k having the property that for every two adjacent vertices of G , the sums of the colors of their neighbors are different in \mathbb{Z}_k . The minimum k for which G has a modular k -coloring is the modular chromatic number of G . Except for some special cases, modular chromatic number of $C_m \square P_n$ is determined.

1. Introduction

For a vertex v of a graph G , let $N_G(v)$, the *neighborhood of v* , denote the set of vertices adjacent to v in G . For a graph G , let $c : V(G) \rightarrow \mathbb{Z}_k$, $k \geq 2$, be a vertex coloring of G where adjacent vertices may be colored the same. The *color sum* $\sigma(v) = \sum_{u \in N_G(v)} c(u)$ of a vertex v of G is the sum of the colors of the vertices in $N_G(v)$. The coloring c is called a *modular k -coloring* of G if $\sigma(x) \neq \sigma(y)$ in \mathbb{Z}_k for all pairs x, y of adjacent vertices in G . The *modular chromatic number* $mc(G)$ of G is the minimum k for which G has a modular k -coloring. This concept was introduced by Okamoto, Salehi and Zhang [2].

The *Cartesian product* $G \square H$ of two graphs G and H has $V(G \square H) = V(G) \times V(H)$, and two vertices (u_1, u_2) and (v_1, v_2) of $G \square H$ are adjacent if and only if either $u_1 = v_1$ and $u_2 v_2 \in E(H)$ or $u_2 = v_2$ and $u_1 v_1 \in E(G)$.

Okamoto, Salehi and Zhang proved in [2] that: every nontrivial connected graph G has a modular k -coloring for some integer $k \geq 2$ and $mc(G) \geq \chi(G)$, where $\chi(G)$ denotes the chromatic number of G ; for the cycle C_n of length n , $mc(C_n)$ is 2 if $n \equiv 0 \pmod{4}$ and it is 3 otherwise; every nontrivial tree has modular chromatic number 2 or 3; for the complete multipartite graph G , $mc(G) = \chi(G)$;

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for the Cartesian product $G = K_r \square K_2$, $mc(G)$ is r if $r \equiv 2 \pmod 4$ and it is $r + 1$ otherwise; for the wheel $W_n = C_n \vee K_1$, $n \geq 3$, $mc(W_n) = \chi(W_n)$, where \vee denotes the join of two graphs; for $n \geq 3$, $mc(C_n \vee K_2^c) = \chi(C_n \vee K_2^c)$, where G^c denotes the complement of G ; and for $n \geq 2$, $mc(P_n \vee K_2) = \chi(P_n \vee K_2)$, where P_n denotes the path of length $n - 1$; and in [3] that: for $m, n \geq 2$, $mc(P_m \square P_n) = 2$.

In this paper, except for some special cases, we compute $mc(C_m \square P_n)$.

2. Result

For the path P_ν on ν vertices, and the cycle C_ν on ν vertices, let $V(P_\nu) = V(C_\nu) = \{1, 2, \dots, \nu\}$, $E(P_\nu) = \{\{i, i + 1\} : i \in \{1, 2, \dots, \nu - 1\}\}$ and $E(C_\nu) = E(P_\nu) \cup \{\{\nu, 1\}\}$. For $m \geq 3$ and $n \geq 2$, $\chi(C_m \square P_n)$ is 2 if m is even and it is 3 if m is odd. In [2], Okamoto, Salehi and Zhang proved that: For every positive integer r , $r \leq mc(K_r \square K_2) \leq r + 1$, and $mc(K_r \square K_2) = r$ if and only if $r \equiv 2 \pmod 4$. Consequently, we have $mc(C_3 \square P_2) = 4$.

Theorem. Let $m \geq 3$ and $n \geq 2$.

(i) $mc(C_3 \square P_2) = 4$.

(ii) If neither

$$m = 3 \text{ and } n \in \{2, 14, 26, 38, \dots, 12r + 2, \dots\} \cup \{16, 28, 40, \dots, 12r + 4, \dots\} \cup \{8, 20, 32, \dots, 12r + 8, \dots\} \cup \{22, 34, \dots, 12r + 10, \dots\},$$

nor

$$m \equiv 2 \pmod 4 \text{ and } n \equiv 1 \pmod 4,$$

then

$$mc(C_m \square P_n) = \chi(C_m \square P_n).$$

(iii) If $m \equiv 2 \pmod 4$ and $n \equiv 1 \pmod 4$, then $mc(C_m \square P_n) \leq 3$.

(iv) If $n \equiv 1 \pmod 4$, then $mc(C_6 \square P_n) = 3$.

Proof. (ii): First we consider $n = 2$; i.e., $C_m \square P_2$. If m is even, then the result follows from Proposition 3.6 (If G is a bipartite graph the degrees of whose vertices are of the same parity, then $mc(G \square K_2) = 2$.) of [2]. By hypothesis, $m \neq 3$. Hence, assume that $m \geq 5$ is odd. Define $c : V(C_m \square P_2) \rightarrow \mathbb{Z}_3$ as follows: $c((i, 1)) = 1$ if $i \in \{1, 3, 5, \dots, m - 4\}$; $c((m - 1, 2)) = 2$; $c((i, j)) = 0$ otherwise.

Then, $\sigma((i, 1)) = 0$ if $i \in \{1, 3, 5, \dots, m - 2\}$;

$$\sigma((m - 3, 1)) = \sigma((m, 1)) = 1;$$

$$\sigma((i, 1)) = 2 \text{ if } i \in \{2, 4, 6, \dots, m - 5\};$$

$$\sigma((m - 1, 1)) = 2;$$

$$\sigma((i, 2)) = 0 \text{ if } i \in \{2, 4, 6, \dots, m - 1\};$$

$$\sigma((i, 2)) = 1 \text{ if } i \in \{1, 3, 5, \dots, m - 4\};$$

$$\sigma((m - 2, 2)) = \sigma((m, 2)) = 2.$$

Hence, c is a modular 3-coloring of $C_m \square P_2$.

This completes the proof for $n = 2$.

Next, we consider $m = 3$; i.e., $C_3 \square P_n$. We prove the result by considering the following 4 cases.

Case 1. $n \equiv 1 \pmod 4$.

Define $c : V(C_3 \square P_n) \rightarrow \mathbb{Z}_3$ as follows:

$c((i, j)) = 1$ if $(i, j) \in (\{1\} \times \{3, 7, 11, \dots, n - 2\}) \cup (\{2\} \times \{1, 5, 9, \dots, n\})$; $c((i, j)) = 2$ if $(i, j) \in (\{2\} \times \{3, 7, 11, \dots, n - 2\}) \cup (\{3\} \times \{1, 5, 9, \dots, n\})$; and $c((i, j)) = 0$ otherwise.

Then, $\sigma((i, j)) = 0$ if $(i, j) \in (\{1\} \times \{1, 5, 9, \dots, n\}) \cup (\{2\} \times \{2, 4, 6, \dots, n - 1\}) \cup (\{3\} \times \{3, 7, 11, \dots, n - 2\})$; $\sigma((i, j)) = 1$ if $(i, j) \in (\{1\} \times \{2, 4, 6, \dots, n - 1\}) \cup (\{2\} \times \{3, 7, 11, \dots, n - 2\}) \cup (\{3\} \times \{1, 5, 9, \dots, n\})$; $\sigma((i, j)) = 2$ if $(i, j) \in (\{1\} \times \{3, 7, 11, \dots, n - 2\}) \cup (\{2\} \times \{1, 5, 9, \dots, n\}) \cup (\{3\} \times \{2, 4, 6, \dots, n - 1\})$. Hence, c is a modular 3-coloring of $C_3 \square P_n$. See Table 1 for colors of the vertices and color sums of the vertices for $C_3 \square P_n$.

color c			color sum σ		
0 0 1 0	0 0 1 0...0 0 1 0	0	0 1 2 1	0 1 2 1...0 1 2 1	0
1 0 2 0	1 0 2 0...1 0 2 0	1	2 0 1 0	2 0 1 0...2 0 1 0	2
2 0 0 0	2 0 0 0...2 0 0 0	2	1 2 0 2	1 2 0 2...1 2 0 2	1

Table 1. $C_3 \square P_n$ ($n \equiv 1 \pmod 4$).

Case 2. $n \equiv 3 \pmod 4$.

Define $c : V(C_3 \square P_n) \rightarrow \mathbb{Z}_3$ as follows:

$c((i, j)) = 1$ if $(i, j) \in (\{1\} \times \{3, 7, 11, \dots, n\}) \cup (\{2\} \times \{1, 5, 9, \dots, n - 2\})$; $c((i, j)) = 2$ if $(i, j) \in (\{2\} \times \{3, 7, 11, \dots, n\}) \cup (\{3\} \times \{1, 5, 9, \dots, n - 2\})$; and $c((i, j)) = 0$ otherwise.

Then, $\sigma((i, j)) = 0$ if $(i, j) \in (\{1\} \times \{1, 5, 9, \dots, n - 2\}) \cup (\{2\} \times \{2, 4, 6, \dots, n - 1\}) \cup (\{3\} \times \{3, 7, 11, \dots, n\})$; $\sigma((i, j)) = 1$ if $(i, j) \in (\{1\} \times \{2, 4, 6, \dots, n - 1\}) \cup (\{2\} \times \{3, 7, 11, \dots, n\}) \cup (\{3\} \times \{1, 5, 9, \dots, n - 2\})$; $\sigma((i, j)) = 2$ if $(i, j) \in (\{1\} \times \{3, 7, 11, \dots, n\}) \cup (\{2\} \times \{1, 5, 9, \dots, n - 2\}) \cup (\{3\} \times \{2, 4, 6, \dots, n - 1\})$. Hence, c is a modular 3-coloring of $C_3 \square P_n$. See Table 2 for colors of the vertices and color sums of the vertices for $C_3 \square P_n$.

color c			color sum σ		
0 0 1 0	0 0 1 0...0 0 1 0	0 0 1	0 1 2 1	0 1 2 1...0 1 2 1	0 1 2
1 0 2 0	1 0 2 0...1 0 2 0	1 0 2	2 0 1 0	2 0 1 0...2 0 1 0	2 0 1
2 0 0 0	2 0 0 0...2 0 0 0	2 0 0	1 2 0 2	1 2 0 2...1 2 0 2	1 2 0

Table 2. $C_3 \square P_n$ ($n \equiv 3 \pmod 4$).

Case 3. $n \equiv 0 \pmod 6$.

Define $c : V(C_3 \square P_n) \rightarrow \mathbb{Z}_3$ as follows:

$c((i, j)) = 1$ if $(i, j) \in (\{1\} \times \{3, 9, 15, \dots, n - 3\}) \cup (\{1\} \times \{4, 10, 16, \dots, n - 2\}) \cup (\{1\} \times \{5, 11, 17, \dots, n - 1\}) \cup (\{2\} \times \{2, 8, 14, \dots, n - 4\})$; $c((i, j)) = 2$ if $(i, j) \in (\{2\} \times \{5, 11, 17, \dots, n - 1\}) \cup (\{3\} \times \{2, 8, 14, \dots, n - 4\}) \cup (\{3\} \times \{3, 9, 15, \dots, n - 3\}) \cup (\{3\} \times \{4, 10, 16, \dots, n - 2\})$; and $c((i, j)) = 0$ otherwise. Then,

$$(\sigma((1, j)), \sigma((2, j)), \sigma((3, j))) = (0, 1, 2) \text{ if } j \in \{1, 3, 5, \dots, n - 1\};$$

$$(\sigma((1, j)), \sigma((2, j)), \sigma((3, j))) = (1, 2, 0) \text{ if } j \in \{2, 4, 6, \dots, n\}.$$

Hence, c is a modular 3-coloring of $C_3 \square P_n$. See Table 3 for colors of the vertices and color sums of the vertices for $C_3 \square P_n$.

color c		color sum σ	
0 0 1 1 1 0	0 0 1 1 1 0...0 0 1 1 1 0	0 1 0 1 0 1	0 1 0 1 0 1...0 1 0 1 0 1
0 1 0 0 2 0	0 1 0 0 2 0...0 1 0 0 2 0	1 2 1 2 1 2	1 2 1 2 1 2...1 2 1 2 1 2
0 2 2 2 0 0	0 2 2 2 0 0...0 2 2 2 0 0	2 0 2 0 2 0	2 0 2 0 2 0...2 0 2 0 2 0

Table 3. $C_3 \square P_n$ ($n \equiv 0 \pmod 6$).

Case 4. $n \in \{4, 10\}$.

See Tables 4 and 5 for colors of the vertices and color sums of the vertices for $C_3 \square P_4$ and $C_3 \square P_{10}$, respectively.

color c	color sum σ
0 1 1 1	1 0 1 0
1 0 0 2	2 1 2 1
2 2 2 0	0 2 0 2

Table 4. $C_3 \square P_4$.

color c	color sum σ
0 0 2 0 0 0 2 0 1 2	0 2 1 2 0 2 1 0 1 2
1 2 1 1 0 0 0 2 0 0	1 0 2 0 1 0 2 1 2 0
2 1 0 2 0 0 1 1 2 1	2 1 0 1 2 1 0 2 0 1

Table 5. $C_3 \square P_{10}$.

Finally, assume that $m \geq 4$ and $n \geq 3$. We consider the following 12 cases.

Case 1. $m \equiv 0 \pmod 2$ and $n \equiv 0 \pmod 4$.

Define $c : V(C_m \square P_n) \rightarrow \mathbb{Z}_2$ as follows: $c((i, j)) = 1$ if $(i, j) \in (\{1, 3, 5, \dots, m - 1\} \times \{3, 7, 11, \dots, n - 1\}) \cup (\{2, 4, 6, \dots, m\} \times \{2, 6, 10, \dots, n - 2\})$; and $c((i, j)) = 0$ otherwise.

Then, $\sigma((i, j)) = 1$ if $(i, j) \in (\{2, 4, 6, \dots, m\} \times \{1, 3, 5, \dots, n - 1\}) \cup (\{1, 3, 5, \dots, m - 1\} \times \{2, 4, 6, \dots, n\})$; and $\sigma((i, j)) = 0$ otherwise. Hence, c is a modular 2-coloring of $C_m \square P_n$. See

Table 6 for colors of the vertices and color sums of the vertices for $C_m \square P_n$.

color c		color sum σ	
0 0 1 0	0 0 1 0...0 0 1 0	0 1 0 1	0 1 0 1...0 1 0 1
0 1 0 0	0 1 0 0...0 1 0 0	1 0 1 0	1 0 1 0...1 0 1 0
0 0 1 0	0 0 1 0...0 0 1 0	0 1 0 1	0 1 0 1...0 1 0 1
0 1 0 0	0 1 0 0...0 1 0 0	1 0 1 0	1 0 1 0...1 0 1 0
⋮ ⋮ ⋮ ⋮	⋮ ⋮ ⋮ ⋮	⋮ ⋮ ⋮ ⋮	⋮ ⋮ ⋮ ⋮
0 0 1 0	0 0 1 0...0 0 1 0	0 1 0 1	0 1 0 1...0 1 0 1
0 1 0 0	0 1 0 0...0 1 0 0	1 0 1 0	1 0 1 0...1 0 1 0

Table 6. $C_m \square P_n$ ($m \equiv 0 \pmod 2$ and $n \equiv 0 \pmod 4$).

Case 2. $m \equiv 0 \pmod 2$ and $n \equiv 2 \pmod 4$.

Define $c : V(C_m \square P_n) \rightarrow \mathbb{Z}_2$ as follows: $c((i, j)) = 1$ if $(i, j) \in (\{1, 3, 5, \dots, m-1\} \times \{2, 6, 10, \dots, n\}) \cup (\{2, 4, 6, \dots, m\} \times \{1, 5, 9, \dots, n-1\})$; and $c((i, j)) = 0$ otherwise.

Then, $\sigma((i, j)) = 1$ if $(i, j) \in (\{1, 3, 5, \dots, m-1\} \times \{1, 3, 5, \dots, n-1\}) \cup (\{2, 4, 6, \dots, m\} \times \{2, 4, 6, \dots, n\})$; and $\sigma((i, j)) = 0$ otherwise. Hence, c is a modular 2-coloring of $C_m \square P_n$. See Table 7 for colors of the vertices and color sums of the vertices for $C_m \square P_n$.

color c				color sum σ			
0 1 0 0	0 1 0 0...	0 1 0 0	0 1	1 0 1 0	1 0 1 0...	1 0 1 0	1 0
1 0 0 0	1 0 0 0...	1 0 0 0	1 0	0 1 0 1	0 1 0 1...	0 1 0 1	0 1
0 1 0 0	0 1 0 0...	0 1 0 0	0 1	1 0 1 0	1 0 1 0...	1 0 1 0	1 0
1 0 0 0	1 0 0 0...	1 0 0 0	1 0	0 1 0 1	0 1 0 1...	0 1 0 1	0 1
⋮ ⋮ ⋮ ⋮	⋮ ⋮ ⋮ ⋮	⋮ ⋮ ⋮ ⋮	⋮ ⋮	⋮ ⋮ ⋮ ⋮	⋮ ⋮ ⋮ ⋮	⋮ ⋮ ⋮ ⋮	⋮ ⋮
0 1 0 0	0 1 0 0...	0 1 0 0	0 1	1 0 1 0	1 0 1 0...	1 0 1 0	1 0
1 0 0 0	1 0 0 0...	1 0 0 0	1 0	0 1 0 1	0 1 0 1...	0 1 0 1	0 1

Table 7. $C_m \square P_n$ ($m \equiv 0 \pmod 2$ and $n \equiv 2 \pmod 4$).

Case 3. $m \equiv 2 \pmod 4$ and $n \equiv 3 \pmod 4$.

Define $c : V(C_m \square P_n) \rightarrow \mathbb{Z}_2$ as follows: $c((i, j)) = 1$ if $(i, j) \in (\{1, 3, 5, \dots, m-1\} \times \{2, 6, 10, \dots, n-1\}) \cup (\{2, 4, 6, \dots, m\} \times \{1, 5, 9, \dots, n-2\})$; and $c((i, j)) = 0$ otherwise.

Then, $\sigma((i, j)) = 1$ if $(i, j) \in (\{1, 3, 5, \dots, m-1\} \times \{1, 3, 5, \dots, n\}) \cup (\{2, 4, 6, \dots, m\} \times \{2, 4, 6, \dots, n-1\})$; and $\sigma((i, j)) = 0$ otherwise. Hence, c is a modular 2-coloring of $C_m \square P_n$. See Table 8 for colors of the vertices and color sums of the vertices for $C_m \square P_n$.

color c				color sum σ			
0 1 0 0	0 1 0 0...	0 1 0 0	0 1 0	1 0 1 0	1 0 1 0...	1 0 1 0	1 0 1
1 0 0 0	1 0 0 0...	1 0 0 0	1 0 0	0 1 0 1	0 1 0 1...	0 1 0 1	0 1 0
0 1 0 0	0 1 0 0...	0 1 0 0	0 1 0	1 0 1 0	1 0 1 0...	1 0 1 0	1 0 1
1 0 0 0	1 0 0 0...	1 0 0 0	1 0 0	0 1 0 1	0 1 0 1...	0 1 0 1	0 1 0
⋮ ⋮ ⋮ ⋮	⋮ ⋮ ⋮ ⋮	⋮ ⋮ ⋮ ⋮	⋮ ⋮	⋮ ⋮ ⋮ ⋮	⋮ ⋮ ⋮ ⋮	⋮ ⋮ ⋮ ⋮	⋮ ⋮
0 1 0 0	0 1 0 0...	0 1 0 0	0 1 0	1 0 1 0	1 0 1 0...	1 0 1 0	1 0 1
1 0 0 0	1 0 0 0...	1 0 0 0	1 0 0	0 1 0 1	0 1 0 1...	0 1 0 1	0 1 0

Table 8. $C_m \square P_n$ ($m \equiv 2 \pmod 4$ and $n \equiv 3 \pmod 4$).

Case 4. $m \equiv 0 \pmod 4$ and $n \equiv 1 \pmod 2$.

Define $c : V(C_m \square P_n) \rightarrow \mathbb{Z}_2$ as follows:

For $n \equiv 1 \pmod 4$, $c((i, j)) = 1$ if $(i, j) \in (\{1, 5, 9, \dots, m-3\} \times \{1, 5, 9, \dots, n\}) \cup (\{3, 7, 11, \dots, m-1\} \times \{3, 7, 11, \dots, n-2\})$; and $c((i, j)) = 0$ otherwise.

For $n \equiv 3 \pmod 4$, $c((i, j)) = 1$ if $(i, j) \in (\{1, 5, 9, \dots, m-3\} \times \{1, 5, 9, \dots, n-2\}) \cup (\{3, 7, 11, \dots, m-1\} \times \{3, 7, 11, \dots, n\})$; and $c((i, j)) = 0$ otherwise.

Then, $\sigma((i, j)) = 1$ if $(i, j) \in (\{2, 4, 6, \dots, m\} \times \{1, 3, 5, \dots, n\}) \cup (\{1, 3, 5, \dots, m-1\} \times \{2, 4, 6, \dots, n-1\})$; and $\sigma((i, j)) = 0$ otherwise. Hence, c is a modular 2-coloring of $C_m \square P_n$. For $n \equiv 1 \pmod 4$ and $n \equiv 3 \pmod 4$, respectively, see Tables 9 and 10 for colors of the vertices and color sums of the vertices for $C_m \square P_n$.

color c				color sum σ			
1 0 0 0	1 0 0 0...	1 0 0 0	1	0 1 0 1	0 1 0 1...	0 1 0 1	0
0 0 0 0	0 0 0 0...	0 0 0 0	0	1 0 1 0	1 0 1 0...	1 0 1 0	1
0 0 1 0	0 0 1 0...	0 0 1 0	0	0 1 0 1	0 1 0 1...	0 1 0 1	0
0 0 0 0	0 0 0 0...	0 0 0 0	0	1 0 1 0	1 0 1 0...	1 0 1 0	1
1 0 0 0	1 0 0 0...	1 0 0 0	1	0 1 0 1	0 1 0 1...	0 1 0 1	0
0 0 0 0	0 0 0 0...	0 0 0 0	0	1 0 1 0	1 0 1 0...	1 0 1 0	1
0 0 1 0	0 0 1 0...	0 0 1 0	0	0 1 0 1	0 1 0 1...	0 1 0 1	0
0 0 0 0	0 0 0 0...	0 0 0 0	0	1 0 1 0	1 0 1 0...	1 0 1 0	1
⋮ ⋮ ⋮ ⋮	⋮ ⋮ ⋮ ⋮	⋮ ⋮ ⋮ ⋮	⋮	⋮ ⋮ ⋮ ⋮	⋮ ⋮ ⋮ ⋮	⋮ ⋮ ⋮ ⋮	⋮
1 0 0 0	1 0 0 0...	1 0 0 0	1	0 1 0 1	0 1 0 1...	0 1 0 1	0
0 0 0 0	0 0 0 0...	0 0 0 0	0	1 0 1 0	1 0 1 0...	1 0 1 0	1
0 0 1 0	0 0 1 0...	0 0 1 0	0	0 1 0 1	0 1 0 1...	0 1 0 1	0
0 0 0 0	0 0 0 0...	0 0 0 0	0	1 0 1 0	1 0 1 0...	1 0 1 0	1

Table 9. $C_m \square P_n$ ($m \equiv 0 \pmod 4$ and $n \equiv 1 \pmod 4$).

color c				color sum σ			
1 0 0 0	1 0 0 0...	1 0 0 0	1 0 0	0 1 0 1	0 1 0 1...	0 1 0 1	0 1 0
0 0 0 0	0 0 0 0...	0 0 0 0	0 0 0	1 0 1 0	1 0 1 0...	1 0 1 0	1 0 1
0 0 1 0	0 0 1 0...	0 0 1 0	0 0 1	0 1 0 1	0 1 0 1...	0 1 0 1	0 1 0
0 0 0 0	0 0 0 0...	0 0 0 0	0 0 0	1 0 1 0	1 0 1 0...	1 0 1 0	1 0 1
1 0 0 0	1 0 0 0...	1 0 0 0	1 0 0	0 1 0 1	0 1 0 1...	0 1 0 1	0 1 0
0 0 0 0	0 0 0 0...	0 0 0 0	0 0 0	1 0 1 0	1 0 1 0...	1 0 1 0	1 0 1
0 0 1 0	0 0 1 0...	0 0 1 0	0 0 1	0 1 0 1	0 1 0 1...	0 1 0 1	0 1 0
0 0 0 0	0 0 0 0...	0 0 0 0	0 0 0	1 0 1 0	1 0 1 0...	1 0 1 0	1 0 1
⋮ ⋮ ⋮ ⋮	⋮ ⋮ ⋮ ⋮	⋮ ⋮ ⋮ ⋮	⋮ ⋮	⋮ ⋮ ⋮ ⋮	⋮ ⋮ ⋮ ⋮	⋮ ⋮ ⋮ ⋮	⋮ ⋮
1 0 0 0	1 0 0 0...	1 0 0 0	1 0 0	0 1 0 1	0 1 0 1...	0 1 0 1	0 1 0
0 0 0 0	0 0 0 0...	0 0 0 0	0 0 0	1 0 1 0	1 0 1 0...	1 0 1 0	1 0 1
0 0 1 0	0 0 1 0...	0 0 1 0	0 0 1	0 1 0 1	0 1 0 1...	0 1 0 1	0 1 0
0 0 0 0	0 0 0 0...	0 0 0 0	0 0 0	1 0 1 0	1 0 1 0...	1 0 1 0	1 0 1

Table 10. $C_m \square P_n$ ($m \equiv 0 \pmod 4$ and $n \equiv 3 \pmod 4$).

Case 5. $m \equiv 1 \pmod 4$ and $n \equiv 1 \pmod 4$.

Define $c : V(C_m \square P_n) \rightarrow \mathbb{Z}_3$ as follows:

$c((i, j)) = 1$ if $(i, j) \in (\{1, 5, 9, \dots, m-4\} \times \{1, 5, 9, \dots, n\}) \cup (\{3, 7, 11, \dots, m-6\} \times \{3, 7, 11, \dots, n-2\})$; $c((m-2, j)) = 2$ if $j \in \{3, 7, 11, \dots, n-2\}$; $c((m-1, j)) = 1$ if $j \in \{3, 7, 11, \dots, n-2\}$; $c((m, j)) = 2$ if $j \in \{1, 5, 9, \dots, n\}$; and $c((i, j)) = 0$ otherwise. Then,

color c				color sum σ			
1 0 0 0	1 0 0 0	... 1 0 0 0	1	2 1 0 1	2 1 0 1	... 2 1 0 1	2
0 0 0 0	0 0 0 0	... 0 0 0 0	0	1 0 1 0	1 0 1 0	... 1 0 1 0	1
0 0 1 0	0 0 1 0	... 0 0 1 0	0	0 1 0 1	0 1 0 1	... 0 1 0 1	0
0 0 0 0	0 0 0 0	... 0 0 0 0	0	1 0 1 0	1 0 1 0	... 1 0 1 0	1
1 0 0 0	1 0 0 0	... 1 0 0 0	1	0 1 0 1	0 1 0 1	... 0 1 0 1	0
0 0 0 0	0 0 0 0	... 0 0 0 0	0	1 0 1 0	1 0 1 0	... 1 0 1 0	1
0 0 1 0	0 0 1 0	... 0 0 1 0	0	0 1 0 1	0 1 0 1	... 0 1 0 1	0
0 0 0 0	0 0 0 0	... 0 0 0 0	0	1 0 1 0	1 0 1 0	... 1 0 1 0	1
1 0 0 0	1 0 0 0	... 1 0 0 0	1	0 1 0 1	0 1 0 1	... 0 1 0 1	0
⋮ ⋮ ⋮ ⋮	⋮ ⋮ ⋮ ⋮	⋮ ⋮ ⋮ ⋮	⋮	⋮ ⋮ ⋮ ⋮	⋮ ⋮ ⋮ ⋮	⋮ ⋮ ⋮ ⋮	⋮
0 0 0 0	0 0 0 0	... 0 0 0 0	0	1 0 1 0	1 0 1 0	... 1 0 1 0	1
0 0 1 0	0 0 1 0	... 0 0 1 0	0	0 1 0 1	0 1 0 1	... 0 1 0 1	0
0 0 0 0	0 0 0 0	... 0 0 0 0	0	1 0 1 0	1 0 1 0	... 1 0 1 0	1
1 0 0 0	1 0 0 0	... 1 0 0 0	1	0 1 0 1	0 1 0 1	... 0 1 0 1	0
0 0 0 0	0 0 0 0	... 0 0 0 0	0	1 0 2 0	1 0 2 0	... 1 0 2 0	1
0 0 2 0	0 0 2 0	... 0 0 2 0	0	0 2 1 2	0 2 1 2	... 0 2 1 2	0
0 0 1 0	0 0 1 0	... 0 0 1 0	0	2 1 2 1	2 1 2 1	... 2 1 2 1	2
2 0 0 0	2 0 0 0	... 2 0 0 0	2	1 2 1 2	1 2 1 2	... 1 2 1 2	1

Table 11. $C_m \square P_n$ ($m \equiv 1 \pmod 4$ and $n \equiv 1 \pmod 4$).

$\sigma((i, j)) = 0$ if $(i, j) \in (\{2, 4, 6, \dots, m-5\} \times \{2, 4, 6, \dots, n-1\}) \cup (\{3, 5, 7, \dots, m-4\} \times \{1, 3, 5, \dots, n\})$;

$\sigma((i, j)) = 1$ if $(i, j) \in (\{2, 4, 6, \dots, m-5\} \times \{1, 3, 5, \dots, n\}) \cup (\{3, 5, 7, \dots, m-4\} \times \{2, 4, 6, \dots, n-1\})$;

$\sigma((1, j)) = 0$ if $j \in \{3, 7, 11, \dots, n-2\}$;

$\sigma((1, j)) = 1$ if $j \in \{2, 4, 6, \dots, n-1\}$;

$\sigma((1, j)) = 2$ if $j \in \{1, 5, 9, \dots, n\}$;

$\sigma((m-3, j)) = 0$ if $j \in \{2, 4, 6, \dots, n-1\}$;

$\sigma((m-3, j)) = 1$ if $j \in \{1, 5, 9, \dots, n\}$;

$\sigma((m-3, j)) = 2$ if $j \in \{3, 7, 11, \dots, n-2\}$;

$\sigma((m-2, j)) = 0$ if $j \in \{1, 5, 9, \dots, n\}$;

$\sigma((m-2, j)) = 1$ if $j \in \{3, 7, 11, \dots, n-2\}$;

$\sigma((m-2, j)) = 2$ if $j \in \{2, 4, 6, \dots, n-1\}$;

$$\begin{aligned} \sigma((m-1, j)) &= 1 \text{ if } j \in \{2, 4, 6, \dots, n-1\}; \\ \sigma((m-1, j)) &= 2 \text{ if } j \in \{1, 3, 5, \dots, n\}; \\ \sigma((m, j)) &= 1 \text{ if } j \in \{1, 3, 5, \dots, n\}; \\ \sigma((m, j)) &= 2 \text{ if } j \in \{2, 4, 6, \dots, n-1\}. \end{aligned}$$

Hence, c is a modular 3-coloring of $C_m \square P_n$. See Table 11 for colors of the vertices and color sums of the vertices for $C_m \square P_n$.

Case 6. $m \equiv 1 \pmod 4$ and $n \equiv 3 \pmod 4$.

Define $c : V(C_m \square P_n) \rightarrow \mathbb{Z}_3$ as follows:

$$\begin{aligned} c((i, j)) &= 1 \text{ if } (i, j) \in (\{1, 5, 9, \dots, m-4\} \times \{1, 5, 9, \dots, n-2\}) \cup (\{3, 7, 11, \dots, m-6\} \times \\ &\{3, 7, 11, \dots, n\}); c((m-2, j)) = 2 \text{ if } j \in \{3, 7, 11, \dots, n\}; c((m-1, j)) = 1 \text{ if } j \in \{3, 7, 11, \dots, n\}; \\ c((m, j)) &= 2 \text{ if } j \in \{1, 5, 9, \dots, n-2\}; \text{ and } c((i, j)) = 0 \text{ otherwise. Then,} \end{aligned}$$

color c			color sum σ		
1 0 0 0	1 0 0 0...1 0 0 0	1 0 0	2 1 0 1	2 1 0 1...2 1 0 1	2 1 0
0 0 0 0	0 0 0 0...0 0 0 0	0 0 0	1 0 1 0	1 0 1 0...1 0 1 0	1 0 1
0 0 1 0	0 0 1 0...0 0 1 0	0 0 1	0 1 0 1	0 1 0 1...0 1 0 1	0 1 0
0 0 0 0	0 0 0 0...0 0 0 0	0 0 0	1 0 1 0	1 0 1 0...1 0 1 0	1 0 1
1 0 0 0	1 0 0 0...1 0 0 0	1 0 0	0 1 0 1	0 1 0 1...0 1 0 1	0 1 0
0 0 0 0	0 0 0 0...0 0 0 0	0 0 0	1 0 1 0	1 0 1 0...1 0 1 0	1 0 1
0 0 1 0	0 0 1 0...0 0 1 0	0 0 1	0 1 0 1	0 1 0 1...0 1 0 1	0 1 0
0 0 0 0	0 0 0 0...0 0 0 0	0 0 0	1 0 1 0	1 0 1 0...1 0 1 0	1 0 1
1 0 0 0	1 0 0 0...1 0 0 0	1 0 0	0 1 0 1	0 1 0 1...0 1 0 1	0 1 0
⋮ ⋮ ⋮ ⋮	⋮ ⋮ ⋮ ⋮	⋮ ⋮ ⋮	⋮ ⋮ ⋮ ⋮	⋮ ⋮ ⋮ ⋮	⋮ ⋮ ⋮
0 0 0 0	0 0 0 0...0 0 0 0	0 0 0	1 0 1 0	1 0 1 0...1 0 1 0	1 0 1
0 0 1 0	0 0 1 0...0 0 1 0	0 0 1	0 1 0 1	0 1 0 1...0 1 0 1	0 1 0
0 0 0 0	0 0 0 0...0 0 0 0	0 0 0	1 0 1 0	1 0 1 0...1 0 1 0	1 0 1
1 0 0 0	1 0 0 0...1 0 0 0	1 0 0	0 1 0 1	0 1 0 1...0 1 0 1	0 1 0
0 0 0 0	0 0 0 0...0 0 0 0	0 0 0	1 0 2 0	1 0 2 0...1 0 2 0	1 0 2
0 0 2 0	0 0 2 0...0 0 2 0	0 0 2	0 2 1 2	0 2 1 2...0 2 1 2	0 2 1
0 0 1 0	0 0 1 0...0 0 1 0	0 0 1	2 1 2 1	2 1 2 1...2 1 2 1	2 1 2
2 0 0 0	2 0 0 0...2 0 0 0	2 0 0	1 2 1 2	1 2 1 2...1 2 1 2	1 2 1

Table 12. $C_m \square P_n$ ($m \equiv 1 \pmod 4$ and $n \equiv 3 \pmod 4$).

$$\begin{aligned} \sigma((i, j)) &= 0 \text{ if } (i, j) \in (\{2, 4, 6, \dots, m-5\} \times \{2, 4, 6, \dots, n-1\}) \cup (\{3, 5, 7, \dots, m-4\} \times \\ &\{1, 3, 5, \dots, n\}); \\ \sigma((i, j)) &= 1 \text{ if } (i, j) \in (\{2, 4, 6, \dots, m-5\} \times \{1, 3, 5, \dots, n\}) \cup (\{3, 5, 7, \dots, m-4\} \times \{2, 4, 6, \dots, \\ &n-1\}); \\ \sigma((1, j)) &= 0 \text{ if } j \in \{3, 7, 11, \dots, n\}; \end{aligned}$$

- $\sigma((1, j)) = 1$ if $j \in \{2, 4, 6, \dots, n - 1\}$;
- $\sigma((1, j)) = 2$ if $j \in \{1, 5, 9, \dots, n - 2\}$;
- $\sigma((m - 3, j)) = 0$ if $j \in \{2, 4, 6, \dots, n - 1\}$;
- $\sigma((m - 3, j)) = 1$ if $j \in \{1, 5, 9, \dots, n - 2\}$;
- $\sigma((m - 3, j)) = 2$ if $j \in \{3, 7, 11, \dots, n\}$;
- $\sigma((m - 2, j)) = 0$ if $j \in \{1, 5, 9, \dots, n - 2\}$;
- $\sigma((m - 2, j)) = 1$ if $j \in \{3, 7, 11, \dots, n\}$;
- $\sigma((m - 2, j)) = 2$ if $j \in \{2, 4, 6, \dots, n - 1\}$;
- $\sigma((m - 1, j)) = 1$ if $j \in \{2, 4, 6, \dots, n - 1\}$;
- $\sigma((m - 1, j)) = 2$ if $j \in \{1, 3, 5, \dots, n\}$;
- $\sigma((m, j)) = 1$ if $j \in \{1, 3, 5, \dots, n\}$;
- $\sigma((m, j)) = 2$ if $j \in \{2, 4, 6, \dots, n - 1\}$.

Hence, c is a modular 3-coloring of $C_m \square P_n$. See Table 12 for colors of the vertices and color sums of the vertices for $C_m \square P_n$.

Case 7. $m \equiv 3 \pmod 4$ and $n \equiv 1 \pmod 4$.

Define $c : V(C_m \square P_n) \rightarrow \mathbb{Z}_3$ as follows:

- $c((i, j)) = 1$ if $(i, j) \in (\{1, 5, 9, \dots, m - 6\} \times \{1, 5, 9, \dots, n\}) \cup (\{3, 7, 11, \dots, m - 4\} \times \{3, 7, 11, \dots, n - 2\})$;
- $c((m - 2, j)) = 2$ if $j \in \{1, 5, 9, \dots, n\}$;
- $c((m, j)) = 2$ if $j \in \{3, 7, 11, \dots, n - 2\}$;
- and $c((i, j)) = 0$ otherwise. Then,

- $\sigma((i, j)) = 0$ if $(i, j) \in (\{2, 4, 6, \dots, m - 5\} \times \{2, 4, 6, \dots, n - 1\}) \cup (\{3, 5, 7, \dots, m - 4\} \times \{1, 3, 5, \dots, n\})$;

- $\sigma((i, j)) = 1$ if $(i, j) \in (\{2, 4, 6, \dots, m - 5\} \times \{1, 3, 5, \dots, n\}) \cup (\{3, 5, 7, \dots, m - 4\} \times \{2, 4, 6, \dots, n - 1\})$;

- $\sigma((1, j)) = 0$ if $j \in \{1, 5, 9, \dots, n\}$;
- $\sigma((1, j)) = 1$ if $j \in \{2, 4, 6, \dots, n - 1\}$;
- $\sigma((1, j)) = 2$ if $j \in \{3, 7, 11, \dots, n - 2\}$;
- $\sigma((m - 3, j)) = 0$ if $j \in \{2, 4, 6, \dots, n - 1\}$;
- $\sigma((m - 3, j)) = 1$ if $j \in \{3, 7, 11, \dots, n - 2\}$;
- $\sigma((m - 3, j)) = 2$ if $j \in \{1, 5, 9, \dots, n\}$;
- $\sigma((m - 2, j)) = 0$ if $j \in \{1, 3, 5, \dots, n\}$;
- $\sigma((m - 2, j)) = 2$ if $j \in \{2, 4, 6, \dots, n - 1\}$;
- $\sigma((m - 1, j)) = 0$ if $j \in \{2, 4, 6, \dots, n - 1\}$;
- $\sigma((m - 1, j)) = 2$ if $j \in \{1, 3, 5, \dots, n\}$;
- $\sigma((m, j)) = 0$ if $j \in \{3, 7, 11, \dots, n - 2\}$;
- $\sigma((m, j)) = 1$ if $j \in \{1, 5, 9, \dots, n\}$;
- $\sigma((m, j)) = 2$ if $j \in \{2, 4, 6, \dots, n - 1\}$.

Hence, c is a modular 3-coloring of $C_m \square P_n$. See Table 13 for colors of the vertices and color sums of the vertices for $C_m \square P_n$.

color c				color sum σ			
1 0 0 0	1 0 0 0	... 1 0 0 0	1	0 1 2 1	0 1 2 1	... 0 1 2 1	0
0 0 0 0	0 0 0 0	... 0 0 0 0	0	1 0 1 0	1 0 1 0	... 1 0 1 0	1
0 0 1 0	0 0 1 0	... 0 0 1 0	0	0 1 0 1	0 1 0 1	... 0 1 0 1	0
0 0 0 0	0 0 0 0	... 0 0 0 0	0	1 0 1 0	1 0 1 0	... 1 0 1 0	1
1 0 0 0	1 0 0 0	... 1 0 0 0	1	0 1 0 1	0 1 0 1	... 0 1 0 1	0
0 0 0 0	0 0 0 0	... 0 0 0 0	0	1 0 1 0	1 0 1 0	... 1 0 1 0	1
0 0 1 0	0 0 1 0	... 0 0 1 0	0	0 1 0 1	0 1 0 1	... 0 1 0 1	0
0 0 0 0	0 0 0 0	... 0 0 0 0	0	1 0 1 0	1 0 1 0	... 1 0 1 0	1
1 0 0 0	1 0 0 0	... 1 0 0 0	1	0 1 0 1	0 1 0 1	... 0 1 0 1	0
⋮ ⋮ ⋮ ⋮	⋮ ⋮ ⋮ ⋮	⋮ ⋮ ⋮ ⋮	⋮	⋮ ⋮ ⋮ ⋮	⋮ ⋮ ⋮ ⋮	⋮ ⋮ ⋮ ⋮	⋮
0 0 0 0	0 0 0 0	... 0 0 0 0	0	1 0 1 0	1 0 1 0	... 1 0 1 0	1
0 0 1 0	0 0 1 0	... 0 0 1 0	0	0 1 0 1	0 1 0 1	... 0 1 0 1	0
0 0 0 0	0 0 0 0	... 0 0 0 0	0	1 0 1 0	1 0 1 0	... 1 0 1 0	1
1 0 0 0	1 0 0 0	... 1 0 0 0	1	0 1 0 1	0 1 0 1	... 0 1 0 1	0
0 0 0 0	0 0 0 0	... 0 0 0 0	0	1 0 1 0	1 0 1 0	... 1 0 1 0	1
0 0 1 0	0 0 1 0	... 0 0 1 0	0	0 1 0 1	0 1 0 1	... 0 1 0 1	0
0 0 0 0	0 0 0 0	... 0 0 0 0	0	2 0 1 0	2 0 1 0	... 2 0 1 0	2
2 0 0 0	2 0 0 0	... 2 0 0 0	2	0 2 0 2	0 2 0 2	... 0 2 0 2	0
0 0 0 0	0 0 0 0	... 0 0 0 0	0	2 0 2 0	2 0 2 0	... 2 0 2 0	2
0 0 2 0	0 0 2 0	... 0 0 2 0	0	1 2 0 2	1 2 0 2	... 1 2 0 2	1

Table 13. $C_m \square P_n$ ($m \equiv 3 \pmod 4$ and $n \equiv 1 \pmod 4$).

Case 8. $m \equiv 3 \pmod 4$ and $n \equiv 3 \pmod 4$.

Define $c : V(C_m \square P_n) \rightarrow \mathbb{Z}_3$ as follows:

$c((i, j)) = 1$ if $(i, j) \in (\{1, 5, 9, \dots, m - 6\} \times \{1, 5, 9, \dots, n - 2\}) \cup (\{3, 7, 11, \dots, m - 4\} \times \{3, 7, 11, \dots, n\})$; $c((m - 2, j)) = 2$ if $j \in \{1, 5, 9, \dots, n - 2\}$; $c((m, j)) = 2$ if $j \in \{3, 7, 11, \dots, n\}$; and $c((i, j)) = 0$ otherwise. Then,

$\sigma((i, j)) = 0$ if $(i, j) \in (\{2, 4, 6, \dots, m - 5\} \times \{2, 4, 6, \dots, n - 1\}) \cup (\{3, 5, 7, \dots, m - 4\} \times \{1, 3, 5, \dots, n\})$;

$\sigma((i, j)) = 1$ if $(i, j) \in (\{2, 4, 6, \dots, m - 5\} \times \{1, 3, 5, \dots, n\}) \cup (\{3, 5, 7, \dots, m - 4\} \times \{2, 4, 6, \dots, n - 1\})$;

$\sigma((1, j)) = 1$ if $j \in \{2, 4, 6, \dots, n - 1\}$;

$\sigma((1, j)) = 0$ if $j \in \{1, 5, 9, \dots, n - 2\}$;

$\sigma((1, j)) = 2$ if $j \in \{3, 7, 11, \dots, n\}$;

$\sigma((m - 3, j)) = 0$ if $j \in \{2, 4, 6, \dots, n - 1\}$;

$\sigma((m - 3, j)) = 1$ if $j \in \{3, 7, 11, \dots, n\}$;

- $\sigma((m - 3, j)) = 2$ if $j \in \{1, 5, 9, \dots, n - 2\}$;
- $\sigma((m - 2, j)) = 0$ if $j \in \{1, 3, 5, \dots, n\}$;
- $\sigma((m - 2, j)) = 2$ if $j \in \{2, 4, 6, \dots, n - 1\}$;
- $\sigma((m - 1, j)) = 0$ if $j \in \{2, 4, 6, \dots, n - 1\}$;
- $\sigma((m - 1, j)) = 2$ if $j \in \{1, 3, 5, \dots, n\}$;
- $\sigma((m, j)) = 2$ if $j \in \{2, 4, 6, \dots, n - 1\}$;
- $\sigma((m, j)) = 0$ if $j \in \{3, 7, 11, \dots, n\}$,
- $\sigma((m, j)) = 1$ if $j \in \{1, 5, 9, \dots, n - 2\}$.

Hence, c is a modular 3-coloring of $C_m \square P_n$. See Table 14 for colors of the vertices and color sums of the vertices for $C_m \square P_n$.

color c			color sum σ		
1 0 0 0	1 0 0 0...1 0 0 0	1 0 0	0 1 2 1	0 1 2 1...0 1 2 1	0 1 2
0 0 0 0	0 0 0 0...0 0 0 0	0 0 0	1 0 1 0	1 0 1 0...1 0 1 0	1 0 1
0 0 1 0	0 0 1 0...0 0 1 0	0 0 1	0 1 0 1	0 1 0 1...0 1 0 1	0 1 0
0 0 0 0	0 0 0 0...0 0 0 0	0 0 0	1 0 1 0	1 0 1 0...1 0 1 0	1 0 1
1 0 0 0	1 0 0 0...1 0 0 0	1 0 0	0 1 0 1	0 1 0 1...0 1 0 1	0 1 0
0 0 0 0	0 0 0 0...0 0 0 0	0 0 0	1 0 1 0	1 0 1 0...1 0 1 0	1 0 1
0 0 1 0	0 0 1 0...0 0 1 0	0 0 1	0 1 0 1	0 1 0 1...0 1 0 1	0 1 0
0 0 0 0	0 0 0 0...0 0 0 0	0 0 0	1 0 1 0	1 0 1 0...1 0 1 0	1 0 1
1 0 0 0	1 0 0 0...1 0 0 0	1 0 0	0 1 0 1	0 1 0 1...0 1 0 1	0 1 0
⋮ ⋮ ⋮ ⋮	⋮ ⋮ ⋮ ⋮	⋮ ⋮ ⋮	⋮ ⋮ ⋮ ⋮	⋮ ⋮ ⋮ ⋮	⋮ ⋮ ⋮
0 0 0 0	0 0 0 0...0 0 0 0	0 0 0	1 0 1 0	1 0 1 0...1 0 1 0	1 0 1
0 0 1 0	0 0 1 0...0 0 1 0	0 0 1	0 1 0 1	0 1 0 1...0 1 0 1	0 1 0
0 0 0 0	0 0 0 0...0 0 0 0	0 0 0	1 0 1 0	1 0 1 0...1 0 1 0	1 0 1
1 0 0 0	1 0 0 0...1 0 0 0	1 0 0	0 1 0 1	0 1 0 1...0 1 0 1	0 1 0
0 0 0 0	0 0 0 0...0 0 0 0	0 0 0	1 0 1 0	1 0 1 0...1 0 1 0	1 0 1
0 0 1 0	0 0 1 0...0 0 1 0	0 0 1	0 1 0 1	0 1 0 1...0 1 0 1	0 1 0
0 0 0 0	0 0 0 0...0 0 0 0	0 0 0	2 0 1 0	2 0 1 0...2 0 1 0	2 0 1
2 0 0 0	2 0 0 0...2 0 0 0	2 0 0	0 2 0 2	0 2 0 2...0 2 0 2	0 2 0
0 0 0 0	0 0 0 0...0 0 0 0	0 0 0	2 0 2 0	2 0 2 0...2 0 2 0	2 0 2
0 0 2 0	0 0 2 0...0 0 2 0	0 0 2	1 2 0 2	1 2 0 2...1 2 0 2	1 2 0

Table 14. $C_m \square P_n$ ($m \equiv 3 \pmod 4$ and $n \equiv 3 \pmod 4$).

Case 9. $m \equiv 1 \pmod 4$ and $n \equiv 0 \pmod 4$.

Define $c : V(C_m \square P_n) \rightarrow \mathbb{Z}_3$ as follows:

- $c((i, j)) = 1$ if $(i, j) \in (\{1, 5, 9, \dots, m-4\} \times \{1, 5, 9, \dots, n-3\}) \cup (\{3, 7, 11, \dots, m-6\} \times \{3, 7, 11, \dots, n-1\}) \cup \{(m-2, n-1)\} \cup (\{m-1\} \times \{3, 7, 11, \dots, n-5\})$;
- $c((i, j)) = 2$ if $(i, j) \in (\{1, 5, 9, \dots, m-4\}$

$\times \{n\}) \cup (\{m - 2\} \times \{3, 7, 11, \dots, n - 5\}) \cup (\{m\} \times \{1, 5, 9, \dots, n - 3\})$; and $c((i, j)) = 0$ otherwise. Then,

color c			color sum σ		
1 0 0 0	1 0 0 0...1 0 0 0	1 0 0 2	2 1 0 1	2 1 0 1...2 1 0 1	2 1 2 0
0 0 0 0	0 0 0 0...0 0 0 0	0 0 0 0	1 0 1 0	1 0 1 0...1 0 1 0	1 0 1 2
0 0 1 0	0 0 1 0...0 0 1 0	0 0 1 0	0 1 0 1	0 1 0 1...0 1 0 1	0 1 0 1
0 0 0 0	0 0 0 0...0 0 0 0	0 0 0 0	1 0 1 0	1 0 1 0...1 0 1 0	1 0 1 2
1 0 0 0	1 0 0 0...1 0 0 0	1 0 0 2	0 1 0 1	0 1 0 1...0 1 0 1	0 1 2 0
0 0 0 0	0 0 0 0...0 0 0 0	0 0 0 0	1 0 1 0	1 0 1 0...1 0 1 0	1 0 1 2
0 0 1 0	0 0 1 0...0 0 1 0	0 0 1 0	0 1 0 1	0 1 0 1...0 1 0 1	0 1 0 1
0 0 0 0	0 0 0 0...0 0 0 0	0 0 0 0	1 0 1 0	1 0 1 0...1 0 1 0	1 0 1 2
1 0 0 0	1 0 0 0...1 0 0 0	1 0 0 2	0 1 0 1	0 1 0 1...0 1 0 1	0 1 2 0
⋮	⋮	⋮	⋮	⋮	⋮
0 0 0 0	0 0 0 0...0 0 0 0	0 0 0 0	1 0 1 0	1 0 1 0...1 0 1 0	1 0 1 2
0 0 1 0	0 0 1 0...0 0 1 0	0 0 1 0	0 1 0 1	0 1 0 1...0 1 0 1	0 1 0 1
0 0 0 0	0 0 0 0...0 0 0 0	0 0 0 0	1 0 1 0	1 0 1 0...1 0 1 0	1 0 1 2
1 0 0 0	1 0 0 0...1 0 0 0	1 0 0 2	0 1 0 1	0 1 0 1...0 1 0 1	0 1 2 0
0 0 0 0	0 0 0 0...0 0 0 0	0 0 0 0	1 0 2 0	1 0 2 0...1 0 2 0	1 0 1 2
0 0 2 0	0 0 2 0...0 0 2 0	0 0 1 0	0 2 1 2	0 2 1 2...0 2 1 2	0 1 0 1
0 0 1 0	0 0 1 0...0 0 1 0	0 0 0 0	2 1 2 1	2 1 2 1...2 1 2 1	2 0 1 0
2 0 0 0	2 0 0 0...2 0 0 0	2 0 0 0	1 2 1 2	1 2 1 2...1 2 1 2	1 2 0 2

Table 15. $C_m \square P_n$ ($m \equiv 1 \pmod 4$ and $n \equiv 0 \pmod 4$).

$\sigma((i, j)) = 0$ if $(i, j) \in (\{2, 4, 6, \dots, m - 5\} \times \{2, 4, 6, \dots, n - 2\}) \cup (\{3, 7, 11, \dots, m - 6\} \times \{1, 3, 5, \dots, n - 1\}) \cup (\{5, 9, 13, \dots, m - 4\} \times \{1, 3, 5, \dots, n - 3\}) \cup (\{5, 9, 13, \dots, m - 4\} \times \{n\})$;

$\sigma((i, j)) = 1$ if $(i, j) \in (\{2, 4, 6, \dots, m - 5\} \times \{1, 3, 5, \dots, n - 1\}) \cup (\{3, 7, 11, \dots, m - 6\} \times \{2, 4, 6, \dots, n\}) \cup (\{5, 9, 13, \dots, m - 4\} \times \{2, 4, 6, \dots, n - 2\})$;

$\sigma((i, n - 1)) = 2$ if $i \in \{5, 9, 13, \dots, m - 4\}$;

$\sigma((i, n)) = 2$ if $i \in \{2, 4, 6, \dots, m - 5\}$;

$\sigma((1, j)) = 0$ if $j \in \{3, 7, 11, \dots, n - 5\}$;

$\sigma((1, n)) = 0$;

$\sigma((1, j)) = 1$ if $j \in \{2, 4, 6, \dots, n - 2\}$;

$\sigma((1, j)) = 2$ if $j \in \{1, 5, 9, \dots, n - 3\}$;

$\sigma((1, n - 1)) = 2$;

$\sigma((m - 3, j)) = 0$ if $j \in \{2, 4, 6, \dots, n - 2\}$;

$\sigma((m - 3, j)) = 1$ if $j \in \{1, 5, 9, \dots, n - 3\}$;

$$\begin{aligned} \sigma((m-3, n-1)) &= 1; \\ \sigma((m-3, j)) &= 2 \text{ if } j \in \{3, 7, 11, \dots, n-5\}; \\ \sigma((m-3, n)) &= 2; \\ \sigma((m-2, j)) &= 0 \text{ if } j \in \{1, 5, 9, \dots, n-3\}; \\ \sigma((m-2, n-1)) &= 0; \\ \sigma((m-2, j)) &= 1 \text{ if } j \in \{3, 7, 11, \dots, n-5\}; \\ \sigma((m-2, n-2)) &= \sigma((m-2, n)) = 1; \\ \sigma((m-2, j)) &= 2 \text{ if } j \in \{2, 4, 6, \dots, n-4\}; \\ \sigma((m-1, n-2)) &= \sigma((m-1, n)) = 0; \\ \sigma((m-1, j)) &= 1 \text{ if } j \in \{2, 4, 6, \dots, n-4\}; \\ \sigma((m-1, n-1)) &= 1; \\ \sigma((m-1, j)) &= 2 \text{ if } j \in \{1, 3, 5, \dots, n-3\}; \\ \sigma((m, n-1)) &= 0; \\ \sigma((m, j)) &= 1 \text{ if } j \in \{1, 3, 5, \dots, n-3\}; \\ \sigma((m, j)) &= 2 \text{ if } j \in \{2, 4, 6, \dots, n\}. \end{aligned}$$

Hence, c is a modular 3-coloring of $C_m \square P_n$. See Table 15 for colors of the vertices and color sums of the vertices for $C_m \square P_n$.

Case 10. $m \equiv 1 \pmod 4$ and $n \equiv 2 \pmod 4$.

Define $c : V(C_m \square P_n) \rightarrow \mathbb{Z}_3$ as follows:

$c((i, j)) = 1$ if $(i, j) \in (\{1, 5, 9, \dots, m-4\} \times \{1, 5, 9, \dots, n-1\}) \cup (\{3, 7, 11, \dots, m-6\} \times \{3, 7, 11, \dots, n-3\}) \cup (\{m-1\} \times \{3, 7, 11, \dots, n-3\})$; $c((i, j)) = 2$ if $(i, j) \in (\{3, 7, 11, \dots, m-6\} \times \{n\}) \cup (\{m-2\} \times \{3, 7, 11, \dots, n-3\}) \cup \{(m-1, n)\} \cup (\{m\} \times \{1, 5, 9, \dots, n-5\})$; and $c((i, j)) = 0$ otherwise. Then,

$$\begin{aligned} \sigma((i, j)) &= 0 \text{ if } (i, j) \in (\{2, 4, 6, \dots, m-5\} \times \{2, 4, 6, \dots, n-2\}) \cup (\{3, 7, 11, \dots, m-6\} \times \{1, 3, 5, \dots, n-3\}) \cup (\{3, 7, 11, \dots, m-6\} \times \{n\}) \cup (\{5, 9, 13, \dots, m-4\} \times \{1, 3, 5, \dots, n-1\}); \\ \sigma((i, j)) &= 1 \text{ if } (i, j) \in (\{2, 4, 6, \dots, m-5\} \times \{1, 3, 5, \dots, n-1\}) \cup (\{3, 7, 11, \dots, m-6\} \times \{2, 4, 6, \dots, n-2\}) \cup (\{5, 9, 13, \dots, m-4\} \times \{2, 4, 6, \dots, n\}); \\ \sigma((i, n-1)) &= 2 \text{ if } i \in \{3, 7, 11, \dots, m-6\}; \\ \sigma((i, n)) &= 2 \text{ if } i \in \{2, 4, 6, \dots, m-5\}; \\ \sigma((1, j)) &= 0 \text{ if } j \in \{3, 7, 11, \dots, n-3\}; \\ \sigma((1, n-1)) &= 0; \\ \sigma((1, j)) &= 1 \text{ if } j \in \{2, 4, 6, \dots, n\}; \\ \sigma((1, j)) &= 2 \text{ if } j \in \{1, 5, 9, \dots, n-5\}; \\ \sigma((m-3, j)) &= 0 \text{ if } j \in \{2, 4, 6, \dots, n\}; \\ \sigma((m-3, j)) &= 1 \text{ if } j \in \{1, 5, 9, \dots, n-1\}; \\ \sigma((m-3, j)) &= 2 \text{ if } j \in \{3, 7, 11, \dots, n-3\}; \\ \sigma((m-2, j)) &= 0 \text{ if } j \in \{1, 5, 9, \dots, n-1\}; \end{aligned}$$

- $\sigma((m - 2, j)) = 1$ if $j \in \{3, 7, 11, \dots, n - 3\}$;
- $\sigma((m - 2, j)) = 2$ if $j \in \{2, 4, 6, \dots, n\}$;
- $\sigma((m - 1, n)) = 0$;
- $\sigma((m - 1, j)) = 1$ if $j \in \{2, 4, 6, \dots, n - 2\}$;
- $\sigma((m - 1, j)) = 2$ if $j \in \{1, 3, 5, \dots, n - 1\}$;
- $\sigma((m, n - 2)) = 0$;
- $\sigma((m, j)) = 1$ if $j \in \{1, 3, 5, \dots, n - 1\}$;
- $\sigma((m, j)) = 2$ if $j \in \{2, 4, 6, \dots, n - 4\}$;
- $\sigma((m, n)) = 2$.

Hence, c is a modular 3-coloring of $C_m \square P_n$. See Table 16 for colors of the vertices and color sums of the vertices for $C_m \square P_n$.

	color c			color sum σ			
1 0 0 0	1 0 0 0...	1 0 0 0	1 0	2 1 0 1	2 1 0 1...	2 1 0 1	0 1
0 0 0 0	0 0 0 0...	0 0 0 0	0 0	1 0 1 0	1 0 1 0...	1 0 1 0	1 2
0 0 1 0	0 0 1 0...	0 0 1 0	0 2	0 1 0 1	0 1 0 1...	0 1 0 1	2 0
0 0 0 0	0 0 0 0...	0 0 0 0	0 0	1 0 1 0	1 0 1 0...	1 0 1 0	1 2
1 0 0 0	1 0 0 0...	1 0 0 0	1 0	0 1 0 1	0 1 0 1...	0 1 0 1	0 1
0 0 0 0	0 0 0 0...	0 0 0 0	0 0	1 0 1 0	1 0 1 0...	1 0 1 0	1 2
0 0 1 0	0 0 1 0...	0 0 1 0	0 2	0 1 0 1	0 1 0 1...	0 1 0 1	2 0
0 0 0 0	0 0 0 0...	0 0 0 0	0 0	1 0 1 0	1 0 1 0...	1 0 1 0	1 2
1 0 0 0	1 0 0 0...	1 0 0 0	1 0	0 1 0 1	0 1 0 1...	0 1 0 1	0 1
⋮ ⋮ ⋮ ⋮	⋮ ⋮ ⋮ ⋮	⋮ ⋮ ⋮ ⋮	⋮ ⋮	⋮ ⋮ ⋮ ⋮	⋮ ⋮ ⋮ ⋮	⋮ ⋮ ⋮ ⋮	⋮ ⋮
0 0 0 0	0 0 0 0...	0 0 0 0	0 0	1 0 1 0	1 0 1 0...	1 0 1 0	1 2
0 0 1 0	0 0 1 0...	0 0 1 0	0 2	0 1 0 1	0 1 0 1...	0 1 0 1	2 0
0 0 0 0	0 0 0 0...	0 0 0 0	0 0	1 0 1 0	1 0 1 0...	1 0 1 0	1 2
1 0 0 0	1 0 0 0...	1 0 0 0	1 0	0 1 0 1	0 1 0 1...	0 1 0 1	0 1
0 0 0 0	0 0 0 0...	0 0 0 0	0 0	1 0 2 0	1 0 2 0...	1 0 2 0	1 0
0 0 2 0	0 0 2 0...	0 0 2 0	0 0	0 2 1 2	0 2 1 2...	0 2 1 2	0 2
0 0 1 0	0 0 1 0...	0 0 1 0	0 2	2 1 2 1	2 1 2 1...	2 1 2 1	2 0
2 0 0 0	2 0 0 0...	2 0 0 0	0 0	1 2 1 2	1 2 1 2...	1 2 1 0	1 2

Table 16. $C_m \square P_n$ ($m \equiv 1 \pmod 4$ and $n \equiv 2 \pmod 4$).

Case 11. $m \equiv 3 \pmod 4$ and $n \equiv 0 \pmod 4$.

Subcase 11.1. $m \geq 7$ and $n = 4$.

Define $c : V(C_m \square P_4) \rightarrow \mathbb{Z}_3$ as follows:

$$c((i, j)) = 1 \text{ if } (i, j) \in (\{3, 7, 11, \dots, m-4\} \times \{1\}) \cup (\{1, 5, 9, \dots, m-6\} \times \{3\}) \cup (\{4, 8, 12, \dots, m-7\}$$

$\times \{4\} \cup \{(m-2, 4)\}$; $c((i, j)) = 2$ if $(i, j) \in (\{2, 6, 10, \dots, m-5\} \times \{1\}) \cup \{(m-3, 4), (m-1, 2)\}$ and $c((i, j)) = 0$ otherwise. Then,

- $(\sigma((1, 1)), \sigma((1, 2)), \sigma((1, 3)), \sigma((1, 4))) = (2, 1, 0, 1)$;
- $(\sigma((i, 1)), \sigma((i, 2)), \sigma((i, 3)), \sigma((i, 4))) = (1, 2, 1, 0)$ if $i \in \{2, 6, 10, \dots, m-5\}$;
- $(\sigma((i, 1)), \sigma((i, 2)), \sigma((i, 3)), \sigma((i, 4))) = (2, 1, 0, 1)$ if $i \in \{3, 7, 11, \dots, m-8\}$;
- $(\sigma((i, 1)), \sigma((i, 2)), \sigma((i, 3)), \sigma((i, 4))) = (1, 0, 2, 0)$ if $i \in \{4, 8, 12, \dots, m-7\}$;
- $(\sigma((i, 1)), \sigma((i, 2)), \sigma((i, 3)), \sigma((i, 4))) = (2, 1, 0, 2)$ if $i \in \{5, 9, 13, \dots, m-6\}$;
- $(\sigma((m-4, 1)), \sigma((m-4, 2)), \sigma((m-4, 3)), \sigma((m-4, 4))) = (2, 1, 0, 2)$;
- $(\sigma((m-3, 1)), \sigma((m-3, 2)), \sigma((m-3, 3)), \sigma((m-3, 4))) = (1, 0, 2, 1)$;
- $(\sigma((m-2, 1)), \sigma((m-2, 2)), \sigma((m-2, 3)), \sigma((m-2, 4))) = (0, 2, 1, 2)$;
- $(\sigma((m-1, 1)), \sigma((m-1, 2)), \sigma((m-1, 3)), \sigma((m-1, 4))) = (2, 0, 2, 1)$;
- $(\sigma((m, 1)), \sigma((m, 2)), \sigma((m, 3)), \sigma((m, 4))) = (0, 2, 1, 0)$.

Hence, c is a modular 3-coloring of $C_m \square P_4$. See Table 17 for colors of the vertices and color sums of the vertices for $C_m \square P_4$.

color c	color sum σ
0 0 1 0	2 1 0 1
2 0 0 0	1 2 1 0
1 0 0 0	2 1 0 1
0 0 0 1	1 0 2 0
0 0 1 0	2 1 0 2
2 0 0 0	1 2 1 0
1 0 0 0	2 1 0 1
0 0 0 1	1 0 2 0
0 0 1 0	2 1 0 2
⋮ ⋮ ⋮ ⋮	⋮ ⋮ ⋮ ⋮
2 0 0 0	1 2 1 0
1 0 0 0	2 1 0 1
0 0 0 1	1 0 2 0
0 0 1 0	2 1 0 2
2 0 0 0	1 2 1 0
1 0 0 0	2 1 0 2
0 0 0 2	1 0 2 1
0 0 0 1	0 2 1 2
0 2 0 0	2 0 2 1
0 0 0 0	0 2 1 0

Table 17. $C_m \square P_4$ ($m \equiv 3 \pmod{4}$).

Subcase 11.2. $m \geq 7$ and $n \geq 8$.

Define $c : V(C_m \square P_n) \rightarrow \mathbb{Z}_3$ as follows:

$c((i, j)) = 1$ if $(i, j) \in (\{1, 5, 9, \dots, m-6\} \times \{3, 7, 11, \dots, n-1\}) \cup (\{3, 7, 11, \dots, m-4\} \times \{1, 5, 9, \dots, n-3\}) \cup (\{4, 8, 12, \dots, m-7\} \times \{n\}) \cup \{(m-2, n)\}$; $c((i, j)) = 2$ if $(i, j) \in \{(1, n-4), (m-3, n), (m-1, n-2)\} \cup (\{m-2\} \times \{3, 7, 11, \dots, n-5\}) \cup (\{m\} \times \{1, 5, 9, \dots, n-7\})$; and $c((i, j)) = 0$ otherwise. Then,

$\sigma((i, j)) = 0$ if $(i, j) \in (\{3, 5, 7, \dots, m-6\} \times \{1, 3, 5, \dots, n-1\}) \cup (\{4, 6, 8, \dots, m-5\} \times \{2, 4, 6, \dots, n\})$;

$\sigma((i, j)) = 1$ if $(i, j) \in (\{3, 7, 11, \dots, m-8\} \times \{2, 4, 6, \dots, n\}) \cup (\{5, 9, 13, \dots, m-6\} \times \{2, 4, 6, \dots, n-2\}) \cup (\{6, 10, 14, \dots, m-5\} \times \{1, 3, 5, \dots, n-1\}) \cup (\{4, 8, 12, \dots, m-7\} \times \{1, 3, 5, \dots, n-3\})$;

$\sigma((i, n-1)) = 2$ if $i \in \{4, 8, 12, \dots, m-7\}$;

$\sigma((i, n)) = 2$ if $i \in \{5, 9, 13, \dots, m-6\}$;

$\sigma((1, j)) = 0$ if $j \in \{3, 7, 11, \dots, n-9\}$;

$\sigma((1, n-1)) = 0$;

$\sigma((1, j)) = 1$ if $j \in \{2, 4, 6, \dots, n\}$;

$\sigma((1, j)) = 2$ if $j \in \{1, 5, 9, \dots, n-3\}$;

$\sigma((1, n-5)) = 2$;

$\sigma((2, j)) = 0$ if $j \in \{2, 4, 6, \dots, n-6\} \cup \{n-2, n\}$;

$\sigma((2, j)) = 1$ if $j \in \{1, 3, 5, \dots, n-1\}$;

$\sigma((2, n-4)) = 2$;

$\sigma((m-4, j)) = 0$ if $j \in \{1, 3, 5, \dots, n-1\}$;

$\sigma((m-4, j)) = 1$ if $j \in \{2, 4, 6, \dots, n-2\}$;

$\sigma((m-4, n)) = 2$;

$\sigma((m-3, j)) = 0$ if $j \in \{2, 4, 6, \dots, n-2\}$;

$\sigma((m-3, j)) = 1$ if $j \in \{1, 5, 9, \dots, n-3\}$;

$\sigma((m-3, n)) = 1$;

$\sigma((m-3, j)) = 2$ if $j \in \{3, 7, 11, \dots, n-1\}$;

$\sigma((m-2, j)) = 0$ if $j \in \{1, 3, 5, \dots, n-3\}$;

$\sigma((m-2, n-1)) = 1$;

$\sigma((m-2, j)) = 2$ if $j \in \{2, 4, 6, \dots, n\}$;

$\sigma((m-1, j)) = 0$ if $j \in \{2, 4, 6, \dots, n-2\}$;

$\sigma((m-1, n)) = 1$;

$\sigma((m-1, j)) = 2$ if $j \in \{1, 3, 5, \dots, n-1\}$;

$\sigma((m, j)) = 0$ if $j \in \{1, 5, 9, \dots, n-3\}$;

$\sigma((m, n)) = 0$;

$\sigma((m, j)) = 1$ if $j \in \{3, 7, 11, \dots, n-1\}$;

$\sigma((m, j)) = 2$ if $j \in \{2, 4, 6, \dots, n-2\}$.

Hence, c is a modular 3-coloring of $C_m \square P_n$. See Table 18 for colors of the vertices and color sums of the vertices for $C_m \square P_n$.

color c				color sum σ			
0 0 1 0	0 0 1 0...0 0 1 0	0 0 1 2	0 0 1 0	2 1 0 1	2 1 0 1...2 1 0 1	2 1 2 1	2 1 0 1
0 0 0 0	0 0 0 0...0 0 0 0	0 0 0 0	0 0 0 0	1 0 1 0	1 0 1 0...1 0 1 0	1 0 1 2	1 0 1 0
1 0 0 0	1 0 0 0...1 0 0 0	1 0 0 0	1 0 0 0	0 1 0 1	0 1 0 1...0 1 0 1	0 1 0 1	0 1 0 1
0 0 0 0	0 0 0 0...0 0 0 0	0 0 0 0	0 0 0 1	1 0 1 0	1 0 1 0...1 0 1 0	1 0 1 0	1 0 2 0
0 0 1 0	0 0 1 0...0 0 1 0	0 0 1 0	0 0 1 0	0 1 0 1	0 1 0 1...0 1 0 1	0 1 0 1	0 1 0 2
0 0 0 0	0 0 0 0...0 0 0 0	0 0 0 0	0 0 0 0	1 0 1 0	1 0 1 0...1 0 1 0	1 0 1 0	1 0 1 0
1 0 0 0	1 0 0 0...1 0 0 0	1 0 0 0	1 0 0 0	0 1 0 1	0 1 0 1...0 1 0 1	0 1 0 1	0 1 0 1
0 0 0 0	0 0 0 0...0 0 0 0	0 0 0 0	0 0 0 1	1 0 1 0	1 0 1 0...1 0 1 0	1 0 1 0	1 0 2 0
0 0 1 0	0 0 1 0...0 0 1 0	0 0 1 0	0 0 1 0	0 1 0 1	0 1 0 1...0 1 0 1	0 1 0 1	0 1 0 2
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
0 0 0 0	0 0 0 0...0 0 0 0	0 0 0 0	0 0 0 0	1 0 1 0	1 0 1 0...1 0 1 0	1 0 1 0	1 0 1 0
1 0 0 0	1 0 0 0...1 0 0 0	1 0 0 0	1 0 0 0	0 1 0 1	0 1 0 1...0 1 0 1	0 1 0 1	0 1 0 1
0 0 0 0	0 0 0 0...0 0 0 0	0 0 0 0	0 0 0 1	1 0 1 0	1 0 1 0...1 0 1 0	1 0 1 0	1 0 2 0
0 0 1 0	0 0 1 0...0 0 1 0	0 0 1 0	0 0 1 0	0 1 0 1	0 1 0 1...0 1 0 1	0 1 0 1	0 1 0 2
0 0 0 0	0 0 0 0...0 0 0 0	0 0 0 0	0 0 0 0	1 0 1 0	1 0 1 0...1 0 1 0	1 0 1 0	1 0 1 0
1 0 0 0	1 0 0 0...1 0 0 0	1 0 0 0	1 0 0 0	0 1 0 1	0 1 0 1...0 1 0 1	0 1 0 1	0 1 0 2
0 0 0 0	0 0 0 0...0 0 0 0	0 0 0 0	0 0 0 2	1 0 2 0	1 0 2 0...1 0 2 0	1 0 2 0	1 0 2 1
0 0 2 0	0 0 2 0...0 0 2 0	0 0 2 0	0 0 0 1	0 2 0 2	0 2 0 2...0 2 0 2	0 2 0 2	0 2 1 2
0 0 0 0	0 0 0 0...0 0 0 0	0 0 0 0	0 2 0 0	2 0 2 0	2 0 2 0...2 0 2 0	2 0 2 0	2 0 2 1
2 0 0 0	2 0 0 0...2 0 0 0	2 0 0 0	0 0 0 0	0 2 1 2	0 2 1 2...0 2 1 2	0 2 1 2	0 2 1 0

Table 18. $C_m \square P_n$ ($m \equiv 3 \pmod 4, m \geq 7$ and $n \equiv 0 \pmod 4, n \geq 8$).

Case 12. $m \equiv 3 \pmod 4$ and $n \equiv 2 \pmod 4$.

Subcase 12.1. $m \geq 7$ and $n \geq 10$.

Define $c : V(C_m \square P_n) \rightarrow \mathbb{Z}_3$ as follows:

$c((i, j)) = 1$ if $(i, j) \in (\{1, 5, 9, \dots, m-6\} \times \{1, 5, 9, \dots, n-1\}) \cup (\{3, 7, 11, \dots, m-4\} \times \{3, 7, 11, \dots, n-3\}) \cup (\{4, 8, 12, \dots, m-7\} \times \{n\}) \cup \{(m-2, n)\}$; $c((i, j)) = 2$ if $(i, j) \in \{(1, n-4), (m-3, n)\} \cup (\{m-2\} \times \{1, 5, 9, \dots, n-5\}) \cup \{(m-1, n-2)\} \cup (\{m\} \times \{3, 7, 11, \dots, n-7\})$; and $c((i, j)) = 0$ otherwise. Then,

$$\sigma((i, j)) = 0 \text{ if } (i, j) \in (\{3, 7, 11, \dots, m-8\} \times \{1, 3, 5, \dots, n-1\});$$

$$\sigma((i, j)) = 1 \text{ if } (i, j) \in (\{3, 7, 11, \dots, m-8\} \times \{2, 4, 6, \dots, n\});$$

$$\sigma((i, j)) = 0 \text{ if } (i, j) \in (\{4, 8, 12, \dots, m-7\} \times \{2, 4, 6, \dots, n\});$$

$$\begin{aligned}
\sigma((i, j)) &= 1 \text{ if } (i, j) \in (\{4, 8, 12, \dots, m-7\} \times \{1, 3, 5, \dots, n-3\}); \\
\sigma((i, j)) &= 2 \text{ if } (i, j) \in (\{4, 8, 12, \dots, m-7\} \times \{n-1\}); \\
\sigma((i, j)) &= 0 \text{ if } (i, j) \in (\{5, 9, 13, \dots, m-6\} \times \{1, 3, 5, \dots, n-1\}); \\
\sigma((i, j)) &= 1 \text{ if } (i, j) \in (\{5, 9, 13, \dots, m-6\} \times \{2, 4, 6, \dots, n-2\}); \\
\sigma((i, j)) &= 2 \text{ if } (i, j) \in (\{5, 9, 13, \dots, m-6\} \times \{n\}); \\
\sigma((i, j)) &= 0 \text{ if } (i, j) \in (\{6, 10, 14, \dots, m-5\} \times \{2, 4, 6, \dots, n\}); \\
\sigma((i, j)) &= 1 \text{ if } (i, j) \in (\{6, 10, 14, \dots, m-5\} \times \{1, 3, 5, \dots, n-1\}); \\
\sigma((1, j)) &= 0 \text{ if } j \in \{1, 5, 9, \dots, n-9\}; \\
\sigma((1, n-1)) &= 0; \\
\sigma((1, j)) &= 1 \text{ if } j \in \{2, 4, 6, \dots, n\}; \\
\sigma((1, j)) &= 2 \text{ if } j \in \{3, 7, 11, \dots, n-3\}; \\
\sigma((1, n-5)) &= 2; \\
\sigma((2, j)) &= 0 \text{ if } j \in \{2, 4, 6, \dots, n-6\}; \\
\sigma((2, n-2)) &= \sigma((2, n)) = 0; \\
\sigma((2, j)) &= 1 \text{ if } j \in \{1, 3, 5, \dots, n-1\}; \\
\sigma((2, n-4)) &= 2; \\
\sigma((m-4, j)) &= 0 \text{ if } j \in \{1, 3, 5, \dots, n-1\}; \\
\sigma((m-4, j)) &= 1 \text{ if } j \in \{2, 4, 6, \dots, n-2\}; \\
\sigma((m-4, n)) &= 2; \\
\sigma((m-3, j)) &= 0 \text{ if } j \in \{2, 4, 6, \dots, n-2\}; \\
\sigma((m-3, j)) &= 1 \text{ if } j \in \{3, 7, 11, \dots, n-3\}; \\
\sigma((m-3, n)) &= 1; \\
\sigma((m-3, j)) &= 2 \text{ if } j \in \{1, 5, 9, \dots, n-1\}; \\
\sigma((m-2, j)) &= 0 \text{ if } j \in \{1, 3, 5, \dots, n-3\}; \\
\sigma((m-2, n-1)) &= 1; \\
\sigma((m-2, j)) &= 2 \text{ if } j \in \{2, 4, 6, \dots, n\}; \\
\sigma((m-1, j)) &= 0 \text{ if } j \in \{2, 4, 6, \dots, n-2\}; \\
\sigma((m-1, n)) &= 1; \\
\sigma((m-1, j)) &= 2 \text{ if } j \in \{1, 3, 5, \dots, n-1\}; \\
\sigma((m, j)) &= 0 \text{ if } j \in \{3, 7, 11, \dots, n-3\}; \\
\sigma((m, n)) &= 0; \\
\sigma((m, j)) &= 1 \text{ if } j \in \{1, 5, 9, \dots, n-1\}; \\
\sigma((m, j)) &= 2 \text{ if } j \in \{2, 4, 6, \dots, n-2\}.
\end{aligned}$$

Hence, c is a modular 3-coloring of $C_m \square P_n$. See Table 19 for colors of the vertices and color sums of the vertices for $C_m \square P_n$.

color c				color sum σ			
1 0 0 0	1 0 0 0...1 0 0 0	1 2 0 0	1 0	0 1 2 1	0 1 2 1...0 1 2 1	2 1 2 1	0 1
0 0 0 0	0 0 0 0...0 0 0 0	0 0 0 0	0 0	1 0 1 0	1 0 1 0...1 0 1 0	1 2 1 0	1 0
0 0 1 0	0 0 1 0...0 0 1 0	0 0 1 0	0 0	0 1 0 1	0 1 0 1...0 1 0 1	0 1 0 1	0 1
0 0 0 0	0 0 0 0...0 0 0 0	0 0 0 0	0 1	1 0 1 0	1 0 1 0...1 0 1 0	1 0 1 0	2 0
1 0 0 0	1 0 0 0...1 0 0 0	1 0 0 0	1 0	0 1 0 1	0 1 0 1...0 1 0 1	0 1 0 1	0 2
0 0 0 0	0 0 0 0...0 0 0 0	0 0 0 0	0 0	1 0 1 0	1 0 1 0...1 0 1 0	1 0 1 0	1 0
0 0 1 0	0 0 1 0...0 0 1 0	0 0 1 0	0 0	0 1 0 1	0 1 0 1...0 1 0 1	0 1 0 1	0 1
0 0 0 0	0 0 0 0...0 0 0 0	0 0 0 0	0 1	1 0 1 0	1 0 1 0...1 0 1 0	1 0 1 0	2 0
1 0 0 0	1 0 0 0...1 0 0 0	1 0 0 0	1 0	0 1 0 1	0 1 0 1...0 1 0 1	0 1 0 1	0 2
⋮ ⋮ ⋮ ⋮	⋮ ⋮ ⋮ ⋮	⋮ ⋮ ⋮ ⋮	⋮ ⋮	⋮ ⋮ ⋮ ⋮	⋮ ⋮ ⋮ ⋮	⋮ ⋮ ⋮ ⋮	⋮ ⋮
0 0 0 0	0 0 0 0...0 0 0 0	0 0 0 0	0 0	1 0 1 0	1 0 1 0...1 0 1 0	1 0 1 0	1 0
0 0 1 0	0 0 1 0...0 0 1 0	0 0 1 0	0 0	0 1 0 1	0 1 0 1...0 1 0 1	0 1 0 1	0 1
0 0 0 0	0 0 0 0...0 0 0 0	0 0 0 0	0 1	1 0 1 0	1 0 1 0...1 0 1 0	1 0 1 0	2 0
1 0 0 0	1 0 0 0...1 0 0 0	1 0 0 0	1 0	0 1 0 1	0 1 0 1...0 1 0 1	0 1 0 1	0 2
0 0 0 0	0 0 0 0...0 0 0 0	0 0 0 0	0 0	1 0 1 0	1 0 1 0...1 0 1 0	1 0 1 0	1 0
0 0 1 0	0 0 1 0...0 0 1 0	0 0 1 0	0 0	0 1 0 1	0 1 0 1...0 1 0 1	0 1 0 1	0 2
0 0 0 0	0 0 0 0...0 0 0 0	0 0 0 0	0 2	2 0 1 0	2 0 1 0...2 0 1 0	2 0 1 0	2 1
2 0 0 0	2 0 0 0...2 0 0 0	2 0 0 0	0 1	0 2 0 2	0 2 0 2...0 2 0 2	0 2 0 2	1 2
0 0 0 0	0 0 0 0...0 0 0 0	0 0 0 2	0 0	2 0 2 0	2 0 2 0...2 0 2 0	2 0 2 0	2 1
0 0 2 0	0 0 2 0...0 0 2 0	0 0 0 0	0 0	1 2 0 2	1 2 0 2...1 2 0 2	1 2 0 2	1 0

Table 19. $C_m \square P_n$ ($m \equiv 3 \pmod 4, m \geq 7$ and $n \equiv 2 \pmod 4, n \geq 10$).

Subcase 12.2. $m \geq 7$ and $n = 6$.

Define $c : V(C_m \square P_6) \rightarrow \mathbb{Z}_3$ as follows:

$c((i, j)) = 1$ if $(i, j) \in (\{1, 5, 9, \dots, m-6\} \times \{1, 5\}) \cup (\{3, 7, 11, \dots, m-4\} \times \{3\}) \cup (\{4, 8, 12, \dots, m-7\} \times \{6\}) \cup \{(m-2, 6)\}$; $c((i, j)) = 2$ if $(i, j) \in (\{1, 5, 9, \dots, m-6\} \times \{2\}) \cup \{(m-3, 6), (m-2, 1), (m-1, 4)\}$; and $c((i, j)) = 0$ otherwise. Then,

- $\sigma((i, j)) = 0$ if $(i, j) \in (\{2, 6, 10, \dots, m-5\} \times \{4, 6\})$;
- $\sigma((i, j)) = 1$ if $(i, j) \in (\{2, 6, 10, \dots, m-5\} \times \{1, 3, 5\})$;
- $\sigma((i, j)) = 2$ if $(i, j) \in (\{2, 6, 10, \dots, m-5\} \times \{2\})$;
- $\sigma((i, j)) = 0$ if $(i, j) \in (\{3, 7, 11, \dots, m-8\} \times \{1, 3, 5\})$;
- $\sigma((i, j)) = 1$ if $(i, j) \in (\{3, 7, 11, \dots, m-8\} \times \{2, 4, 6\})$;
- $\sigma((i, j)) = 0$ if $(i, j) \in (\{4, 8, 12, \dots, m-7\} \times \{4, 6\})$;
- $\sigma((i, j)) = 1$ if $(i, j) \in (\{4, 8, 12, \dots, m-7\} \times \{1, 3\})$;
- $\sigma((i, j)) = 2$ if $(i, j) \in (\{4, 8, 12, \dots, m-7\} \times \{2, 5\})$;
- $\sigma((i, j)) = 0$ if $(i, j) \in (\{5, 9, 13, \dots, m-6\} \times \{5\})$;

color c	color sum σ
1 2 0 0 1 0	2 1 2 1 0 1
0 0 0 0 0 0	1 2 1 0 1 0
0 0 1 0 0 0	0 1 0 1 0 1
0 0 0 0 0 1	1 2 1 0 2 0
1 2 0 0 1 0	2 1 2 1 0 2
0 0 0 0 0 0	1 2 1 0 1 0
0 0 1 0 0 0	0 1 0 1 0 1
0 0 0 0 0 1	1 2 1 0 2 0
1 2 0 0 1 0	2 1 2 1 0 2
: : : : : :	: : : : : :
0 0 0 0 0 0	1 2 1 0 1 0
0 0 1 0 0 0	0 1 0 1 0 1
0 0 0 0 0 1	1 2 1 0 2 0
1 2 0 0 1 0	2 1 2 1 0 2
0 0 0 0 0 0	1 2 1 0 1 0
0 0 1 0 0 0	0 1 0 1 0 2
0 0 0 0 0 2	2 0 1 0 2 1
2 0 0 0 0 1	0 2 0 2 1 2
0 0 0 2 0 0	2 0 2 0 2 1
0 0 0 0 0 0	1 2 0 2 1 0

Table 20. $C_m \square P_6$ ($m \equiv 3 \pmod 4, m \geq 7$).

- $\sigma((i, j)) = 1$ if $(i, j) \in (\{5, 9, 13, \dots, m - 6\} \times \{2, 4\})$;
- $\sigma((i, j)) = 2$ if $(i, j) \in (\{5, 9, 13, \dots, m - 6\} \times \{1, 3, 6\})$;
- $\sigma((1, 5)) = 0$;
- $\sigma((1, 2)) = \sigma((1, 4)) = \sigma((1, 6)) = 1$;
- $\sigma((1, 1)) = \sigma((1, 3)) = 2$;
- $\sigma((m - 4, 1)) = \sigma((m - 4, 3)) = \sigma((m - 4, 5)) = 0$;
- $\sigma((m - 4, 2)) = \sigma((m - 4, 4)) = 1$;
- $\sigma((m - 4, 6)) = 2$;
- $\sigma((m - 3, 2)) = \sigma((m - 3, 4)) = 0$;
- $\sigma((m - 3, 3)) = \sigma((m - 3, 6)) = 1$;
- $\sigma((m - 3, 1)) = \sigma((m - 3, 5)) = 2$;
- $\sigma((m - 2, 1)) = \sigma((m - 2, 3)) = 0$;
- $\sigma((m - 2, 5)) = 1$;
- $\sigma((m - 2, 2)) = \sigma((m - 2, 4)) = \sigma((m - 2, 6)) = 2$;

$$\begin{aligned} \sigma((m-1, 2)) &= \sigma((m-1, 4)) = 0; \\ \sigma((m-1, 6)) &= 1; \\ \sigma((m-1, 1)) &= \sigma((m-1, 3)) = \sigma((m-1, 5)) = 2; \\ \sigma((m, 3)) &= \sigma((m, 6)) = 0; \\ \sigma((m, 1)) &= \sigma((m, 5)) = 1; \\ \sigma((m, 2)) &= \sigma((m, 4)) = 2. \end{aligned}$$

Hence, c is a modular 3-coloring of $C_m \square P_6$. See Table 20 for colors of the vertices and color sums of the vertices for $C_m \square P_6$.

This completes the proof. □

Proof. (iii): By hypothesis, $m \equiv 2 \pmod 4$ and $n \equiv 1 \pmod 4$.

Define $c : V(C_m \square P_n) \rightarrow \mathbb{Z}_3$ as follows:

$c((i, j)) = 1$ if $(i, j) \in (\{1, 5, 9, \dots, m-5\} \times \{1, 5, 9, \dots, n\}) \cup (\{3, 7, 11, \dots, m-3\} \times \{3, 7, 11, \dots, n-2\}) \cup (\{m-1\} \times \{1, 3, 5, \dots, n\})$; and $c((i, j)) = 0$ otherwise. Then,

color c				color sum σ			
1 0 0 0	1 0 0 0...1 0 0 0	1		0 1 0 1	0 1 0 1...0 1 0 1	0	
0 0 0 0	0 0 0 0...0 0 0 0	0		1 0 1 0	1 0 1 0...1 0 1 0	1	
0 0 1 0	0 0 1 0...0 0 1 0	0		0 1 0 1	0 1 0 1...0 1 0 1	0	
0 0 0 0	0 0 0 0...0 0 0 0	0		1 0 1 0	1 0 1 0...1 0 1 0	1	
1 0 0 0	1 0 0 0...1 0 0 0	1		0 1 0 1	0 1 0 1...0 1 0 1	0	
0 0 0 0	0 0 0 0...0 0 0 0	0		1 0 1 0	1 0 1 0...1 0 1 0	1	
0 0 1 0	0 0 1 0...0 0 1 0	0		0 1 0 1	0 1 0 1...0 1 0 1	0	
0 0 0 0	0 0 0 0...0 0 0 0	0		1 0 1 0	1 0 1 0...1 0 1 0	1	
⋮ ⋮ ⋮ ⋮	⋮ ⋮ ⋮ ⋮	⋮		⋮ ⋮ ⋮ ⋮	⋮ ⋮ ⋮ ⋮	⋮	
1 0 0 0	1 0 0 0...1 0 0 0	1		0 1 0 1	0 1 0 1...0 1 0 1	0	
0 0 0 0	0 0 0 0...0 0 0 0	0		1 0 1 0	1 0 1 0...1 0 1 0	1	
0 0 1 0	0 0 1 0...0 0 1 0	0		0 1 0 1	0 1 0 1...0 1 0 1	0	
0 0 0 0	0 0 0 0...0 0 0 0	0		1 0 1 0	1 0 1 0...1 0 1 0	1	
1 0 0 0	1 0 0 0...1 0 0 0	1		0 1 0 1	0 1 0 1...0 1 0 1	0	
0 0 0 0	0 0 0 0...0 0 0 0	0		1 0 1 0	1 0 1 0...1 0 1 0	1	
0 0 1 0	0 0 1 0...0 0 1 0	0		0 1 0 1	0 1 0 1...0 1 0 1	0	
0 0 0 0	0 0 0 0...0 0 0 0	0		1 0 2 0	1 0 2 0...1 0 2 0	1	
1 0 1 0	1 0 1 0...1 0 1 0	1		0 2 0 2	0 2 0 2...0 2 0 2	0	
0 0 0 0	0 0 0 0...0 0 0 0	0		2 0 1 0	2 0 1 0...2 0 1 0	2	

Table 21. $C_m \square P_n$ ($m \equiv 2 \pmod 4$ and $n \equiv 1 \pmod 4$).

$\sigma((i, j)) = 0$ if $(i, j) \in (\{1, 3, 5, \dots, m-1\} \times \{1, 3, 5, \dots, n\}) \cup (\{2, 4, 6, \dots, m\} \times \{2, 4, 6, \dots, n-1\})$;

$$\begin{aligned} \sigma((i, j)) &= 1 \text{ if } (i, j) \in (\{1, 3, 5, \dots, m-3\} \times \{2, 4, 6, \dots, n-1\}) \cup \\ &(\{2, 4, 6, \dots, m-4\} \times \{1, 3, 5, \dots, n\}) \cup (\{m-2\} \times \{1, 5, 9, \dots, n\}) \cup (\{m\} \times \{3, 7, 9, \dots, n-2\}); \\ \sigma((i, j)) &= 2 \text{ if } (i, j) \in (\{m-2\} \times \{3, 7, 11, \dots, n-2\}) \cup \\ &(\{m-1\} \times \{2, 4, 6, \dots, n-1\}) \cup (\{m\} \times \{1, 5, 9, \dots, n\}). \end{aligned}$$

See Table 21 for colors of the vertices and color sums of the vertices for $C_m \square P_n$. □

Proof. (iv): By (iii), $mc(C_6 \square P_n) \leq 3$. Hence, we have to show that $mc(C_6 \square P_n) \geq 3$. Assume, to the contrary, that there exists a modular 2-coloring c of $C_6 \square P_n$. By the symmetry of the graph $C_6 \square P_n$, we may assume that $\sigma((i, j)) = 0$ for (i, j) such that i and j are of different parity and $\sigma((i, j)) = 1$ for (i, j) such that i and j are of same parity. $\sigma((1, 1)) = 1$ implies that $c((1, 2)) = c((2, 1)) = c((6, 1)) = 1$, or $c((1, 2)) = 1$ and $c((2, 1)) = c((6, 1)) = 0$, or $c((2, 1)) = 1$ and $c((1, 2)) = c((6, 1)) = 0$, or $c((6, 1)) = 1$ and $c((1, 2)) = c((2, 1)) = 0$. The case $c((6, 1)) = 1$ and $c((1, 2)) = c((2, 1)) = 0$ is similar to the case $c((2, 1)) = 1$ and $c((1, 2)) = c((6, 1)) = 0$.

Case 1. $c((1, 2)) = c((2, 1)) = c((6, 1)) = 1$.

$\sigma((3, 1)) = 1$ implies that $c((3, 2)) = c((4, 1))$.

Subcase 1.1. $c((3, 2)) = c((4, 1)) = 0$.

$$\begin{aligned} \sigma((5, 1)) = 1 &\Rightarrow c((5, 2)) = 0. \text{ Hence,} \\ (c((2, 1)), c((4, 1)), c((6, 1))) &= (1, 0, 1) \text{ and } (c((1, 2)), c((3, 2)), c((5, 2))) = (1, 0, 0). \\ (\sigma((2, 2)), \sigma((4, 2)), \sigma((6, 2))) &= (1, 1, 1) \Rightarrow (c((2, 3)), c((4, 3)), c((6, 3))) = (1, 1, 1). \\ (\sigma((1, 3)), \sigma((3, 3)), \sigma((5, 3))) &= (1, 1, 1) \Rightarrow (c((1, 4)), c((3, 4)), c((5, 4))) = (0, 1, 1). \\ (\sigma((2, 4)), \sigma((4, 4)), \sigma((6, 4))) &= (1, 1, 1) \Rightarrow (c((2, 5)), c((4, 5)), c((6, 5))) = (1, 0, 1). \end{aligned}$$

If $n = 5$, then $\sigma((1, 5)) = 0$, a contradiction. Hence assume that $n \geq 9$.

$$\begin{aligned} (\sigma((1, 5)), \sigma((3, 5)), \sigma((5, 5))) &= (1, 1, 1) \Rightarrow (c((1, 6)), c((3, 6)), c((5, 6))) = (1, 1, 1). \\ (\sigma((2, 6)), \sigma((4, 6)), \sigma((6, 6))) &= (1, 1, 1) \Rightarrow (c((2, 7)), c((4, 7)), c((6, 7))) = (0, 1, 0). \\ (\sigma((1, 7)), \sigma((3, 7)), \sigma((5, 7))) &= (1, 1, 1) \Rightarrow (c((1, 8)), c((3, 8)), c((5, 8))) = (0, 1, 1). \\ (\sigma((2, 8)), \sigma((4, 8)), \sigma((6, 8))) &= (1, 1, 1) \Rightarrow (c((2, 9)), c((4, 9)), c((6, 9))) = (0, 0, 0). \end{aligned}$$

If $n = 9$, then $\sigma((1, 9)) = 0$, a contradiction. Hence assume that $n \geq 13$.

$$\begin{aligned} (\sigma((1, 9)), \sigma((3, 9)), \sigma((5, 9))) &= (1, 1, 1) \Rightarrow (c((1, 10)), c((3, 10)), c((5, 10))) = (1, 0, 0). \\ (\sigma((2, 10)), \sigma((4, 10)), \sigma((6, 10))) &= (1, 1, 1) \Rightarrow (c((2, 11)), c((4, 11)), c((6, 11))) = (0, 1, 0). \\ (\sigma((1, 11)), \sigma((3, 11)), \sigma((5, 11))) &= (1, 1, 1) \Rightarrow (c((1, 12)), c((3, 12)), c((5, 12))) = (0, 0, 0). \\ (\sigma((2, 12)), \sigma((4, 12)), \sigma((6, 12))) &= (1, 1, 1) \Rightarrow (c((2, 13)), c((4, 13)), c((6, 13))) = (1, 0, 1). \end{aligned}$$

If $n = 13$, then $\sigma((1, 13)) = 0$, a contradiction. Hence assume that $n \geq 17$.

$$(\sigma((1, 13)), \sigma((3, 13)), \sigma((5, 13))) = (1, 1, 1) \Rightarrow (c((1, 14)), c((3, 14)), c((5, 14))) = (1, 0, 0).$$

Observe that

$$\begin{aligned} (c((2, 13)), c((4, 13)), c((6, 13))) &= (1, 0, 1) = (c((2, 1)), c((4, 1)), c((6, 1))) \text{ and} \\ (c((1, 14)), c((3, 14)), c((5, 14))) &= (1, 0, 0) = (c((1, 2)), c((3, 2)), c((5, 2))). \end{aligned}$$

This implies that

$$(c((2, j)), c((4, j)), c((6, j))) = (c((2, j+12)), c((4, j+12)), c((6, j+12))) \text{ for odd } j \geq 3$$

and

$(c((1, j)), c((3, j)), c((5, j))) = (c((1, j + 12)), c((3, j + 12)), c((5, j + 12)))$ for even $j \geq 4$.

As we have obtained contradictions for $n = 5, 9, 13$, we have contradictions for $n = 17, 21, 25$, and so on.

Subcase 1.2. $c((3, 2)) = c((4, 1)) = 1$.

$\sigma((5, 1)) = 1 \Rightarrow c((5, 2)) = 1$. Hence,
 $(c((2, 1)), c((4, 1)), c((6, 1))) = (1, 1, 1)$ and $(c((1, 2)), c((3, 2)), c((5, 2))) = (1, 1, 1)$.
 $(\sigma((2, 2)), \sigma((4, 2)), \sigma((6, 2))) = (1, 1, 1) \Rightarrow (c((2, 3)), c((4, 3)), c((6, 3))) = (0, 0, 0)$.
 $(\sigma((1, 3)), \sigma((3, 3)), \sigma((5, 3))) = (1, 1, 1) \Rightarrow (c((1, 4)), c((3, 4)), c((5, 4))) = (0, 0, 0)$.
 $(\sigma((2, 4)), \sigma((4, 4)), \sigma((6, 4))) = (1, 1, 1) \Rightarrow (c((2, 5)), c((4, 5)), c((6, 5))) = (1, 1, 1)$.

If $n = 5$, then $\sigma((1, 5)) = 0$, a contradiction. Hence assume that $n \geq 9$.
 $(\sigma((1, 5)), \sigma((3, 5)), \sigma((5, 5))) = (1, 1, 1) \Rightarrow (c((1, 6)), c((3, 6)), c((5, 6))) = (1, 1, 1)$.

Observe that
 $(c((2, 5)), c((4, 5)), c((6, 5))) = (1, 1, 1) = (c((2, 1)), c((4, 1)), c((6, 1)))$ and
 $(c((1, 6)), c((3, 6)), c((5, 6))) = (1, 1, 1) = (c((1, 2)), c((3, 2)), c((5, 2)))$.

This implies that

$(c((2, j)), c((4, j)), c((6, j))) = (c((2, j + 4)), c((4, j + 4)), c((6, j + 4)))$ for odd $j \geq 3$
 and
 $(c((1, j)), c((3, j)), c((5, j))) = (c((1, j + 4)), c((3, j + 4)), c((5, j + 4)))$ for even $j \geq 4$.

As we have obtained a contradiction for $n = 5$, we have a contradiction for $n = 9$, and so on.

Case 2. $c((1, 2)) = 1$ and $c((2, 1)) = c((6, 1)) = 0$.

$\sigma((3, 1)) = 1$ implies that $c((3, 2)) \neq c((4, 1))$.

Subcase 2.1. $c((3, 2)) = 0$ and $c((4, 1)) = 1$.

$\sigma((5, 1)) = 1 \Rightarrow c((5, 2)) = 0$. Hence,
 $(c((2, 1)), c((4, 1)), c((6, 1))) = (0, 1, 0)$ and $(c((1, 2)), c((3, 2)), c((5, 2))) = (1, 0, 0)$.
 $(\sigma((2, 2)), \sigma((4, 2)), \sigma((6, 2))) = (1, 1, 1) \Rightarrow (c((2, 3)), c((4, 3)), c((6, 3))) = (0, 0, 0)$.
 $(\sigma((1, 3)), \sigma((3, 3)), \sigma((5, 3))) = (1, 1, 1) \Rightarrow (c((1, 4)), c((3, 4)), c((5, 4))) = (0, 1, 1)$.
 $(\sigma((2, 4)), \sigma((4, 4)), \sigma((6, 4))) = (1, 1, 1) \Rightarrow (c((2, 5)), c((4, 5)), c((6, 5))) = (0, 1, 0)$.

If $n = 5$, then $\sigma((1, 5)) = 0$, a contradiction. Hence assume that $n \geq 9$.
 $(\sigma((1, 5)), \sigma((3, 5)), \sigma((5, 5))) = (1, 1, 1) \Rightarrow (c((1, 6)), c((3, 6)), c((5, 6))) = (1, 1, 1)$.
 $(\sigma((2, 6)), \sigma((4, 6)), \sigma((6, 6))) = (1, 1, 1) \Rightarrow (c((2, 7)), c((4, 7)), c((6, 7))) = (1, 0, 1)$.
 $(\sigma((1, 7)), \sigma((3, 7)), \sigma((5, 7))) = (1, 1, 1) \Rightarrow (c((1, 8)), c((3, 8)), c((5, 8))) = (0, 1, 1)$.
 $(\sigma((2, 8)), \sigma((4, 8)), \sigma((6, 8))) = (1, 1, 1) \Rightarrow (c((2, 9)), c((4, 9)), c((6, 9))) = (1, 1, 1)$.

If $n = 9$, then $\sigma((1, 9)) = 0$, a contradiction. Hence assume that $n \geq 13$.
 $(\sigma((1, 9)), \sigma((3, 9)), \sigma((5, 9))) = (1, 1, 1) \Rightarrow (c((1, 10)), c((3, 10)), c((5, 10))) = (1, 0, 0)$.
 $(\sigma((2, 10)), \sigma((4, 10)), \sigma((6, 10))) = (1, 1, 1) \Rightarrow (c((2, 11)), c((4, 11)), c((6, 11))) = (1, 0, 1)$.
 $(\sigma((1, 11)), \sigma((3, 11)), \sigma((5, 11))) = (1, 1, 1) \Rightarrow (c((1, 12)), c((3, 12)), c((5, 12))) = (0, 0, 0)$.
 $(\sigma((2, 12)), \sigma((4, 12)), \sigma((6, 12))) = (1, 1, 1) \Rightarrow (c((2, 13)), c((4, 13)), c((6, 13))) = (0, 1, 0)$.

If $n = 13$, then $\sigma((1, 13)) = 0$, a contradiction. Hence assume that $n \geq 17$.
 $(\sigma((1, 13)), \sigma((3, 13)), \sigma((5, 13))) = (1, 1, 1) \Rightarrow (c((1, 14)), c((3, 14)), c((5, 14))) = (1, 0, 0)$.

Observe that

$$(c((2, 13)), c((4, 13)), c((6, 13))) = (0, 1, 0) = (c((2, 1)), c((4, 1)), c((6, 1))) \text{ and} \\ (c((1, 14)), c((3, 14)), c((5, 14))) = (1, 0, 0) = (c((1, 2)), c((3, 2)), c((5, 2))).$$

This implies that

$$(c((2, j)), c((4, j)), c((6, j))) = (c((2, j + 12)), c((4, j + 12)), c((6, j + 12))) \text{ for odd } j \geq 3 \\ \text{and}$$

$$(c((1, j)), c((3, j)), c((5, j))) = (c((1, j + 12)), c((3, j + 12)), c((5, j + 12))) \text{ for even } j \geq 4.$$

As we have obtained contradictions for $n = 5, 9, 13$, we have contradictions for $n = 17, 21, 25$, and so on.

Subcase 2.2. $c((3, 2)) = 1$ and $c((4, 1)) = 0$.

$\sigma((5, 1)) = 1 \Rightarrow c((5, 2)) = 1$. Hence,

$$(c((2, 1)), c((4, 1)), c((6, 1))) = (0, 0, 0) \text{ and } (c((1, 2)), c((3, 2)), c((5, 2))) = (1, 1, 1). \\ (\sigma((2, 2)), \sigma((4, 2)), \sigma((6, 2))) = (1, 1, 1) \Rightarrow (c((2, 3)), c((4, 3)), c((6, 3))) = (1, 1, 1). \\ (\sigma((1, 3)), \sigma((3, 3)), \sigma((5, 3))) = (1, 1, 1) \Rightarrow (c((1, 4)), c((3, 4)), c((5, 4))) = (0, 0, 0). \\ (\sigma((2, 4)), \sigma((4, 4)), \sigma((6, 4))) = (1, 1, 1) \Rightarrow (c((2, 5)), c((4, 5)), c((6, 5))) = (0, 0, 0).$$

If $n = 5$, then $\sigma((1, 5)) = 0$, a contradiction. Hence assume that $n \geq 9$.

$$(\sigma((1, 5)), \sigma((3, 5)), \sigma((5, 5))) = (1, 1, 1) \Rightarrow (c((1, 6)), c((3, 6)), c((5, 6))) = (1, 1, 1).$$

Observe that

$$(c((2, 5)), c((4, 5)), c((6, 5))) = (0, 0, 0) = (c((2, 1)), c((4, 1)), c((6, 1))) \text{ and} \\ (c((1, 6)), c((3, 6)), c((5, 6))) = (1, 1, 1) = (c((1, 2)), c((3, 2)), c((5, 2))).$$

This implies that

$$(c((2, j)), c((4, j)), c((6, j))) = (c((2, j + 4)), c((4, j + 4)), c((6, j + 4))) \text{ for odd } j \geq 3 \\ \text{and}$$

$$(c((1, j)), c((3, j)), c((5, j))) = (c((1, j + 4)), c((3, j + 4)), c((5, j + 4))) \text{ for even } j \geq 4.$$

As we have obtained a contradiction for $n = 5$, we have a contradiction for $n = 9$, and so on.

Case 3. $c((2, 1)) = 1$ and $c((1, 2)) = c((6, 1)) = 0$.

$\sigma((3, 1)) = 1$ implies that $c((3, 2)) = c((4, 1))$.

Subcase 3.1. $c((3, 2)) = c((4, 1)) = 0$.

$\sigma((5, 1)) = 1 \Rightarrow c((5, 2)) = 1$. Hence,

$$(c((2, 1)), c((4, 1)), c((6, 1))) = (1, 0, 0) \text{ and } (c((1, 2)), c((3, 2)), c((5, 2))) = (0, 0, 1). \\ (\sigma((2, 2)), \sigma((4, 2)), \sigma((6, 2))) = (1, 1, 1) \Rightarrow (c((2, 3)), c((4, 3)), c((6, 3))) = (0, 0, 0). \\ (\sigma((1, 3)), \sigma((3, 3)), \sigma((5, 3))) = (1, 1, 1) \Rightarrow (c((1, 4)), c((3, 4)), c((5, 4))) = (1, 1, 0). \\ (\sigma((2, 4)), \sigma((4, 4)), \sigma((6, 4))) = (1, 1, 1) \Rightarrow (c((2, 5)), c((4, 5)), c((6, 5))) = (1, 0, 0).$$

If $n = 5$, then $\sigma((1, 5)) = 0$, a contradiction. Hence assume that $n \geq 9$.

$$(\sigma((1, 5)), \sigma((3, 5)), \sigma((5, 5))) = (1, 1, 1) \Rightarrow (c((1, 6)), c((3, 6)), c((5, 6))) = (1, 1, 1). \\ (\sigma((2, 6)), \sigma((4, 6)), \sigma((6, 6))) = (1, 1, 1) \Rightarrow (c((2, 7)), c((4, 7)), c((6, 7))) = (0, 1, 1). \\ (\sigma((1, 7)), \sigma((3, 7)), \sigma((5, 7))) = (1, 1, 1) \Rightarrow (c((1, 8)), c((3, 8)), c((5, 8))) = (1, 1, 0). \\ (\sigma((2, 8)), \sigma((4, 8)), \sigma((6, 8))) = (1, 1, 1) \Rightarrow (c((2, 9)), c((4, 9)), c((6, 9))) = (1, 1, 1).$$

If $n = 9$, then $\sigma((5, 9)) = 0$, a contradiction. Hence assume that $n \geq 13$.

$$\begin{aligned}
 (\sigma((1, 9)), \sigma((3, 9)), \sigma((5, 9))) &= (1, 1, 1) \Rightarrow (c((1, 10)), c((3, 10)), c((5, 10))) = (0, 0, 1). \\
 (\sigma((2, 10)), \sigma((4, 10)), \sigma((6, 10))) &= (1, 1, 1) \Rightarrow (c((2, 11)), c((4, 11)), c((6, 11))) = (0, 1, 1). \\
 (\sigma((1, 11)), \sigma((3, 11)), \sigma((5, 11))) &= (1, 1, 1) \Rightarrow (c((1, 12)), c((3, 12)), c((5, 12))) = (0, 0, 0). \\
 (\sigma((2, 12)), \sigma((4, 12)), \sigma((6, 12))) &= (1, 1, 1) \Rightarrow (c((2, 13)), c((4, 13)), c((6, 13))) = (1, 0, 0).
 \end{aligned}$$

If $n = 13$, then $\sigma((1, 13)) = 0$, a contradiction. Hence assume that $n \geq 17$.

$$(\sigma((1, 13)), \sigma((3, 13)), \sigma((5, 13))) = (1, 1, 1) \Rightarrow (c((1, 14)), c((3, 14)), c((5, 14))) = (0, 0, 1).$$

Observe that

$$\begin{aligned}
 (c((2, 13)), c((4, 13)), c((6, 13))) &= (1, 0, 0) = (c((2, 1)), c((4, 1)), c((6, 1))) \text{ and} \\
 (c((1, 14)), c((3, 14)), c((5, 14))) &= (0, 0, 1) = (c((1, 2)), c((3, 2)), c((5, 2))).
 \end{aligned}$$

This implies that

$$(c((2, j)), c((4, j)), c((6, j))) = (c((2, j + 12)), c((4, j + 12)), c((6, j + 12))) \text{ for odd } j \geq 3$$

and

$$(c((1, j)), c((3, j)), c((5, j))) = (c((1, j + 12)), c((3, j + 12)), c((5, j + 12))) \text{ for even } j \geq 4.$$

As we have obtained contradictions for $n = 5, 9, 13$, we have contradictions for $n = 17, 21, 25$, and so on.

Subcase 3.2. $c((3, 2)) = c((4, 1)) = 1$.

$$\sigma((5, 1)) = 1 \Rightarrow c((5, 2)) = 0. \text{ Hence,}$$

$$\begin{aligned}
 (c((2, 1)), c((4, 1)), c((6, 1))) &= (1, 1, 0) \text{ and } (c((1, 2)), c((3, 2)), c((5, 2))) = (0, 1, 0). \\
 (\sigma((2, 2)), \sigma((4, 2)), \sigma((6, 2))) &= (1, 1, 1) \Rightarrow (c((2, 3)), c((4, 3)), c((6, 3))) = (1, 1, 1). \\
 (\sigma((1, 3)), \sigma((3, 3)), \sigma((5, 3))) &= (1, 1, 1) \Rightarrow (c((1, 4)), c((3, 4)), c((5, 4))) = (1, 0, 1). \\
 (\sigma((2, 4)), \sigma((4, 4)), \sigma((6, 4))) &= (1, 1, 1) \Rightarrow (c((2, 5)), c((4, 5)), c((6, 5))) = (1, 1, 0).
 \end{aligned}$$

If $n = 5$, then $\sigma((1, 5)) = 0$, a contradiction. Hence assume that $n \geq 9$.

$$\begin{aligned}
 (\sigma((1, 5)), \sigma((3, 5)), \sigma((5, 5))) &= (1, 1, 1) \Rightarrow (c((1, 6)), c((3, 6)), c((5, 6))) = (1, 1, 1). \\
 (\sigma((2, 6)), \sigma((4, 6)), \sigma((6, 6))) &= (1, 1, 1) \Rightarrow (c((2, 7)), c((4, 7)), c((6, 7))) = (0, 0, 1). \\
 (\sigma((1, 7)), \sigma((3, 7)), \sigma((5, 7))) &= (1, 1, 1) \Rightarrow (c((1, 8)), c((3, 8)), c((5, 8))) = (1, 0, 1). \\
 (\sigma((2, 8)), \sigma((4, 8)), \sigma((6, 8))) &= (1, 1, 1) \Rightarrow (c((2, 9)), c((4, 9)), c((6, 9))) = (0, 0, 0).
 \end{aligned}$$

If $n = 9$, then $\sigma((3, 9)) = 0$, a contradiction. Hence assume that $n \geq 13$.

$$\begin{aligned}
 (\sigma((1, 9)), \sigma((3, 9)), \sigma((5, 9))) &= (1, 1, 1) \Rightarrow (c((1, 10)), c((3, 10)), c((5, 10))) = (0, 1, 0). \\
 (\sigma((2, 10)), \sigma((4, 10)), \sigma((6, 10))) &= (1, 1, 1) \Rightarrow (c((2, 11)), c((4, 11)), c((6, 11))) = (0, 0, 1). \\
 (\sigma((1, 11)), \sigma((3, 11)), \sigma((5, 11))) &= (1, 1, 1) \Rightarrow (c((1, 12)), c((3, 12)), c((5, 12))) = (0, 0, 0). \\
 (\sigma((2, 12)), \sigma((4, 12)), \sigma((6, 12))) &= (1, 1, 1) \Rightarrow (c((2, 13)), c((4, 13)), c((6, 13))) = (1, 1, 0).
 \end{aligned}$$

If $n = 13$, then $\sigma((3, 13)) = 0$, a contradiction. Hence assume that $n \geq 17$.

$$(\sigma((1, 13)), \sigma((3, 13)), \sigma((5, 13))) = (1, 1, 1) \Rightarrow (c((1, 14)), c((3, 14)), c((5, 14))) = (0, 1, 0).$$

Observe that

$$\begin{aligned}
 (c((2, 13)), c((4, 13)), c((6, 13))) &= (1, 1, 0) = (c((2, 1)), c((4, 1)), c((6, 1))) \text{ and} \\
 (c((1, 14)), c((3, 14)), c((5, 14))) &= (0, 1, 0) = (c((1, 2)), c((3, 2)), c((5, 2))).
 \end{aligned}$$

This implies that

$$(c((2, j)), c((4, j)), c((6, j))) = (c((2, j + 12)), c((4, j + 12)), c((6, j + 12))) \text{ for odd } j \geq 3$$

and

$(c((1, j)), c((3, j)), c((5, j))) = (c((1, j + 12)), c((3, j + 12)), c((5, j + 12)))$ for even $j \geq 4$.

As we have obtained contradictions for $n = 5, 9, 13$, we have contradictions for $n = 17, 21, 25$, and so on.

This completes the proof. □

3. Conclusion

For the left over cases, we conjecture that: (i) if $n \in \{14, 26, 38, \dots, 12r+2, \dots\} \cup \{16, 28, 40, \dots, 12r+4, \dots\} \cup \{8, 20, 32, \dots, 12r+8, \dots\} \cup \{22, 34, \dots, 12r+10, \dots\}$, then $mc(C_3 \square P_n) = 3$, and (ii) if $m \equiv 2 \pmod{4}$, $m \geq 10$, and $n \equiv 1 \pmod{4}$, $n \geq 5$, then $mc(C_m \square P_n) = 3$.

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