TWO-OUT DEGREE EQUITABLE DOMINATION IN GRAPHS

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Abstract. An equitable domination has interesting application in the context of social networks. In a network, nodes with nearly equal capacity may interact with each other in a better way. In the society persons with nearly equal status, tend to be friendly. In this paper, we introduce new variant of equitable domination of a graph. Basic properties and some interesting results have been obtained.

1. Introduction

By a graph $G = (V, E)$ we mean a finite, undirected graph with neither loops nor multiple edges. The order and size of $G$ are denoted by $n$ and $m$, respectively. For graph theoretic terminology we refer to Chartrand and Lesnaik [2]. Let $G = (V, E)$ be a graph and let $v \in V$. The open neighborhood and the closed neighborhood of $v$ are denoted by $N(v) = \{u \in V : uv \in E\}$ and $N[v] = N(v) \cup \{v\}$, respectively. If $S \subseteq V$ then $N(S) = \bigcup_{v \in S} N(v)$ and $N[S] = N(S) \cup S$. A subset $S$ of $V$ is called a dominating set if $N[S] = V$. The minimum (maximum) cardinality of a minimal dominating set of $G$ is called the domination number (upper domination number) of $G$ and is denoted by $\gamma(G)$ ($\Gamma(G)$). An excellent treatment of the fundamentals of domination is given in the book by Haynes et al. [4]. A survey of several advanced topics in domination is given in the book edited by Haynes et al. [5]. Various types of domination have been defined and studied by several authors and more than 75 models of domination are listed in the appendix of Haynes et al. [4]. A double star is the tree obtained from two disjoint stars $K_{1,n}$ and $K_{1,m}$ by connecting their centers.

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Definition 1.1. Let $G = (V, E)$ be a graph, $D \subseteq V(G)$ and $v$ be any vertex in $D$. The out degree of $v$ with respect to $D$ denoted by $od_D(v)$, is defined as $od_D(v) = |N(v) \cap (V - D)|$.

Definition 1.2[1]. Let $D$ be a dominating set of a graph $G = (V, E)$. For $v \in D$, let $od_D(v) = |N(v) \cap (V - D)|$. Then $D$ is called an equitable dominating set of type 1 if $|od_D(v_1) - od_D(v_2)| \leq 1$ for all $v_1, v_2 \in D$. The minimum cardinality of such a dominating set is denoted by $\gamma_{eq1}(G)$ and is called the 1- equitable domination number of $G$.

In this paper we make the equitable dominating set of a graph.

2. Two-Out Degree Equitable Domination Number in Graphs

Definition 2.1. A dominating set $D$ in a graph $G$ is called a two-out degree equitable dominating set if for any two vertices $u, v \in D$, $|od_D(u) - od_D(v)| \leq 2$. The minimum cardinality of a two-out degree equitable dominating set is called the two-out degree equitable domination number of $G$, and is denoted by $\gamma_{2oe}(G)$. A subset $D$ of $V$ is a minimal two-out degree equitable dominating set if no proper subset of $D$ is a two-out degree equitable dominating set.

It is obvious that any two-out degree dominating set in a graph $G$ is also a dominating set, and thus we obtain the obvious bound $\gamma(G) \leq \gamma_{2oe}(G)$. Also, it is easy to see that, $\gamma_{2oe}(G) = 1$ if and only if $\gamma(G) = 1$.

The following results are straightforward.

Proposition 2.2.

1. For the complete bipartite graph $K_{n,m}$, $1 < m \leq n$, the two-out degree equitable domination number is:

$$
\gamma_{2oe}(K_{n,m}) = \begin{cases} 
2, & \text{if } n - m \leq 2; \\
r, & \text{if } n - m = r, \ 3 \leq r < m; \\
m, & \text{if } n - m = r, \ 3 \leq m \leq r.
\end{cases}
$$

2. For the double star $S_{n,m}$, the two-out degree equitable domination number is:

$$
\gamma_{2oe}(S_{n,m}) = \begin{cases} 
2, & \text{if } |n - m| \leq 2; \\
n + m - 1, & \text{if } |n - m| \geq 3, \ n \text{ or } m = 1; \\
n + m - 2, & \text{if } |n - m| \geq 3, \ n, m \geq 2.
\end{cases}
$$

Theorem 2.3. For any connected graph $G$, if $\Delta - \delta \leq 2$, then $\gamma_{2oe}(G) = \gamma(G)$.

Proof. Let $G$ be a connected graph such that $\Delta - \delta \leq 2$ and let $D$ be a minimum dominating set of $G$. Then $|D| = \gamma(G)$. Since $\Delta - \delta \leq 2$, it follows that for any two vertices $u, v \in D$, $|od_D(u) - od_D(v)| \leq 2$. Hence $\gamma_{2oe}(G) = \gamma(G)$. □
Theorem 2.4. Let $D$ be a two-out degree equitable dominating set of a graph $G$. Then $D$ is a minimal two-out degree equitable dominating set of $G$ if and only if one of the following holds:

1. $D$ is minimal dominating set.
2. For any vertex $v \in D$, the set $U_v$ is nonempty, where $U_v = \{x, y \in D, |\text{od}_D(x) - \text{od}_D(y)| = 2, v \in D\}$, and $v$ is adjacent to $x$ but not adjacent to $y$.

Proof. Suppose that $D$ is a minimal two-out degree equitable dominating set of $G$. Then for any $v \in D$, $D - \{v\}$ is not two-out degree equitable dominating set. If $D$ is a minimal dominating set, then we are done. If not, then for any $v \in D$, let $U_v = \{x, y \in D, |\text{od}_D(x) - \text{od}_D(y)| = 2, v \in D\}$, and $v$ is adjacent to $x$ but not adjacent to $y$. Then for any $x, y \in D - \{v\}$ such that $|\text{od}_{D - \{v\}}(x) - \text{od}_{D - \{v\}}(y)| > 2$. If both $x, y$ are adjacent to $v$, then $|\text{od}_{D - \{v\}}(x) - \text{od}_{D - \{v\}}(y)| = |\text{od}_D(x) - \text{od}_D(y)| < 2$, a contradiction. If both $x, y$ are not adjacent to $v$, then $|\text{od}_{D - \{v\}}(x) - \text{od}_{D - \{v\}}(y)| = |\text{od}_D(x) - \text{od}_D(y)| \leq 2$, a contradiction. So, $v$ is adjacent to precisely one vertex of $\{x, y\}$. Without loss of generality, assume that $v$ is adjacent to $x$ and $v$ not adjacent to $y$.

Then,

$$2 < |\text{od}_{D - \{v\}}(x) - \text{od}_{D - \{v\}}(y)| = |\text{od}_D(x) + 1 - \text{od}_D(y)| \leq |\text{od}_D(x) - \text{od}_D(y)| + 1$$

So, $|\text{od}_D(x) - \text{od}_D(y)| > 1$. But $|\text{od}_D(x) - \text{od}_D(y)| \leq 2$.

So, $|\text{od}_D(x) - \text{od}_D(y)| = 2$. Hence $U_v$ is not empty.

Conversely, let $D$ be a two-out degree equitable dominating set and suppose that $D$ is a minimal two-out degree equitable dominating set. Suppose to the contrary $D$ is not a minimal two-out degree equitable dominating set. Then for every $v \in D$, $D - \{v\}$ is a two-out degree equitable dominating set. So, $D$ is not a minimal dominating set, a contradiction. Next, suppose that $D$ is a two-out degree equitable dominating set and (2) holds. Then for every $v \in D$, $U_v$ is not empty. So, for every $v \in D$, there exist $x, y \in D$ such that $v$ is adjacent to precisely one vertex of $\{x, y\}$, and $|\text{od}_D(x) - \text{od}_D(y)| = 2$.

Suppose to the contrary $D$ is not a minimal two-out degree equitable dominating set. Then for every $v \in D$, $D - \{v\}$ is a two-out degree equitable dominating set. So, $2 < |\text{od}_{D - \{v\}}(x) - \text{od}_{D - \{v\}}(y)| \leq 2$ and thus

$$2 \geq |\text{od}_{D - \{v\}}(x) - \text{od}_{D - \{v\}}(y)| = |\text{od}_D(x) - \text{od}_D(y)| \leq |\text{od}_D(x) - \text{od}_D(y)| + 1 = 3$$

Since $D - \{v\}$ is a two-out degree equitable dominating set, we have $|\text{od}_{D - \{v\}}(x) - \text{od}_{D - \{v\}}(y)| = 2$. Then $|\text{od}_{D - \{v\}}(x) - \text{od}_{D - \{v\}}(y)| = |\text{od}_D(x) - \text{od}_D(y)|$, either $\{x, y\} \subseteq N(v)$, or $\{x, y\} \cap N(v) = \phi$. □

Theorem 2.5. Let $G$ be a graph. Then $G$ has a unique minimal two-out degree equitable dominating set if and only if $G = \overline{K}_n$.

Proof. Suppose that $D$ is a unique minimal two-out degree equitable dominating set of $G$. Suppose $G \neq \overline{K}_n$, then there exists $u \in D$ such that $\text{deg}(u) \geq 1$. Then $V - \{u\}$ is a two-out degree equitable dominating set of $G$. Hence there exists $D' \subseteq V - \{u\}$ such that $D'$ is a minimal two-out degree equitable dominating set. Since $u \notin D'$, $D \neq D'$. Hence $G$ has two minimal two-out degree equitable
dominating sets, a contradiction. Thus $G = K_n$ and $V(G)$ is the only two-out degree equitable dominating set of $G$. \hfill \Box$

**Theorem 2.6.** Let $G$ be a graph of order $n$. Let $u, v \in V(G)$ such that $N(u) \neq \emptyset$, $N(v) \neq \emptyset$ and $N[u] \cap N[v] = \emptyset$. Then $\gamma_{2oe}(G) \leq n - 2$.

Proof. Let $D = V - \{u, v\}$. Since $N[u] \cap N[v] = \emptyset$, $u$ and $v$ are not adjacent vertices. Since $u$ and $v$ are not isolated, there exist two distinct vertices $x, y \in D$ such that $x$ is adjacent to $u$ but not adjacent to $v$ and $y$ is adjacent to $v$ but not adjacent to $u$. Clearly the out degree of any vertex of $D$ is either 0 or 1. Hence $D$ is a two-out degree equitable dominating set of $G$. Thus $\gamma_{2oe}(G) \leq n - 2$. \hfill \Box

**Lemma 2.7.** Let $G = (V, E)$ be a connected graph and let $D = \{u, v\}$ be a subset of $V$ such that $N(u) \cap N(v) = \emptyset$ and $|\text{od}_D(u) - \text{od}_D(v)| \leq 2$. Then $\gamma_{2oe}(\overline{G}) = 2$.

Proof. Let $D = \{u, v\}$ and let $x$ be any vertex of $V - D$. Since $u, v \in V(G)$ such that $N(u) \cap N(v) = \emptyset$, we consider the following cases.

Case 1: $x$ is adjacent to precisely one vertex of $\{u, v\}$ in $G$. Then $x$ is adjacent to precisely one vertex of $\{u, v\}$ in $\overline{G}$ also.

Case 2: $x$ is not adjacent to both $u$ and $v$ in $G$. Then $x$ is adjacent to both $u$ and $v$ in $\overline{G}$.

Since $G$ is connected, it follows from the above two cases that $\{u, v\}$ is a dominating set of $\overline{G}$ and $|\text{od}_D(u) - \text{od}_D(v)| \leq 2$ of $\overline{G}$. Hence there is no vertex with full degree and hence $\{u, v\}$ is a minimum two-out degree equitable dominating set. Thus $\gamma_{2oe}(\overline{G}) = 2$. \hfill \Box

**Theorem 2.8.** Let $G = (V, E)$ be a connected graph, let $u$ and $v$ be any two vertices of $V(G)$ such that $N(u) \cap N(v) = \emptyset$ and $|\text{deg}(u) - \text{deg}(v)| \leq 2$. Then

1. $\gamma_{2oe}(G) + \gamma_{2oe}(\overline{G}) \leq n$.
2. $\gamma_{2oe}(G)\gamma_{2oe}(\overline{G}) \leq 2(n - 2)$.

Proof. Let $G = (V, E)$ be a connected graph. By Lemma 2.7, $\gamma_{2oe}(\overline{G}) = 2$ and by Theorem 2.6, $\gamma_{2oe}(G) \leq n - 2$. Hence $\gamma_{2oe}(G) + \gamma_{2oe}(\overline{G}) \leq n$ and $\gamma_{2oe}(G)\gamma_{2oe}(\overline{G}) \leq 2(n - 2)$. \hfill \Box

**Theorem 2.9.** Let $G$ be an isolate-free graph of order $n$ and let $D$ be a maximum independent set of $G$ such that for $u \in V - D$, $|N(u) \cap D| \leq 2$. Then $\gamma_{2oe}(G) \leq n - \beta$.

Proof. Let $G$ be an isolate-free graph of order $n$. Since $D$ is the maximum independent set of $G$, $V - D$ is dominating set of $G$. Then for any $u, v \in V - D$, $|\text{od}_{V - D}(u) - \text{od}_{V - D}(v)| \leq 2$. Hence $V - D$ is a two-out degree equitable dominating set of $G$. Thus $\gamma_{2oe}(G) \leq |V - D| \leq n - |D| \leq n - \beta$. \hfill \Box

**Theorem 2.10.** For any graph $G$ of order $n$ and size $m$, $\gamma_{2oe}(G) = n - m$ if and only if $G = \bigcup_{i=1}^{t} K_{n_i,r_i}$ such that $|r_i - r_j| \leq 2$, $1 \leq i, j \leq \gamma_{2oe}(G)$.

Proof. Let $\gamma_{2oe}(G) = n - m$. Suppose that $G$ has $t$-components. The minimum number of edges in each component is $n_i - 1$, where $n_i$ is the number of vertices in that component. Since any dominating set of $G$ has at least one vertex from each component of $G$, $t \leq \gamma_{2oe}(G)$. But $m \geq n_1 - 1 + n_2 - 1 + \cdots + n_t - 1$.\hfill \Box
So, \( m \geq n - t \). Hence \( t \geq \gamma_{2oe}(G) \). Thus \( t = \gamma_{2oe}(G) \). If \( G \) is not a forest, then \( G \) contains a component \( G_1 \), say which is cyclic. Then \( m \geq n \), so that \( \gamma_{2oe}(G) \leq 0 \) which is not possible. Hence \( G \) is a forest. Since \( t = \gamma_{2oe}(G) \), it follows that each component is a star. That is \( G = \bigcup_{i=1}^{j} K_{1,r_i} \). Since the centers of the stars constitute a minimum two-out degree equitable dominating set, it follows that if \( G_i = K_{1,r_i} \) and \( G_j = K_{1,r_j} \), then \( |r_i - r_j| \leq 2 \).

**Theorem 2.11.** For any positive integer \( m \), there exists a graph \( G \) such that \( \gamma_{2oe}(G) = \left\lfloor \frac{n}{\Delta + 1} \right\rfloor = m \), where \( \lfloor x \rfloor \) denotes the greatest integer not exceeding \( x \).

*Proof.* For \( m = 1 \), take \( G = K_{4,4} \), \( \gamma_{2oe}(G) = \left\lfloor \frac{n}{\Delta + 1} \right\rfloor = 2 - 1 = 1 \).

For \( m = 2 \), take \( G = K_{3,6} \), \( \gamma_{2oe}(G) = \left\lfloor \frac{n}{\Delta + 1} \right\rfloor = 3 - 1 = 2 \).

For \( m \geq 3 \), take \( G = S_{r,s} \), where \( r + s = m + 3 \), \( s \geq r + 3 \)

\[
\gamma_{2oe}(G) = r + s - 2 = m + 1,
\]

\[
\lfloor \frac{n}{\Delta + 1} \rfloor = \left\lfloor \frac{r+s+2}{r+s+2} \right\rfloor = 1,
\]

\[
\gamma_{2oe}(G) = \left\lfloor \frac{n}{\Delta + 1} \right\rfloor = r + s - 3 = m.
\]

**Theorem 2.12.** Let \( G \) be a graph of order \( n \) having \( p_0 \) isolated vertices. Then \( \gamma_{2oe}(G) \geq \frac{n + 2p_0}{3} \).

*Proof.* Let \( D \) be any minimum two-out degree equitable dominating set of \( G \). Then for any \( v \in D \), \( v \) is dominating at most two vertices of \( V - D \). Let \( |D'| = |D| - p_0 \). Then \( 2|D'| \geq |V - D| \). It follows that, \( 2|D'| + |D| \geq n \). Then \( \gamma_{2oe}(G) \geq \frac{n + 2p_0}{3} \).

The bound is sharp for \( \overline{K_n} \).

**Theorem 2.13.** Let \( G \) be a graph and let \( D \) be a minimum two-out degree equitable dominating set of \( G \) containing \( t \) pendant vertices such that every vertex of \( V - D \) is a pendant vertex. Then \( \gamma_{2oe}(G) \geq \frac{n + t}{3} \).

*Proof.* Let \( D \) be any minimum two-out degree equitable dominating set of \( G \) containing \( t \) pendant vertices such that every vertex \( v \in V - D \) is a pendant vertex. Then \( 2|D| - t \geq |V - D| \). It follows that, \( 3|D| - t \geq n \). Hence \( \gamma_{2oe}(G) \geq \frac{n + t}{3} \).

The bound is sharp for \( lK_2, l \geq 1 \).

**Theorem 2.14.** Let \( G \) be a graph, and \( u, v, w \in V(G) \) such that \( u, v, w \) are not isolates and \((uv, vw, uw) \notin E(G)\). Then \( V - \{u, v, w\} \) is a two-out degree equitable dominating set if and only if one of the following conditions hold:

1. \( N(u) \cap N(v) \cap N(w) = \phi \).
2. \( N(u) \cap N(v) \cap N(w) \neq \phi \) and \( N[u] \cup N[v] \cup N[w] = V \).

*Proof.* Let \( D = V - \{u, v, w\} \). Suppose one of the conditions (1),(2) holds. Since \( u, v, w \) are not isolates and \((uv, vw, uw) \notin E(G)\), then \( D \) is a dominating set. If (1) holds, then any vertex \( x \in D \) is adjacent to at most two of \( u, v, w \), such that the out degree of \( x \) is at most two. Hence \( D \) is a two-out degree equitable dominating set. If (2) holds, then there exists \( x \in N(u) \cap N(v) \cap N(w) \) such that
It follows that \( \text{od}_D(x) \leq 3 \). Since \( N[u] \cup N[v] \cup N[w] = V \), \( \text{od}_D(y) \geq 1 \), for each \( y \in D \). Hence \( D \) is a two-out degree equitable dominating set.

Conversely, suppose \( D = V - \{u, v, w\} \) is a two-out degree equitable dominating set. If (1) hold, then we are done. If not then there exists \( x \in N(u) \cap N(v) \cap N(w) \). i.e, \( x \in D \), and \( \text{od}_D(x) = 3 \). Since \( D \) is a two-out degree equitable dominating set, then for any \( y \in D \), \( 1 \leq \text{od}_D(y) \leq 3 \).

Hence \( N(u) \cap N(v) \cap N(w) \neq \emptyset \) and \( N[u] \cup N[v] \cup N[w] = V \). \( \square \)

### 3. Two-Out Degree Equitable Domatic Number in Graphs

**Definition 3.1.** A partition \( P = \{V_1, V_2, \ldots, V_l\} \) of \( V(G) \) is called a two-out degree equitable domatic partition if \( V_i \) is a two-out degree equitable dominating set for every \( 1 \leq i \leq l \).

**Example:**

\[
\begin{array}{c}
1 & 2 \\
3 & 4 \\
\end{array}
\]

Figure 1

\( \{1, 2\}, \{3, 4\} \) is a two-out degree equitable domatic partition of \( G \).

**Definition 3.2.** The two-out degree equitable domatic number of \( G \) is the maximum cardinality of a two-out degree equitable domatic partition of \( G \) and is denoted by \( d_{2oe}(G) \).

We now proceed to compute \( d_{2oe}(G) \) for some standard graphs. It can be easily verified that

1. For the complete graph \( K_n \), \( d_{2oe}(K_n) = n \).
2. For the cycle \( C_n \), \( n \geq 4 \), \( d_{2oe}(C_n) = 2 \).
3. For the path \( P_n \), \( d_{2oe}(P_n) = 2 \).
4. For the star \( K_{1,n} \), \( d_{2oe}(K_{1,n}) = 2 \).
5. For the wheel \( W_n \) with \( n \) vertices, \( d_{2oe}(W_n) = 2 \).
6. For the complete bipartite graph \( K_{n,m} \), \( m \leq n \) we have

\[
d_{2oe}(K_{n,m}) = \begin{cases} 
m, & \text{if } n - m \leq 2, \ n, m \geq 2; \\
2, & \text{otherwise.} \\
\end{cases}
\]

**Theorem 3.3.** For any graph \( G \), \( d_{2oe}(G) \leq \delta(G) + 1 \).
Proof. Let $D$ be any two-out degree equitable dominating set of $G$. Then for any $v \in V(G)$, $D \cap N[v] \neq \emptyset$. Let $v \in V(G)$ such that $\text{deg}(v) = \delta(G)$ and $N[v] = \{v, u_1, u_2, \ldots, u_\delta\}$. If $d_{2\text{oe}}(G) > \delta(G) + 1$, then there exist at least $(\delta(G) + 2)$ sets of a two-out degree equitable domatic partition of $G$, each containing at least one element of $N[v]$. Then $\text{deg}(v) \geq \delta(G) + 1$, a contradiction. Hence $d_{2\text{oe}}(G) \leq \delta(G) + 1$. □

Theorem 3.4. For any graph $G$ of order $n$, $d_{2\text{oe}}(G) \leq \frac{n}{\gamma_{2\text{oe}}(G)}$.

Proof. Suppose that $d_{2\text{oe}}(G) = t$, for some positive integer $t$. Let $P = \{D_1, D_2, \ldots, D_t\}$ be the two-out degree equitable domatic partition of $G$. Obviously, $|V(G)| = \sum_{i=1}^{t} |D_i|$ and from definition of the two-out degree equitable domination number $\gamma_{2\text{oe}}(G)$, we have $\gamma_{2\text{oe}}(G) \leq |D_i|, i = 1, 2, \ldots, t$. Hence $n = \sum_{i=1}^{t} |D_i| \geq t \gamma_{2\text{oe}}(G)$. Thus $d_{2\text{oe}}(G) \leq \frac{n}{\gamma_{2\text{oe}}(G)}$. □

4. Conclusion

We can generalize the concept of a two-out degree equitable domination as follows: Let $G = (V, E)$ be a graph with dominating set $D$, then $D$ is called a $k$-out degree equitable dominating set if for any two vertices $u$ and $v$ in $D$, $|\text{od}_D(u) - \text{od}_D(v)| \leq k$. The similar results can be obtained from the two-out degree equitable domination.

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