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TWO-OUT DEGREE EQUITABLE DOMINATION IN GRAPHS

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ABSTRACT. An equitable domination has interesting application in the context of social networks. In a network, nodes with nearly equal capacity may interact with each other in a better way. In the society persons with nearly equal status, tend to be friendly. In this paper, we introduce new variant of equitable domination of a graph. Basic properties and some interesting results have been obtained.

1. **Introduction**

By a graph $G = (V, E)$ we mean a finite, undirected graph with neither loops nor multiple edges. The order and size of G are denoted by n and m, respectively. For graph theoretic terminology we refer to Chartrand and Lesnaik [2]. Let $G = (V, E)$ be a graph and let $v \in V$. The open neighborhood and the closed neighborhood of v are denoted by $N(v) = \{u \in V : uv \in E\}$ and $N[v] = N(v) \cup \{v\}$, respectively. If $S \subseteq V$ then $N(S) = \cup_{v \in S} N(v)$ and $N[S] = N(S) \cup S$. A subset S of V is called a dominating set if $N[S] = V$. The minimum (maximum) cardinality of a minimal dominating set of G is called the domination number (upper domination number) of G and is denoted by $\gamma(G)$ $(\Gamma(G))$. An excellent treatment of the fundamentals of domination is given in the book by Haynes et al. [4]. A survey of several advanced topics in domination is given in the book edited by Haynes et al. [5]. Various types of domination have been defined and studied by several authors and more than 75 models of domination are listed in the appendix of Haynes et al. [4]. A double star is the tree obtained from two disjoint stars $K_{1,n}$ and $K_{1,m}$ by connecting their centers.

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Definition 1.1. *Let $G = (V, E)$ be a graph, $D \subseteq V(G)$ and v be any vertex in D. The out degree of v with respect to D denoted by $od_D(v)$, is defined as $od_D(v) = |N(v) \cap (V - D)|$.*

Definition 1.2[1]. Let D be a dominating set of a graph $G = (V, E)$. For $v \in D$, let $od_D(v) = |N(v) \cap (V - D)|$. Then D is called an equitable dominating set of type 1 if $|od_D(v_1) - od_D(v_2)| \leq 1$ for all $v_1, v_2 \in D$. The minimum cardinality of such a dominating set is denoted by $\gamma_{eq1}(G)$ and is called the 1- equitable domination number of G.

In this paper we make the equitable dominating set of a graph.

2. Two-Out Degree Equitable Domination Number in Graphs

Definition 2.1. *A dominating set D in a graph G is called a two-out degree equitable dominating set if for any two vertices $u, v \in D$, $|od_D(u) - od_D(v)| \leq 2$. The minimum cardinality of a two-out degree equitable dominating set is called the two-out degree equitable domination number of G, and is denoted by $\gamma_{2oe}(G)$. A subset D of V is a minimal two-out degree equitable dominating set if no proper subset of D is a two-out degree equitable dominating set.*

It is obvious that any two-out degree dominating set in a graph G is also a dominating set, and thus we obtain the obvious bound $\gamma(G) \leq \gamma_{2oe}(G)$. Also, it is easy to see that, $\gamma_{2oe}(G) = 1$ if and only if $\gamma(G) = 1$.

The following results are straightforward.

Proposition 2.2.

(1) For the complete bipartite graph $K_{n,m}$, $1 < m \leq n$, the two-out degree equitable domination number is:

$$\gamma_{2oe}(K_{n,m}) = \begin{cases} 2, & \text{if } n - m \leq 2; \\ r, & \text{if } n - m = r,\ 3 \leq r < m; \\ m, & \text{if } n - m = r,\ 3 \leq m \leq r. \end{cases}$$

(2) For the double star $S_{n,m}$, the two-out degree equitable domination number is:

$$\gamma_{2oe}(S_{n,m}) = \begin{cases} 2, & \text{if } |n - m| \leq 2\ ; \\ n + m - 1, & \text{if } |n - m| \geq 3,\ n \textit{ or } m = 1; \\ n + m - 2, & \text{if } |n - m| \geq 3,\ n, m \geq 2. \end{cases}$$

Theorem 2.3. *For any connected graph G, if $\Delta - \delta \leq 2$, then $\gamma_{2oe}(G) = \gamma(G)$.*

Proof. Let G be a connected graph such that $\Delta - \delta \leq 2$ and let D be a minimum dominating set of G. Then $|D| = \gamma(G)$. Since $\Delta - \delta \leq 2$, it follows that for any two vertices $u, v \in D$, $|od_D(u) - od_D(v)| \leq 2$. Hence $\gamma_{2oe}(G) = \gamma(G)$. □

Theorem 2.4. *Let D be a two-out degree equitable dominating set of a graph G. Then D is a minimal two-out degree equitable dominating set of G if and only if one of the following holds:*

(1) *D is minimal dominating set.*

(2) *For any vertex $v \in D$, the set U_v is nonempty, where $U_v = \{x, y \in D, |od_D(x) - od_D(y)| = 2$, and v is adjacent to x but not adjacent to $y\}$.*

Proof. Suppose that D is a minimal two-out degree equitable dominating set of G. Then for any $v \in D$, $D - \{v\}$ is not two-out degree equitable dominating set. If D is a minimal dominating set, then we are done. If not, then for any $v \in D$, let $U_v = \{x, y \in D, |od_D(x) - od_D(y)| = 2$, and v is adjacent to x but not adjacent to $y\}$. There exist $x, y \in D - \{v\}$ such that $|od_{D-\{v\}}(x) - od_{D-\{v\}}(y)| > 2$. If both x, y are adjacent to v, then $|od_{D-\{v\}}(x) - od_{D-\{v\}}(y)| = |od_D(x) - od_D(y)| \leq 2$, a contradiction. If both x, y are not adjacent to v, then $|od_{D-\{v\}}(x) - od_{D-\{v\}}(y)| = |od_D(x) - od_D(y)| \leq 2$, a contradiction. So, v is adjacent to precisely one vertex of $\{x, y\}$. Without loss of generality, assume that v is adjacent to x and v not adjacent to y.
Then,

$$2 < |od_{D-\{v\}}(x) - od_{D-\{v\}}(y)| = |od_D(x) + 1 - od_D(y)| \quad \leq \quad |od_D(x) - od_D(y)| + 1$$

So, $|od_D(x) - od_D(y)| > 1$. But $|od_D(x) - od_D(y)| \leq 2$.

So, $|od_D(x) - od_D(y)| = 2$. Hence U_v is not empty.
Conversely, let D be a two-out degree equitable dominating set and suppose that D is a minimal two-out degree equitable dominating set. Suppose to the contrary D is not a minimal two-out degree equitable dominating set. Then for every $v \in D$, $D - \{v\}$ is a two-out degree equitable dominating set. So, D is not a minimal dominating set, a contradiction. Next, suppose that D is a two-out degree equitable dominating set and (2) holds. Then for every $v \in D$, U_v is not empty. So, for every $v \in D$, there exist $x, y \in D$ such that v is adjacent to precisely one vertex of $\{x, y\}$, and $|od_D(x) - od_D(y)| = 2$. Suppose to the contrary D is not a minimal two-out degree equitable dominating set. Then for every $v \in D$, $D - \{v\}$ is a two-out degree equitable dominating set. So, $2 < |od_{D-\{v\}}(x) - od_{D-\{v\}}(y)| \leq 2$ and thus

$$2 \geq |od_{D-\{v\}}(x) - od_{D-\{v\}}(y)| = |od_D(x) - od_D(y)| \quad \leq \quad |od_D(x) - od_D(y)| + 1 = 3$$

Since $D - \{v\}$ is a two-out degree equitable dominating set, we have $|od_{D-\{v\}}(x) - od_{D-\{v\}}(y)| = 2$. Then $|od_{D-\{v\}}(x) - od_{D-\{v\}}(y)| = |od_D(x) - od_D(y)|$, either $\{x, y\} \subseteq N(v)$, or $\{x, y\} \cap N(v) = \phi$. □

Theorem 2.5. *Let G be a graph. Then G has a unique minimal two-out degree equitable dominating set if and only if $G = \overline{K}_n$.*

Proof. Suppose that D is a unique minimal two-out degree equitable dominating set of G. Suppose $G \neq \overline{K}_n$, then there exists $u \in D$ such that $deg(u) \geq 1$. Then $V - \{u\}$ is a two-out degree equitable dominating set of G. Hence there exists $D' \subseteq V - \{u\}$ such that D' is a minimal two-out degree equitable dominating set. Since $u \notin D'$, $D \neq D'$. Hence G has two minimal two-out degree equitable dominating sets, a contradiction. Thus $G = \overline{K}_n$ and $V(G)$ is the only two-out degree equitable dominating set of G. □

Theorem 2.6. *Let G be a graph of order n. Let $u, v \in V(G)$ such that $N(u) \neq \phi$, $N(v) \neq \phi$ and $N[u] \cap N[v] = \phi$. Then $\gamma_{2oe}(G) \leq n-2$.*

Proof. Let $D = V - \{u, v\}$. Since $N[u] \cap N[v] = \phi$, u and v are not adjacent vertices. Since u and v are not isolated, there exist two distinct vertices $x, y \in D$ such that x is adjacent to u but not adjacent to v and y is adjacent to v but not adjacent to u. Clearly the out degree of any vertex of D is either 0 or 1. Hence D is a two-out degree equitable dominating set of G. Thus $\gamma_{2oe}(G) \leq n-2$. □

Lemma 2.7. *Let $G = (V, E)$ be a connected graph and let $D = \{u, v\}$ be a subset of V such that $N(u) \cap N(v) = \phi$ and $|od_D(u) - od_{V-D}(v)| \leq 2$. Then $\gamma_{2oe}(\overline{G}) = 2$.*

Proof. Let $D = \{u, v\}$ and let x be any vertex of $V - D$. Since $u, v \in V(G)$ such that $N(u) \cap N(v) = \phi$, we consider the following cases.

Case 1: x is adjacent to precisely one vertex of $\{u, v\}$ in G. Then x is adjacent to precisely one vertex of $\{u, v\}$ in $\overline{G}$ also.

Case 2: x is not adjacent to both u and v in G. Then x is adjacent to both u and v in $\overline{G}$.

Since G is connected, it follows from the above two cases that $\{u, v\}$ is a dominating set of $\overline{G}$ and $|od_D(u) - od_D(v)| \leq 2$ of $\overline{G}$. Hence there is no vertex with full degree and hence $\{u, v\}$ is a minimum two-out degree equitable dominating set. Thus $\gamma_{2oe}(\overline{G}) = 2$. □

Theorem 2.8. *Let $G = (V, E)$ be a connected graph, let u and v be any two vertices of $V(G)$ such that $N(u) \cap N(v) = \phi$ and $|deg(u) - deg(v)| \leq 2$. Then*

(1) $\gamma_{2oe}(G) + \gamma_{2oe}(\overline{G}) \leq n$.
(2) $\gamma_{2oe}(G)\gamma_{2oe}(\overline{G}) \leq 2(n-2)$.

Proof. Let $G = (V, E)$ be a connected graph. By Lemma 2.7, $\gamma_{2oe}(\overline{G}) = 2$ and by Theorem 2.6, $\gamma_{2oe}(G) \leq n-2$. Hence $\gamma_{2oe}(G) + \gamma_{2oe}(\overline{G}) \leq n$ and $\gamma_{2oe}(G)\gamma_{2oe}(\overline{G}) \leq 2(n-2)$. □

Theorem 2.9. *Let G be an isolate-free graph of order n and let D be a maximum independent set of G such that for $u \in V - D$, $|N(u) \cap D| \leq 2$. Then $\gamma_{2oe}(G) \leq n - \beta$.*

Proof. Let G be an isolate-free graph of order n. Since D is a maximum independent set of G, $V - D$ is dominating set of G. Then for any $u, v \in V - D$, $|od_{V-D}(u) - od_{V-D}(v)| \leq 2$. Hence $V - D$ is a two-out degree equitable dominating set of G. Thus $\gamma_{2oe}(G) \leq |V - D| \leq n - |D| \leq n - \beta$. □

Theorem 2.10. *For any graph G of order n and size m, $\gamma_{2oe}(G) = n - m$ if and only if $G = \cup_{i=1}^{\gamma_{2oe}} K_{1,r_i}$ such that $|r_i - r_j| \leq 2$, $1 \leq i, j \leq \gamma_{2oe}(G)$.*

Proof. Let $\gamma_{2oe}(G) = n - m$. Suppose that G has t-components. The minimum number of edges in each component is $n_i - 1$, where n_i is the number of vertices in that component. Since any dominating set of G has at least one vertex from each component of G, $t \leq \gamma_{2oe}(G)$. But $m \geq n_1 - 1 + n_2 - 1 + \cdots + n_t - 1$.

So, $m \geq n-t$. Hence $t \geq \gamma_{2oe}(G)$. Thus $t = \gamma_{2oe}(G)$. If G is not a forest, then G contains a component G_1, say which is cyclic. Then $m \geq n$, so that $\gamma_{2oe}(G) \leq 0$ which is not possible. Hence G is a forest. Since $t = \gamma_{2oe}(G)$, it follows that each component is a star. That is $G = \cup_{i=1}^{\gamma_{2oe}} K_{1,r_i}$. Since the centers of the stars constitute a minimum two-out degree equitable dominating set, it follows that if $G_i = K_{1,r_i}$ and $G_j = K_{1,r_j}$, then $|r_i - r_j| \leq 2$. □

Theorem 2.11. *For any positive integer m, there exists a graph G such that $\gamma_{2oe}(G) - \lfloor \frac{n}{\Delta+1} \rfloor = m$, where $\lfloor x \rfloor$ denotes the greatest integer not exceeding x.*

Proof. For $m = 1$, take $G = K_{4,4}$, $\gamma_{2oe}(G) - \lfloor \frac{n}{\Delta+1} \rfloor = 2 - 1 = 1$.
For $m = 2$, take $G = K_{3,6}$, $\gamma_{2oe}(G) - \lfloor \frac{n}{\Delta+1} \rfloor = 3 - 1 = 2$.
For $m \geq 3$, take $G = S_{r,s}$, where $r + s = m + 3$, $s \geq r + 3$
$\gamma_{2oe}(G) = r + s - 2 = m + 1$,
$\lfloor \frac{n}{\Delta+1} \rfloor = \lfloor \frac{r+s+2}{s+2} \rfloor = 1$,
$\gamma_{2oe}(G) - \lfloor \frac{n}{\Delta+1} \rfloor = r + s - 3 = m$. □

Theorem 2.12. *Let G be a graph of order n having p_0 isolated vertices. Then $\gamma_{2oe}(G) \geq \frac{n+2p_0}{3}$.*

Proof. Let D be any minimum two-out degree equitable dominating set of G. Then for any $v \in D$, v is dominating at most two vertices of $V - D$. Let $|D'| = |D| - p_0$. Then $2|D'| \geq |V - D|$. It follows that, $2|D'| + |D| \geq n$. Then $\gamma_{2oe}(G) \geq \frac{n+2p_0}{3}$. □

The bound is sharp for $\overline{K_n}$.

Theorem 2.13. *Let G be a graph and let D be a minimum two-out degree equitable dominating set of G containing t pendant vertices such that every vertex of $V - D$ is a pendant vertex. Then $\gamma_{2oe}(G) \geq \frac{n+t}{3}$.*

Proof. Let D be any minimum two-out degree equitable dominating set of G containing t pendant vertices such that every vertex $v \in V - D$ is a pendant vertex. Then $2|D| - t \geq |V - D|$. It follows that, $3|D| - t \geq n$. Hence $\gamma_{2oe}(G) \geq \frac{n+t}{3}$. □

The bound is sharp for lK_2, $l \geq 1$.

Theorem 2.14. *Let G be a graph, and $u, v, w \in V(G)$ such that u, v, w are not isolates and $(uv, vw, uw) \notin E(G)$. Then $V - \{u, v, w\}$ is a two-out degree equitable dominating set if and only if one of the following conditions hold:*

(1) $N(u) \cap N(v) \cap N(w) = \phi$.
(2) $N(u) \cap N(v) \cap N(w) \neq \phi$ *and* $N[u] \cup N[v] \cup N[w] = V$.

Proof. Let $D = V - \{u, v, w\}$. Suppose one of the conditions (1),(2) holds. Since u, v, w are not isolates and $(uv, vw, uw) \notin E(G)$, then D is a dominating set. If (1) holds, then any vertex $x \in D$ is adjacent to at most two of u, v, w, such that the out degree of x is at most two. Hence D is a two-out degree equitable dominating set. If (2) holds, then there exists $x \in N(u) \cap N(v) \cap N(w)$ such that

$x \in D$. It follows that $od_D(x) \leq 3$. Since $N[u] \cup N[v] \cup N[w] = V$, $od_D(y) \geq 1$, for each $y \in D$. Hence D is a two-out degree equitable dominating set.

Conversely, suppose $D = V - \{u, v, w\}$ is a two-out degree equitable dominating set. If (1) hold, then we are done. If not then there exists $x \in N(u) \cap N(v) \cap N(w)$. i.e, $x \in D$, and $od_D(x) = 3$. Since D is a two-out degree equitable dominating set, then for any $y \in D$, $1 \leq od_D(y) \leq 3$.

Hence $N(u) \cap N(v) \cap N(w) \neq \phi$ and $N[u] \cup N[v] \cup N[w] = V$. □

3. Two-Out Degree Equitable Domatic Number in Graphs

Definition 3.1. *A partition $P = \{V_1, V_2, \ldots, V_l\}$ of $V(G)$ is called a two-out degree equitable domatic partition if V_i is a two-out degree equitable dominating set for every $1 \leq i \leq l$.*

Example :

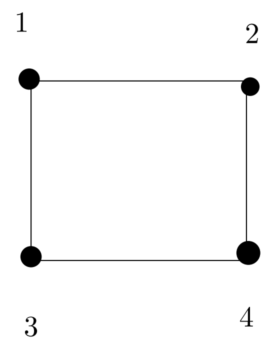

Figure 1

$\{\{1, 2\}, \{3, 4\}\}$ is a two-out degree equitable domatic partition of G.

Definition 3.2. *The two-out degree equitable domatic number of G is the maximum cardinality of a two-out degree equitable domatic partition of G and is denoted by $d_{2oe}(G)$.*

We now proceed to compute $d_{2oe}(G)$ for some standard graphs. It can be easily verified that

(1) For the complete graph K_n, $d_{2oe}(K_n) = n$.
(2) For the cycle C_n, $n \geq 4$, $d_{2oe}(C_n) = 2$.
(3) For the path P_n, $d_{2oe}(P_n) = 2$.
(4) For the star $K_{1,n}$, $d_{2oe}(K_{1,n}) = 2$.
(5) For the wheel W_n with n vertices, $d_{2oe}(W_n) = 2$.
(6) For the complete bipartite graph $K_{n,m}$, $m \leq n$ we have

$$d_{2oe}(K_{n,m}) = \begin{cases} m, & \text{if } n - m \leq 2,\ n, m \geq 2; \\ 2, & \text{otherwise.} \end{cases}$$

Theorem 3.3. *For any graph G, $d_{2oe}(G) \leq \delta(G) + 1$.*

Proof. Let D be any two-out degree equitable dominating set of G. Then for any $v \in V(G)$, $D \cap N[v] \neq \phi$. Let $v \in V(G)$ such that $deg(v) = \delta(G)$ and $N[v] = \{v, u_1, u_2, \ldots, u_\delta\}$. If $d_{2oe}(G) > \delta(G) + 1$, then there exist at least $(\delta(G)+2)$ sets of a two-out degree equitable domatic partition of G, each containing at least one element of $N[v]$. Then $deg(v) \geq \delta(G) + 1$, a contradiction. Hence $d_{2oe}(G) \leq \delta(G) + 1$. □

Theorem 3.4. *For any graph G of order n, $d_{2oe}(G) \leq \frac{n}{\gamma_{2oe}(G)}$.*

Proof. Suppose that $d_{2oe}(G) = t$, for some positive integer t. Let $P = \{D_1, D_2, \ldots, D_t\}$ be the two-out degree equitable domatic partition of G. Obviously, $|V(G)| = \sum_{i=1}^{t} |D_i|$ and from definition of the two-out degree equitable domination number $\gamma_{_{2oe}}(G)$, we have $\gamma_{_{2oe}}(G) \leq |D_i|$, $i = 1, 2, \ldots, t$. Hence $n = \sum_{i=1}^{t} |D_i| \geq t\gamma_{_{2oe}}(G)$. Thus $d_{2e}(G) \leq \frac{n}{\gamma_{_{2oe}}(G)}$. □

4. Conclusion

We can generalize the concept of a two-out degree equitable domination as follows: Let $G = (V, E)$ be a graph with dominating set D, then D is called a k-out degree equitable dominating set if for any two vertices u and v in D, $|od_{_D}(u) - od_{_D}(v)| \leq k$. The similar results can be obtained from the two-out degree equitable domination.

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