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ON THE NOMURA ALGEBRAS OF FORMALLY SELF-DUAL ASSOCIATION SCHEMES OF CLASS 2

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ABSTRACT. In this paper, the type-II matrices on (negative) Latin square graphs are considered and it is proved that, under certain conditions, the Nomura algebras of such type-II matrices are trivial. In addition, we construct type-II matrices on doubly regular tournaments and show that the Nomura algebras of such matrices are also trivial.

1. Introduction

In [3], Chan and Hosoya have considered the type-II matrices in the Bose-Mesner algebra of conference graphs and have proved that the Nomura algebras of such matrices are trivial when $n > 9$. In this paper, we show that the Nomura algebras obtained from some of the (negative) Latin square graphs are trivial. Moreover, we determine the type-II matrices attached to doubly regular tournaments. Then we show that the Nomura algebras obtained from these type-II matrices are trivial.

In the rest of the section, we remind some concepts and notations about type-II matrices, strongly regular graphs and association schemes.

1.1. Type-II matrices. In this subsection, we drive some notations and concepts in the type-II matrices and the Nomura algebras according to [2]. Let I_n and J denote the identity matrix of order n and the matrix of order n whose entries are all 1, respectively. Denote by $\mathbf{M}_n(\mathbb{C})$ the set of $n \times n$

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complex matrices. Let $W = (w_{ij})$ be an $n \times n$ complex matrix whose entries are all nonzero. We can define an associated $n \times n$ matrix by $W^{(-)} = (w_{ij}^{-1})$. An $n \times n$ complex matrix W is called *type-II* if

$$WW^{(-)t} = nI.$$

As an example, the matrix $I+xJ$ of order $n \geq 2$ is a type-II where x is one of the roots of $nx^2+nx+1 = 0$, i. e., $x = \frac{1}{2}(2 - n \pm \sqrt{n^2 - 4n})$. Such a matrix is called the *Potts model*.

Let X be a nonempty finite set with $|X| = n$. Let $W = (w_{ij})$ be a complex matrix whose rows and columns are indexed by X and whose entries are all nonzero. For each $u, v = 1, \dots, n$, we define a column vector \mathbf{e}_{uv} whose i -th entry is

$$\mathbf{e}_{uv}(i) = \frac{w_{iu}}{w_{iv}}.$$

The *Nomura algebra* of W is defined by

$$\mathcal{N}_W = \{M \in \mathbf{M}_n(\mathbb{C}) \mid \mathbf{e}_{uv} \text{ is an eigenvector of } M, \forall u, v \in X\}.$$

From [2, Corollary 4.1] it follows that the Nomura algebra of any type-II matrix is the Bose-Mesner algebra of a commutative association scheme. Since \mathcal{N}_W contains I it is nonempty. The Nomura algebra \mathcal{N}_W is said to be *trivial* if $\dim \mathcal{N}_W = 2$. For $M \in \mathcal{N}_W$, we define the *matrix of eigenvalues* of M denoted by $\Theta_W(M)$ to be an $n \times n$ matrix whose (u, v) -th entry is equal to the eigenvalue of W on \mathbf{e}_{uv} .

The following lemma determines whether or not two eigenvectors \mathbf{e}_{uv} 's belong to the same eigenspace of \mathcal{N}_W .

Lemma 1.1. [2, Lemma 3.2] *Let $\mathbf{e}_{uv}^t \mathbf{e}_{uz} \neq 0$. Then $(\Theta_W(M))_{vu} = (\Theta_W(M))_{uz}$ where $M \in \mathcal{N}_W$.*

The following lemma gives a sufficient condition for the Nomura algebra of a type-II matrix being trivial.

Lemma 1.2. *Let W be a type-II matrix. If $\mathbf{e}_{uv}^t \mathbf{e}_{uz} \neq 0$ for any distinct $u, v \in X$ and for all $z \neq u$, then \mathcal{N}_W is trivial.*

Proof. By Lemma 1.1 we see that $(\Theta_W(M))_{vu} = (\Theta_W(M))_{uz}$ for any matrix M in \mathcal{N}_W . Put $\lambda := (\Theta_W(M))_{vu}$. Then \mathbf{e}_{vu} and \mathbf{e}_{uz} belong to the same eigenspace denoted by \mathcal{E}_λ for all $z \neq u$. Put $V_\lambda := \{\mathbf{e}_{uz} \mid z \neq u\}$. Clearly, $V_\lambda \subseteq \mathcal{E}_\lambda$. From the definition of W we conclude that all vectors in V_λ are in $\mathbf{1}^\perp$. From [2, Lemma 2.1] it follows that V_λ is the set of $n - 1$ linearly independent vectors. Hence, $\dim(\mathcal{E}_\lambda)$ is either $n - 1$ or n . If $\dim(\mathcal{E}_\lambda) = n - 1$ then the Hermitian space \mathbb{C}^n has the form

$$\mathbf{1} \oplus \mathcal{E}_\lambda.$$

Thus, the Bose-Mesner algebra \mathcal{N}_W has only two principle idempotents which implies that its rank is equal to 2.

If $\dim(\mathcal{E}_\lambda) = n$, then each vector in \mathbb{C}^n is an eigenvector of M , especially $(1, 0, 0, \dots, 0)^t$. This implies that $M = \lambda I_n$. Hence, $\mathcal{N}_W = \{I_n\}$ which implies that $n = 1$. □

Theorem 1.3. [2, Theorem 6.4] *Suppose W is a type-II matrix of the form $J + (t - 1)A$, where A is the incidence matrix of a symmetric $(n, k, \lambda) - \text{BIBD}$ and*

$$(1.1) \quad t = \frac{1}{2(k - \lambda)} (2(k - \lambda) - n \pm \sqrt{n(n - 4(k - \lambda))}).$$

If $n > 3$ and $t \neq -1$, then the Nomura algebra of W is trivial.

1.2. Strongly regular graphs. A *strongly regular graph* Γ with parameters (n, k, λ, μ) is a simple graph on n vertices which is regular with valency k such that if the vertices α and β are adjacent then there are exactly λ vertices adjacent to both α and β and otherwise μ there are. It is known that if Γ is connected then it has three distinct eigenvalues k, θ and τ . The strongly regular graph Γ is called (*positive*) *Latin square* (resp. *negative Latin square*) if

$$n = m^2, \quad k = g(m - \epsilon), \quad \lambda = \epsilon m + g^2 - 3\epsilon g, \quad \mu = g(g - \epsilon)$$

for some positive integers m and g where $\epsilon = 1$ (resp. $\epsilon = -1$). For such a graph $\theta = m - g$ and $\tau = -g$ (resp. $\theta = g$ and $\tau = g - m$).

In [2, Section 8], Chan and Godsil constructed the type-II matrices on the strongly regular graphs. Moreover, they showed that if Γ is formally self-dual then there are at most six type-II matrices, up to equivalence, in the Bose-Mesner algebra of Γ . In this paper, we consider one of the six cases and investigate its Nomura algebras. So we assume that Γ is a (negative) Latin square graph. Let A_1 be the adjacency matrix of Γ and let A_2 be its complement adjacency matrix. Again in [2, Section 8], it was shown that a matrix $W := I + xA_1 + yA_2$ is type-II if

$$(1.2) \quad \begin{aligned} x &= \frac{1}{2\tau} (\theta^2 - \tau^2 + 2\theta + \epsilon_1 \sqrt{(\theta - \tau)(\theta - \tau + 2)(\theta + \tau)(\theta + \tau + 2)}) \quad \text{and} \\ y &= \frac{1}{2(\theta + 1)} (\theta^2 - \tau^2 + 2(\theta + 1) + \epsilon_2 \sqrt{(\theta - \tau)(\theta - \tau + 2)(\theta + \tau)(\theta + \tau + 2)}), \end{aligned}$$

or

$$(1.3) \quad \begin{aligned} x &= \frac{1}{2\theta} (\tau^2 - \theta^2 + 2\tau + \epsilon_3 \sqrt{(\theta - \tau)(\theta - \tau - 2)(\theta + \tau)(\theta + \tau + 2)}) \quad \text{and} \\ y &= \frac{1}{2(\tau + 1)} (\tau^2 - \theta^2 + 2(\tau + 1) + \epsilon_4 \sqrt{(\theta - \tau)(\theta - \tau - 2)(\theta + \tau)(\theta + \tau + 2)}), \end{aligned}$$

where $\epsilon_i = \pm 1$ for each i . We show that the Nomura algebras obtained from such a matrix are trivial when $m - 2 \nmid \lambda(g - 1), \lambda(g - 2)$ and $m \geq 3g$ with $g > 2$. We shall assume that $\epsilon = \epsilon_i = \epsilon_{i+1} = \pm 1$ in (1.2) and (1.3) for $i = 1, 3$ and prove the main theorem. Substituting $\theta = m - g$ and $\tau = -g$ into (1.2), we can write x and y in terms of m and g as follows.

$$(1.4) \quad x = \frac{e + \epsilon\sqrt{c}}{2g}, \quad y = \frac{gx - 1}{g - m - 1},$$

where $e = 2mg - m^2 + 2g - 2m$ and $c = m(m + 2)(m - 2g)(m - 2g + 2)$. Similarly, (1.3) can be written as the form

$$(1.5) \quad x = \frac{e' + \epsilon'\sqrt{c'}}{2(g - m)}, \quad y = \frac{(g - m)x - 1}{g - 1},$$

where $e' = -2mg + m^2 + 2g$ and $c' = m(m-2)(m-2g)(m-2g+2)$. In (1.4) and (1.5), if we set $m = 2g - 1$ then we see that x and y are complex and so Γ is a conference graph whose Nomura algebras investigated in [3], otherwise they are real. Therefore, the type-II matrices constructed on (negative) Latin square graphs are real when $m \geq 3g$.

Our main result is as follows:

Theorem 1.4. *Suppose that $W = I + xA_1 + yA_2$ is a type-II matrix attached to a Latin square graph where x and y satisfy (1.4) or (1.5). Let $m \geq 3g$ and $m - 2 \nmid \lambda(g - 1), \lambda(g - 2)$ where $g > 2$. Then \mathcal{N}_W is trivial.*

1.3. Association schemes. Let X be a nonempty finite set. Define $\Delta(X) := \{(x, x) \mid x \in X\}$. For each subset R of $X \times X$, we define by R^t the set of all pairs (x, y) with $(y, x) \in R$. Let $\mathcal{R} = \{R_0, R_1, \dots, R_d\}$ be a partition of $X \times X$ and put $R_0 := \Delta(X)$. Then $\mathcal{X} = (X, \mathcal{R})$ is called an *association scheme* of class d if it satisfies the following conditions **(a)** $R_i^t \in \mathcal{R}$ for each $R_i \in \mathcal{R}$; we denote R_i^t by $R_{i'}$, **(b)** for each $R_i, R_j, R_k \in \mathcal{R}$ there exists a number p_{ij}^k called the *intersection number* such that $|R_i(x) \cap R_{j'}(y)| = p_{ij}^k$ for all $(x, y) \in R_k$ where $R(x) := \{y \in X \mid (x, y) \in R\}$. An association scheme is called *symmetric* if $R_i = R_{i'}$, for all i .

It is known that each of the basis relations R_i is associated with a matrix A_i called the *adjacency matrix*. A subalgebra of $\mathbf{M}_n(\mathbb{C})$ spanned by $\mathcal{B} = \{I = A_0, A_1, \dots, A_d\}$ is called the *Bose-Mesner algebra* of an association scheme if it satisfies the following conditions **(a)** $A_{i'} \in \mathcal{B}$ for each $A_i \in \mathcal{B}$, **(b)** the sum of the elements of \mathcal{B} is J , **(c)** \mathcal{B} is a basis for a $(d + 1)$ -dimensional semisimple subalgebra of $\mathbf{M}_n(\mathbb{C})$ whose structure constants are nonnegative.

Let \mathcal{X} be an association scheme of class d with the basis relations R_i 's and the primitive idempotents E_i 's. Then from [1, (3.6) and (3.8)] we have

$$A_i = \sum_{j=0}^d p_i(j)E_j, \quad E_i = \frac{1}{n} \sum_{j=0}^d q_i(j)A_j.$$

The matrices $P = (p_i(j))$ and $Q = (q_i(j))$ are called the first and the second eigenmatrices of \mathcal{X} , respectively, in which j and i denote row and column, respectively. An association scheme is said to be *formally self-dual* if $P = Q$ for some ordering of the primitive idempotents if necessary.

Association schemes of class 2 arise in connection with some combinatorial structures. The basis relations of a symmetric association scheme of class 2 are the edge sets of complementary strongly regular graphs. Conversely, each of the edge set of a strongly regular graph and the edge set of its complement forms a symmetric association scheme of class 2. The Bose-Mesner algebra of a strongly regular graph Γ is formally self-dual if and only if $n = (\theta - \tau)^2$ if and only if Γ is a conference graph, a Latin square graph, or a negative Latin square graph, see [4].

A *doubly regular tournament* is a loopless directed graph of order $2k + 1$ and of valency k whose adjacency matrix A satisfies $A + A^t + I = J$ and $A^2 = \frac{k-1}{2}A + \frac{k+1}{2}A^t$. It follows that k must be an odd number. This definition follows that A is the adjacency matrix of a doubly regular tournament

if and only if so is A^t . It is known that the subalgebra of spanned by $\{I, A, A^t\}$ is the *Bose-Mesner algebra* of a nonsymmetric association scheme of class 2. Conversely, any nonsymmetric association scheme of class 2 arises from a doubly regular tournament. It is easy to show that the Bose-Mesner algebra of a doubly regular tournament is formally self-dual.

2. Nomura algebras

2.1. The Nomura algebras constructed on some of the Latin square graphs. In this subsection, we follow the notations in subsection 1.2 and show that under certain conditions the Nomura algebras of the type-II matrices attached to (negative) Latin square graphs are trivial. To do so, we first show that c and c' cannot be perfect squares when $m \geq 3g$.

Lemma 2.1. *If $m \geq 3g$, then c and c' both cannot be perfect squares.*

Proof. Since $m \geq 3g$, we have $m > 3g - 1$. Then $m = 3g + i - 1$ for some positive integer i . Therefore, we have

$$\begin{aligned} c &= m(m+2)(m-2g)(m-2g+2) \\ &= (3g+i-1)(3g+i+1)(g+i-1)(g+i+1) \\ &= ((3g+i)^2-1)((g+i)^2-1) \end{aligned}$$

and

$$\begin{aligned} c' &= m(m-2)(m-2g)(m-2g+2) \\ &= (3g+i-1)(3g+i-3)(g+i-1)(g+i+1) \\ &= ((3g+i-2)^2-1)((g+i)^2-1). \end{aligned}$$

Let $X_1 = 3g + i, X_2 = 3g + i - 2$ and $Y = g + i$. Then $c = (X_1^2 - 1)(Y^2 - 1)$ and $c' = (X_2^2 - 1)(Y^2 - 1)$. Let $X \in \{X_1, X_2\}$. We show that

$$(2.1) \quad (XY - 2)^2 < (X^2 - 1)(Y^2 - 1) < (XY - 1)^2.$$

It implies that since $XY - 1$ and $XY - 2$ are consecutive natural numbers, $(X^2 - 1)(Y^2 - 1)$ is not perfect square and hence c and c' are not both perfect squares. We have

$$(X^2 - 1)(Y^2 - 1) < (XY - 1)^2 \iff 0 < (X - Y)^2.$$

Since $X \neq Y$, the last inequality is always true. Then the right-hand side of (2.1) is also true. On the other hand,

$$(XY - 2)^2 < (X^2 - 1)(Y^2 - 1) \iff 0 < 4XY - X^2 - Y^2 - 3.$$

Let $T = 4XY - X^2 - Y^2 - 3$. If $X = X_1$, then $T = 2g^2 + 12ig + 2i^2 - 3$ and if $X = X_2$, then $T = 2g^2 + 4g + 8ig + 2i^2 - 4i - 7$. Clearly, in each case $T > 0$ and hence $(XY - 2)^2 < (X^2 - 1)(Y^2 - 1)$. This shows that c and c' both lie between the squares of two consecutive integers. Therefore, they cannot be perfect squares. □

Let $W = (w_{st})$ be a matrix with entries $\{1, x, y\}$ whose rows and columns are indexed by the set X . Let $\Lambda = \{u, v, z\}$ be a subset of X with $|\Lambda| = 3$. Define $\Omega_{uvz} = \{s \in X \mid w_{us} = w_{vs} = w_{zs} = x\}$. Similarly, for instance, if we replace u by \bar{u} , then we may define the subset $\Omega_{\bar{u}vz}$ of X as the elements

s in X such that $w_{us} = y$ and $w_{vs} = w_{zs} = x$. Therefore, by the similar way one can define $\Omega_{u\bar{v}z}$, $\Omega_{uv\bar{z}}$ or $\Omega_{\bar{u}\bar{v}z}$ and so on. Also, we define Ω_s to be the set of elements t of X such that $w_{st} = x$.

The proof of Theorem 1.4. Let w_{ij} denote the (i, j) -th entry of W . Suppose that Γ is a Latin square graph with parameters

$$n = m^2, \quad k = g(m - 1), \quad \lambda = g^2 - 3g + m, \quad \mu = g(g - 1)$$

for some positive integers m and g . By definition of \mathbf{e}_{uv} 's, we have

$$\mathbf{e}_{uv}(t)\mathbf{e}_{uz}(t) = \frac{w_{tu}^2}{w_{tv}w_{tz}}$$

for each $t \in X$ and then we have

$$\mathbf{e}_{uv}(t)\mathbf{e}_{uz}(t) = \begin{cases} 1 & \text{if } t \in \Omega_{uvz} \cup \Omega_{\bar{u}\bar{v}\bar{z}} \\ xy^{-1} & \text{if } t \in \Omega_{u\bar{v}z} \cup \Omega_{uv\bar{z}} \\ yx^{-1} & \text{if } t \in \Omega_{\bar{u}\bar{v}z} \cup \Omega_{\bar{u}v\bar{z}} \\ x^2y^{-2} & \text{if } t \in \Omega_{u\bar{v}\bar{z}} \\ y^2x^{-2} & \text{if } t \in \Omega_{\bar{u}vz} \end{cases}$$

for each $t \in X \setminus \Lambda$. Since W is symmetric and the entries on its main diagonal are 1, the Hermitian product of the vectors \mathbf{e}_{uv} and \mathbf{e}_{uz} can be written as the following form

$$\begin{aligned} \mathbf{e}_{uv}^t \mathbf{e}_{uz} &= \sum_{t \in X} \mathbf{e}_{uv}(t)\mathbf{e}_{uz}(t) \\ (2.2) \quad &= (w_{uv}w_{uz})^{-1} + (w_{uv}^2 + w_{uz}^2)w_{vz}^{-1} + |\Omega_{uvz} \cup \Omega_{\bar{u}\bar{v}\bar{z}}| + |\Omega_{u\bar{v}z} \cup \Omega_{uv\bar{z}}|xy^{-1} + \\ & \quad |\Omega_{\bar{u}\bar{v}z} \cup \Omega_{\bar{u}v\bar{z}}|yx^{-1} + |\Omega_{u\bar{v}\bar{z}}|x^2y^{-2} + |\Omega_{\bar{u}vz}|y^2x^{-2}. \end{aligned}$$

Let $\hat{u} \in \{u, \bar{u}\}$, $\hat{v} \in \{v, \bar{v}\}$, $\hat{z} \in \{z, \bar{z}\}$ and $s, t \in \Lambda$. Define $\Lambda_t = \{h \in \Lambda \mid w_{ht} = x\}$ and $\Lambda_{st} = \{h \in \Lambda \mid w_{ht} = w_{hs} = x\}$. By the definitions of Ω_t 's and $\Omega_{\hat{u}\hat{v}\hat{z}}$'s we have the following sets.

$$\begin{aligned} \Omega_u &= \bigcup_{\hat{v}, \hat{z}} \Omega_{u\hat{v}\hat{z}} \cup \Lambda_u, & \Omega_u \cap \Omega_v &= \bigcup_{\hat{z}} \Omega_{uv\hat{z}} \cup \Lambda_{uv}, & X &= \bigcup_{\hat{u}, \hat{v}, \hat{z}} \Omega_{\hat{u}\hat{v}\hat{z}} \cup \Lambda, \\ \Omega_v &= \bigcup_{\hat{u}, \hat{z}} \Omega_{\hat{u}v\hat{z}} \cup \Lambda_v, & \Omega_u \cap \Omega_z &= \bigcup_{\hat{v}} \Omega_{u\hat{v}z} \cup \Lambda_{uz}, \\ \Omega_z &= \bigcup_{\hat{u}, \hat{v}} \Omega_{\hat{u}\hat{v}z} \cup \Lambda_z, & \Omega_v \cap \Omega_z &= \bigcup_{\hat{u}} \Omega_{\hat{u}vz} \cup \Lambda_{vz}. \end{aligned}$$

By definition, $|\Omega_s \cap \Omega_t|$ is equal to λ if $w_{st} = x$ and μ otherwise. Then we have

$$(2.3) \quad \begin{cases} \sum_{\hat{v}, \hat{z}} |\Omega_{u\hat{v}\hat{z}}| + |\Lambda_u| = k, & \sum_{\hat{z}} |\Omega_{uv\hat{z}}| + |\Lambda_{uv}| \in \{\lambda, \mu\}, & \sum_{\hat{u}, \hat{v}, \hat{z}} |\Omega_{\hat{u}\hat{v}\hat{z}}| = n - 3, \\ \sum_{\hat{u}, \hat{z}} |\Omega_{\hat{u}v\hat{z}}| + |\Lambda_v| = k, & \sum_{\hat{v}} |\Omega_{u\hat{v}z}| + |\Lambda_{uz}| \in \{\lambda, \mu\}, \\ \sum_{\hat{u}, \hat{v}} |\Omega_{\hat{u}\hat{v}z}| + |\Lambda_z| = k, & \sum_{\hat{u}} |\Omega_{\hat{u}vz}| + |\Lambda_{vz}| \in \{\lambda, \mu\}. \end{cases}$$

There are eight distinct cases to consider depending on whether w_{st} is x or y . In each case, we calculate $\mathbf{e}_{uv}^t \mathbf{e}_{uz}$. This calculation shows that if x and y satisfy (1.4) and (1.5), $\mathbf{e}_{uv}^t \mathbf{e}_{uz}$ can be written in the form $a + \epsilon b \sqrt{c}$ and $a' + \epsilon b' \sqrt{c'}$, respectively for some real numbers a, a', b and b' . It follows from Lemma 2.1 that to show that $\mathbf{e}_{uv}^t \mathbf{e}_{uz}$ is nonzero it is sufficient to prove that one of a, a', b and b' is nonzero. In what follows, we describe a method of computing $\mathbf{e}_{uv}^t \mathbf{e}_{uz}$ for one case. The other cases are done similarly. By hypothesis, $m = 3g + i$ for some nonnegative integer i . Also, let $|\Omega_{uvz}| = f$.

Suppose first that $w_{uv} = w_{uz} = w_{vz} = x$. Then $|\Omega_s \cap \Omega_t| = \lambda$ for all $s, t \in \Lambda$. Also, we have $\Lambda_u = \{v, z\}, \Lambda_v = \{u, z\}, \Lambda_z = \{u, v\}, \Lambda_{uv} = \{z\}, \Lambda_{vz} = \{u\}$ and $\Lambda_{uz} = \{v\}$. We substitute these back into (2.3) and solve it to obtain

$$\begin{aligned} |\Omega_{\bar{u}vz}| &= |\Omega_{u\bar{v}z}| = |\Omega_{uv\bar{z}}| = \lambda - f - 1, & |\Omega_{\bar{u}\bar{v}\bar{z}}| &= n + 3(\lambda - k) - f, \\ |\Omega_{u\bar{v}\bar{z}}| &= |\Omega_{\bar{u}v\bar{z}}| = |\Omega_{\bar{u}\bar{v}z}| = k - 2\lambda + f. \end{aligned}$$

Substituting into (2.2), it follows that

$$\begin{aligned} \mathbf{e}_{uv}^t \mathbf{e}_{uz} &= x^{-2} + 2x + n - 3(k - \lambda) + (\lambda - f - 1)(2xy^{-1} + y^2x^{-2}) + \\ &\quad (k - 2\lambda + f)(2yx^{-1} + x^2y^{-2}). \end{aligned}$$

We substitute x and y from (1.4) and (1.5) into the equation above and get

$$a = \frac{mc}{2g^2(m - g + 1)}, \quad a' = \frac{mc'}{2(m - g)^2(g - 1)}.$$

Clearly, $a, a' \neq 0$. Similarly, if $w_{uv} = w_{uz} = x$ and $w_{vz} = y$, then

$$\begin{aligned} \mathbf{e}_{uv}^t \mathbf{e}_{uz} &= x^{-2} + 2x^2y^{-1} + n - 3k + 2\lambda + \mu + 2(\lambda - f)xy^{-1} + (\mu - f - 1)y^2x^{-2} + \\ &\quad 2(k - \lambda - \mu + f)yx^{-1} + (k - 2\lambda + f - 2)x^2y^{-2}. \end{aligned}$$

It follows that

$$a = \frac{m(m - g + 2)c}{2g^2(m - g + 1)^2}, \quad a' = \frac{m(g - 2)c'}{2(m - g)^2(g - 1)^2}.$$

Clearly, $a, a' \neq 0$. If $w_{uv} = w_{vz} = x$ and $w_{uz} = y$ or $w_{uv} = y$ and $w_{uz} = w_{vz} = x$, we have

$$\begin{aligned} \mathbf{e}_{uv}^t \mathbf{e}_{uz} &= x^{-1}y^{-1} + y^2x^{-1} + x + n - 3k + 2\lambda + \mu + (2k - 3\lambda - \mu + 2f - 2)yx^{-1} + \\ &\quad (\lambda + \mu - 2f - 1)xy^{-1} + (k - \lambda - \mu + f)x^2y^{-2} + (\lambda - f)y^2x^{-2}. \end{aligned}$$

It implies that

$$\begin{aligned} b &= \frac{m(m^2 + 3mg^2 - m^2g - 5mg - 2f - mf + 2m - g^3 + 5g^2 - 6g)}{g^2(m - g + 1)^2}, \\ b' &= \frac{m(g^3 - 5g^2 + mg - mf - 2m + 6g + 2f)}{(m - g)^2(g - 1)^2}. \end{aligned}$$

If $b = 0$, then $f = -\frac{5mg + m^2g - 3mg^2 - m^2 + 6g - 5g^2 + g^3 - 2m}{m + 2}$. We substitute $m = 3g + i$ into the last equality to obtain $f = \frac{g^3 + g^2 + 3ig^2 - ig - 2i + i^2g - i^2}{-3g - i - 2}$. Let d denote the numerator of this fraction. Since $g > 1$, we see that $3g^2 > g + 2$. Then

$$\begin{aligned} d &= g^3 + g^2 + 3ig^2 - ig - 2i + i^2g - i^2 \\ &= g^3 + g^2 + \underbrace{i(3g^2 - g - 2)}_{>0} + \underbrace{i^2(g - 1)}_{>0}. \end{aligned}$$

This shows that $d > 0$ and so $f < 0$, a contradiction. Hence, $b \neq 0$. If $b' = 0$, then $f = \frac{\lambda(g-2)}{m-2}$. This contradicts the hypothesis that $m - 2 \nmid \lambda(g - 2)$. Thus, $b' \neq 0$. If $w_{uv} = x$ and $w_{uz} = w_{vz} = y$ or $w_{uv} = w_{vz} = y$ and $w_{uz} = x$, then

$$\begin{aligned} \mathbf{e}_{uv}^t \mathbf{e}_{uz} &= x^{-1}y^{-1} + x^2y^{-1} + y + n - 3k + \lambda + 2\mu - 1 + (\lambda + \mu - 2f)xy^{-1} + \\ &\quad (2k - \lambda - 3\mu + 2f - 1)yx^{-1} + (k - \lambda - \mu + f - 1)x^2y^{-2} + (\mu - f)y^2x^{-2}. \end{aligned}$$

From this we deduce that

$$\begin{aligned} b &= \frac{m(4g^2 - mf - g^3 + 3mg^2 - 3mg - m^2g - 3g - 2f)}{g^2(m - g + 1)^2}, \\ b' &= \frac{m(g^3 - 4g^2 - m - mf + 3g + mg + 2f)}{(m - g)^2(g - 1)^2}. \end{aligned}$$

If $b = 0$, then $f = -\frac{g(m^2 - 3mg + 3m + g^2 - 4g + 3)}{m + 2}$. Substituting $m = 3g + i$ into the last equality, we get $f = -\frac{g(i^2 + g^2 + 5g + 3i + 3ig + 3)}{3g + i + 2} < 0$ which is a contradiction and hence $b \neq 0$. If $b' = 0$, then $f = \frac{\lambda(g-1)}{m-2}$ which contradicts the hypothesis of the theorem. If $w_{vz} = x$ and $w_{uv} = w_{uz} = y$, then

$$\begin{aligned} \mathbf{e}_{uv}^t \mathbf{e}_{uz} &= y^{-2} + 2y^2x^{-1} + n - 3k + \lambda + 2\mu - 1 + 2(\mu - f)xy^{-1} + (\lambda - f)y^2x^{-2} + \\ &\quad 2(k - \lambda - \mu + f - 1)yx^{-1} + (k - 2\mu + f)x^2y^{-2}. \end{aligned}$$

It follows that

$$a = \frac{m(g + 1)c}{2g^2(m - g + 1)^2}, \quad a' = \frac{m(m - g - 1)c'}{2(m - g)^2(g - 1)^2}.$$

Clearly, $a, a' \neq 0$. If $w_{uv} = w_{uz} = w_{vz} = y$, then

$$\begin{aligned} \mathbf{e}_{uv}^t \mathbf{e}_{uz} &= y^{-2} + 2y + n - 3k + 3\mu - 3 + (\mu - f)(2xy^{-1} + y^2x^{-2}) + \\ &\quad (k - 2\mu + f)(2yx^{-1} + x^2y^{-2}). \end{aligned}$$

It implies that

$$a = \frac{mc}{2g(m - g + 1)^2}, \quad a' = \frac{mc'}{2(m - g)(g - 1)^2}.$$

It follows that $a, a' \neq 0$. In each case, we see that one of a, a', b and b' is nonzero and so $\mathbf{e}_{uv}^t \mathbf{e}_{uz} \neq 0$ by Lemma 2.1. Now from Lemma 1.2 we imply that \mathcal{N}_W is trivial. This completes the proof of the theorem. \square

From the proof of Theorem 1.4, we conclude that if x and y only satisfy (1.4) we can eliminate the hypotheses $m - 2 \nmid \lambda(g - 1), \lambda(g - 2)$ and $g > 2$ from the theorem. Therefore, we can state the following corollary.

Corollary 2.2. *Suppose that $W = I + xA_1 + yA_2$ is a type-II matrix attached to a Latin square graph where x and y satisfy (1.4). If $m \geq 3g$, then \mathcal{N}_W is trivial.*

The following corollary is a consequence of Theorem 1.4 for negative Latin square graphs.

Corollary 2.3. *Let $W = I + xA_1 + yA_2$ be a type-II matrix attached to a negative Latin square graph and let $m \geq 3g$. If x and y satisfy (1.5) or x and y satisfy (1.4) with $m + 2 \nmid \lambda(g + 1), \lambda(g + 2)$, then \mathcal{N}_W is trivial.*

Proof. Let Γ be a negative Latin square graph with parameters

$$(m^2, g(m + 1), g^2 + 3g - m, g(g + 1))$$

for some positive integers m and g . It is known that by replacing m and g by their opposites, the parameters of a negative Latin square graph obtain from a (positive) Latin square graph. Therefore, by replacing m and g by their opposites and substituting into (1.4) and (1.5), we can express (1.2) and (1.3) in terms of m and g for a negative Latin square graph. It follows that (1.2) (resp. (1.3)) for a negative Latin square graph when $\epsilon = \pm 1$ is the same (1.5) (resp. (1.4)) which we compute for a Latin square graph when $\epsilon = \mp 1$. This means that to show that $\mathbf{e}_{uv}^t \mathbf{e}_{uz} \neq 0$, it is sufficient to replace m and g by their opposites in a, a', b and b' given in the proof of Theorem 1.4 and using a similar argument it can be shown that a, a', b or b' are nonzero in each case. Note that, using techniques similar to those used in the proof of Lemma 2.1 we can show that c and c' both cannot be perfect squares by replacing m and g by their opposites when $m \geq 3g$. This completes the proof of the corollary. \square

2.2. The Nomura algebras constructed on doubly regular tournaments. In this subsection, we first determine type-II matrices attached to nonsymmetric association schemes of class 2. Then, we show that the Nomura algebras of these type-II matrices are trivial.

Theorem 2.4. *Let A be the adjacency matrix of a doubly regular tournament on n vertices with valency k . Suppose that*

$$W = I + xA + y(J - I - A).$$

Then W is a type-II matrix if and only if one of the following holds:

- (1) $x = y = \frac{1}{2}(2 - n \pm \sqrt{n^2 - 4n})$ and W is the Potts model,
- (2) $x = 1, y = -\frac{k \pm i\sqrt{2k+1}}{k+1},$
- (3) $x = -\frac{k \pm i\sqrt{2k+1}}{k+1}, y = 1.$

Proof. We first note that the eigenvalues of a doubly regular tournament Γ are k, α and $\bar{\alpha}$, where $\alpha = \frac{1}{2}(-1 + i\sqrt{2k+1})$. Using the first eigenmatrix of Γ , we see that $WW^{(-)t} = nI$ is equivalent to $(1 + xp_1(j) + yp_2(j))(1 + x^{-1}p_1(j') + y^{-1}p_2(j')) = n$ for all $j = 0, 1$ and 2 . Then,

$$\begin{aligned} (1 + kx + ky)(1 + kx^{-1} + ky^{-1}) &= n, \\ (1 + \alpha x + \bar{\alpha}y)(1 + \bar{\alpha}x^{-1} + \alpha y^{-1}) &= n, \\ (1 + \bar{\alpha}x + \alpha y)(1 + \alpha x^{-1} + \bar{\alpha}y^{-1}) &= n, \end{aligned}$$

where $n = 2k + 1$. Using Maple, these equations may be solved for x and y in terms of k to obtain one of the following cases that may arise.

- (1) x and y are one of the roots of $z^2 + (2k - 1)z + 1 = 0$, that is, $x, y \in \{\frac{1}{2}(1 - 2k - \sqrt{4k^2 - 4k - 3}), \frac{1}{2}(1 - 2k + \sqrt{4k^2 - 4k - 3})\}$. If $x \neq y$, then W is not type-II and so $x = y$. Using $n = 2k + 1$, we can express x and y in terms of n , i. e., $x = y = \frac{1}{2}(2 - n \pm \sqrt{n^2 - 4n})$.
- (2) $x = 1$ and y is one of the roots of $(1 + k)z^2 + 2kz + 1 + k = 0$, i. e., $y = -\frac{k \pm i\sqrt{2k+1}}{k+1}.$

(3) $y = 1$ and x is one of the roots of $(1+k)z^2 + 2kz + 1+k = 0$, i. e., $x = -\frac{k \pm i\sqrt{2k+1}}{k+1}$.

This completes the proof. \square

Theorem 2.5. *Let W be a type-II matrix attached to doubly regular tournaments obtained in Theorem 2.4 with $k > 1$. Then \mathcal{N}_W is trivial.*

Proof. Let x and y satisfy case (1) of Theorem 2.4. It is well known that the Nomura algebra of a Potts model of order $n \geq 5$ is trivial. Let x and y satisfy two other cases of Theorem 2.4 and let $\lambda \geq 1$. In [5], it has been proved that there exists a doubly regular tournament of order $4\lambda + 3$ if and only if there exists a skew Hadamard matrix of order $4\lambda + 4$. On the other hand, in [6], it has been proved that there exists a Hadamard matrix of order $4\lambda + 4$ if and only if there exists a symmetric $(4\lambda + 3, 2\lambda + 1, \lambda)$ -BIBD. Therefore, the adjacency matrix of a doubly regular tournament is the incidence matrix of a symmetric $(4\lambda + 3, 2\lambda + 1, \lambda)$ -BIBD and so $n = 2k + 1$ and $\lambda = \frac{k-1}{2}$. Substituting into (1.1), we see that $t = -\frac{k \pm i\sqrt{2k+1}}{k+1}$. If x and y satisfy case (2) of Theorem 2.4, then we have

$$\begin{aligned} W &= I + xA + y(J - I - A) \\ &= I + A + t(J - I - A) \\ &= J + (t - 1)A^t. \end{aligned}$$

If x and y satisfy case (3) of Theorem 2.4, then we have

$$\begin{aligned} W &= I + xA + y(J - I - A) \\ &= I + tA + (J - I - A) \\ &= J + (t - 1)A. \end{aligned}$$

Therefore, in each case we see that the conditions of Theorem 2.4 satisfies the hypotheses of Theorem 1.3 and so \mathcal{N}_W is trivial. \square

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