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BOUNDING THE RAINBOW DOMINATION NUMBER OF A TREE IN TERMS OF ITS ANNIHILATION NUMBER

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Abstract. A 2-*rainbow dominating function* (2RDF) of a graph G is a function f from the vertex set $V(G)$ to the set of all subsets of the set $\{1,2\}$ such that for any vertex $v \in V(G)$ with $f(v) = \emptyset$ the condition $\bigcup_{u\in N(v)} f(u) = \{1,2\}$ is fulfilled, where $N(v)$ is the open neighborhood of v. The *weight* of a 2RDF f is the value $\omega(f) = \sum_{v\in V} |f(v)|$. The 2-*rainbow domination number* of a graph G, denoted by $\gamma_{r2}(G)$, is the minimum weight of a 2RDF of G. The *annihilation number* $a(G)$ is the largest integer k such that the sum of the first k terms of the non-decreasing degree sequence of G is at most the number of edges in G. In this paper, we prove that for any tree T with at least two vertices, $\gamma_{r2}(T) \le a(T)+1$.

1. **Introduction**

In this paper, G is a simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The order $|V|$ of G is denoted by $n = n(G)$. For every vertex $v \in V(G)$, the *open neighborhood* $N_G(v) = N(v)$ is the set $\{u \in V(G) \mid uv \in E(G)\}$ and the *closed neighborhood* of v is the set $N_G[v] = N[v] = N(v) \cup \{v\}$. The *degree* of a vertex $v \in V$ is $\deg_G(v) = \deg(v) = |N(v)|$. The *minimum* and *maximum degree* of a graph G are denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively. The *open neighborhood* of a set $S \subseteq V$ is the set $N(S) = \bigcup_{v\in S} N(v)$, and the *closed neighborhood* of S is the set $N[S] = N(S) \cup S$. We write P_n for a path of order n. For a subset $S \subseteq V(G)$, we let

$$\sum(S,G) = \sum_{v\in S} \deg_G(v).$$

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A *leaf* of a tree T is a vertex of degree 1, a *support vertex* is a vertex adjacent to a leaf and a *strong support vertex* is a vertex adjacent to at least two leaves. For $r, s \geq 1$, a double star $S(r, s)$ is a tree with exactly two vertices that are not leaves, with one adjacent to r leaves and the other to s leaves. For a vertex v in a rooted tree T, let $C(v)$ denote the set of children of v. Let $D(v)$ denote the set of descendants of v and $D[v] = D(v) \cup \{v\}$. The *maximal subtree* at v is the subtree of T induced by $D[v]$, and is denoted by T_v.

For a positive integer k, a *k-rainbow dominating function* (kRDF) of a graph G is a function f from the vertex set $V(G)$ to the set of all subsets of the set $\{1, 2, \ldots, k\}$ such that for any vertex $v \in V(G)$ with $f(v) = \emptyset$ the condition $\bigcup_{u\in N(v)} f(u) = \{1, 2, \ldots, k\}$ is fulfilled. The *weight* of a kRDF f is the value $\omega(f) = \sum_{v\in V} |f(v)|$. The *k-rainbow domination number* of a graph G, denoted by $\gamma_{rk}(G)$, is the minimum weight of a kRDF of G. A $\gamma_{rk}(G)$*-function* is a k-rainbow dominating function of G with weight $\gamma_{rk}(G)$. Note that $\gamma_{r1}(G)$ is the classical domination number $\gamma(G)$. The k-rainbow domination number was introduced by Brešar, Henning, and Rall [2] and has been studied by several authors (see for example [3, 4, 5, 8, 11, 12, 13]).

Let $d_1, d_2, \ldots, d_n$ be the degree sequence of a graph G arranged in non-decreasing order, and so $d_1 \leq d_2 \leq \ldots \leq d_n$. Pepper [9] defined the annihilation number of G, denoted $a(G)$, to be the largest integer k such that the sum of the first k terms of the degree sequence is at most half the sum of the degrees in the sequence. Equivalently, the annihilation number is the largest integer k such that

$$\sum_{i=1}^{k} d_i \leq \sum_{i=k+1}^{n} d_i.$$

We observe that if G has m edges and annihilation number k, then $\sum_{i=1}^{k} d_i \leq m$.

The relation between annihilation number and independence number and some domination parameters have been studied by several authors (see for example [1, 6, 7, 10]).

Our purpose in this paper is to establish an upper bound on the 2-rainbow domination number of a tree in terms of its annihilation number. We prove that for any tree T with at least 2 vertices, $\gamma_{r2}(T) \leq a(T)+1$. The following results show that for a path P_n with at least two vertices, $\gamma_{r2}(P_n) \leq a(P_n) + 1$.

Proposition A. ([3]) For $n \geq 1$,

$$\gamma_{r2}(P_n) = \left\lfloor \frac{n}{2} \right\rfloor + 1.$$

Proposition B. For $n \geq 2$,

$$a(P_n) = \left\lceil \frac{n}{2} \right\rceil.$$

Corollary 1.1. For $n \geq 2$, $\gamma_{r2}(P_n) \leq a(P_n) + 1$.

2. Main result

A *subdivision* of an edge uv is obtained by replacing the edge uv with a path uwv, where w is a new vertex. The *subdivision graph* $S(G)$ is the graph obtained from G by subdividing each edge of G.

The subdivision star $S(K_{1,t})$ for $t \geq 2$, is called a *healthy spider* S_t. A *wounded spider* S_t is the graph formed by subdividing at most $t-1$ of the edges of a star $K_{1,t}$ for $t \geq 2$. Note that stars are wounded spiders. A *spider* is a healthy or wounded spider.

Lemma 2.1. If T is a spider, then $\gamma_{r2}(T) \leq a(T)+1$ with equality if and only if $T = P_4$.

Proof. First let $T = S_t$ be a healthy spider for some $t \geq 2$. Then obviously $\gamma_{r2}(T) = t+1$ and $a(T) = t + \lfloor \frac{t}{2} \rfloor$ and hence $\gamma_{r2}(T) \leq a(T)$.

Now let T be a wounded spider obtained from $K_{1,t}$ $(t \geq 2)$ by subdividing $0 \leq s \leq t-1$ edges. If $(t,s) = (1,2)$, then $T = P_4$, $\gamma_{r2}(T) = 3$ and $a(T) = 2$, hence $\gamma_{r2}(T) = a(T)+1$. If $s = 0$, then T is a star and we have $\gamma_{r2}(T) = 2$ and $a(T) = t$. Hence $\gamma_{r2}(T) \leq a(T)$. Suppose $s > 0$. Then $\gamma_{r2}(T) = 2+s$ and $a(T) = t + \lfloor \frac{s}{2} \rfloor$. It follows that $\gamma_{r2}(T) \leq a(T)$ if $(t,s) \neq (2,1)$ and the proof is complete. □

Observation 2.2. Let T be a tree. If there is a path $x_3x_2x_1$ in T with $\deg(x_2) = 2$ and $\deg(x_1) = 1$, then T has a $\gamma_{r2}(T)$-function f such that $|f(x_1)| = 1$, $|f(x_3)| \geq 1$ and $f(x_1) \neq f(x_3)$.

Proof. Suppose g is a $\gamma_{r2}(T)$-function. Consider three cases.

Case 1. $g(x_1) = \emptyset$.
Then $g(x_2) = \{1,2\}$ and the function $f : V(G) \to \mathcal{P}(\{1,2\})$ defined by $f(x_1) = \{1\}, f(x_2) = \emptyset, f(x_3) = g(x_3) \cup \{2\}$ and $f(x) = g(x)$ for $x \in V(T) - \{x_1,x_2,x_3\}$ is a $\gamma_{r2}(T)$-function with desired property.

Case 2. $|g(x_1)| = 1$.
We may assume without loss of generality that $g(x_1) = \{1\}$. If $g(x_2) = \emptyset$, then we must have $2 \in g(x_3)$ and the result follows. Let $|g(x_2)| \geq 1$. Then obviously we may assume that $g(x_1) \neq g(x_2)$. Now the function $f : V(G) \to \mathcal{P}(\{1,2\})$ defined by $f(x_2) = \emptyset, f(x_3) = g(x_2) \cup g(x_3)$ and $f(x) = g(x)$ for $x \in V(T) - \{x_2,x_3\}$ is a $\gamma_{r2}(T)$-function with desired property.

Case 3. $g(x_1) = \{1,2\}$.
Then the function $f : V(G) \to \mathcal{P}(\{1,2\})$ defined by $f(x_3) = \{1\} \cup g(x_3), f(x_1) = \{2\}, f(x_2) = \emptyset$ and $f(x) = g(x)$ for $x \in V(T) - \{x_1,x_2,x_3\}$ is a $\gamma_{r2}(T)$-function with $|f(x_1)| = 1$ and $f(x_3)| \geq 1$, as desired. □

Theorem 2.3. If T is a tree of order $n \geq 2$, then $\gamma_{r2}(T) \leq a(T)+1$, and this bound is sharp.

Proof. The proof is by induction on n. The statement holds for all trees of order $n = 2,3,4$. For the inductive hypothesis, let $n \geq 5$ and suppose that for every nontrivial tree T of order less than n the result is true. Let T be a tree of order n. We may assume that T is not a path for otherwise the result follows by Corollary 1.1. If $\operatorname{diam}(T) = 2$, then T is a star and hence $\gamma_{r2}(T) \leq a(T)$ by Lemma 2.1. If $\operatorname{diam}(T) = 3$, then T is a double star $S(r,s)$. In this case, $a(T) = r+s$ and $\gamma_{r2}(T) \leq 4$. If $r+s = 3$, then $\gamma_{r2}(T) = 3$ and so $\gamma_{r2}(T) = a(T)$. If $r+s \geq 4$, then $\gamma_{r2}(T) \leq 4$ and we have $\gamma_{r2}(T) \leq a(T)$. Hence we may assume that $\operatorname{diam}(T) \geq 4$.

In what follows, we will consider trees T' formed from T by removing a set of vertices. For such a tree T' of order n', let $d'_1, d'_2, \ldots, d'_{n'}$ be a non-decreasing degree sequence of T', and let S' be a set of vertices which corresponds to the first $a(T')$ terms in the degree sequence of T'. In fact, if $u_1, u_2, \ldots, u_{n'}$

are the vertices of T' such that $\deg(u_i) = d'_i$ for each $1 \le i \le n'$, then $S' = \{u_1, u_2, \ldots, u_{a(T')}\}$. We denote the size of T' by m'. We proceed further with a series of claims that we may assume satisfied by the tree.

Claim 1. T has no strong support vertex such as u that the graph obtained from T by removing u and the leaves adjacent to u is connected.

Let T have a strong support vertex u such that the graph obtained from T by removing u and the leaves adjacent to u is connected. Suppose w is a vertex in T with maximum distance from u. Root T at w and let v be the parent of u. Assume $T' = T - T_u$. Then every $\gamma_{r2}(T')$-function f, can be extended to a 2-rainbow dominating function of T by defining $f(u) = \{1, 2\}$ and $f(x) = \emptyset$ for each leaf x adjacent to u. Hence $\gamma_{r2}(T) \le \gamma_{r2}(T') + 2$. If $v \notin S'$, then $\sum(S', T) = \sum(S', T')$ and if $v \in S'$, then $\sum(S', T) = \sum(S', T') + 1$. Thus, $\sum(S', T) - 1 \le \sum(S', T') \le m' \le m - 3$. Let z_1, z_2 be two leaves adjacent to u and assume $S = S' \cup \{z_1, z_2\}$. Then $\sum(S, T) = \sum(S', T) + 2 \le m$, implying that $a(T) \ge a(T') + 2$. By inductive hypothesis, we obtain

$$\gamma_{r2}(T) \le \gamma_{r2}(T') + 2 \le a(T') + 3 \le a(T) + 1$$

as desired. (■)

Let $v_1 v_2 \ldots v_D$ be a diametral path in T and root T at v_D. If $\mathrm{diam}(T) = 4$, then it follows from Claim 1 that T is a spider and so $\gamma_{r2}(T) \le a(T)$ by Lemma 2.1. Assume $\mathrm{diam}(T) \ge 5$. It follows from Claim 1 that T_{v_3} is a spider.

Claim 2. $\deg_T(v_3) \le 3$.

Suppose $\deg_T(v_3) \ge 4$. Let $T' = T - \{v_1, v_2\}$. It is easy to see that T' has a $\gamma_{r2}(T')$-function f such that $f(v_3) \ne \emptyset$ and hence $\gamma_{r2}(T) \le \gamma_{r2}(T') + 1$. If $v_3 \notin S'$, then $\sum(S', T) = \sum(S', T')$ and if $v_3 \in S'$, then $\sum(S', T) = \sum(S', T') + 1$. Thus, $\sum(S', T) \le \sum(S', T') + 1 \le m' + 1 = m - 1$. Let $S = S' \cup \{v_1\}$. Then $\sum(S, T) = \sum(S', T) + \deg_T(v_1) \le m$, implying that $a(T) \ge |S| = |S'| + 1 = a(T') + 1$. By inductive hypothesis, we obtain

$$\gamma_{r2}(T) \le \gamma_{r2}(T') + 1 \le a(T') + 2 \le a(T) + 1.$$

Claim 3. $\deg_T(v_3) = 2$.

Assume $\deg_T(v_3) = 3$. First let v_3 be adjacent to a support vertex z_2 not in $\{v_2, v_4\}$. Suppose z_1 is the leaf adjacent to z_2 and let $T' = T - T_{v_3}$. Then every $\gamma_{r2}(T')$-function can be extended to a 2RDF of T by assigning $\{1\}$ to v_1, z_1, $\emptyset$ to v_2, z_2 and $\{2\}$ to v_3. Thus $\gamma_{r2}(T) \le \gamma_{r2}(T') + 3$. As above we have $\sum(S', T) \le \sum(S', T') + 1 \le m' + 1 = m - 4$. Let $S = S' \cup \{v_1, v_2, z_1\}$. Then $\sum(S, T) = \sum(S', T) + \deg_T(v_1) + \deg_T(v_2) + \deg_T(z_1) \le m$. Therefore, $a(T) \ge |S| = |S'| + 3 = a(T') + 3$. It follows from inductive hypothesis that

$$\gamma_{r2}(T) \le \gamma_{r2}(T') + 3 \le (a(T') + 1) + 3 \le a(T) + 1.$$

Now let v_3 be adjacent to a leaf w. By Claims 1, 2 and the first part of Claim 3, we consider the following cases.

Case 3.1. $\deg_T(v_4) \geq 4$.

Let $T' = T - T_{v_3}$. Then every $\gamma_{r2}(T')$-function can be extended to a 2RDF of T by assigning $\{1\}$ to v_1, $\emptyset$ to v_2, w and $\{1,2\}$ to v_3. Hence $\gamma_{r2}(T) \leq \gamma_{r2}(T') + 3$. Suppose that $v_4 \notin S'$. In this case, let $S = S' \cup \{v_1, v_2, w\}$. Then $\sum(S,T) = \sum(S',T) + \deg_T(v_1) + \deg_T(v_2) + \deg_T(w) = \sum(S',T') + 4 \leq m' + 4 = m$, implying that $a(T) \geq a(T') + 3$ and the result follows by inductive hypothesis as above.

Now let $v_4 \in S'$. Let $S = (S' - \{v_4\}) \cup \{v_1, v_2, v_3, w\}$. Then $\sum(S,T) = \sum(S',T') - \deg_{T'}(v_4) + \deg_T(v_1) + \deg_T(v_2) + \deg_T(v_3) + \deg_T(w) \leq m$. Therefore, $a(T) \geq |S| = |S'| + 3 = a(T') + 3$. By inductive hypothesis, we obtain $\gamma_{r2}(T) \leq \gamma_{r2}(T') + 3 \leq (a(T') + 1) + 3 \leq a(T) + 1$.

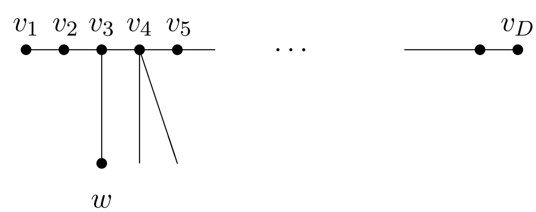

FIGURE 1. Case 3.1

Case 3.2. $\deg_T(v_4) = 2$.

Let $T' = T - T_{v_4}$. Then every $\gamma_{r2}(T')$-function can be extended to a 2RDF of T by assigning $\{1\}$ to v_1, $\emptyset$ to v_2, v_4, w and $\{1,2\}$ to v_3. Hence $\gamma_{r2}(T) \leq \gamma_{r2}(T') + 3$. It is easy to see that $\sum(S',T) \leq \sum(S',T') + 1 \leq m' + 1 = m - 4$. Let $S = S' \cup \{v_1, v_2, w\}$. Then $\sum(S,T) = \sum(S',T) + \deg_T(v_1) + \deg_T(v_2) + \deg_T(w) \leq m$, implying that $a(T) \geq |S| = |S'| + 3 = a(T') + 3$. By inductive hypothesis, we have $\gamma_{r2}(T) \leq \gamma_{r2}(T') + 3 \leq (a(T') + 1) + 3 \leq (a(T) - 3 + 1) + 3 = a(T) + 1$.

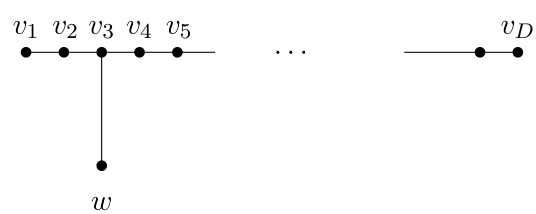

FIGURE 2. Case 3.2

Case 3.3. $\deg_T(v_4) = 3$ and there exists a path $v_4w_3w_2w_1$ in T such that $\deg_T(w_3) = \deg_T(w_2) = 2$, $\deg_T(w_1) = 1$ and $w_3 \neq v_5$.

Let $T' = T - T_{v_4}$. Then every $\gamma_{r2}(T')$-function can be extended to a 2RDF of T by assigning $\{1\}$ to w_1, w, v_3, $\emptyset$ to v_2, v_4, w_2 and $\{2\}$ to v_1, w_3. Hence $\gamma_{r2}(T) \leq \gamma_{r2}(T') + 5$. Clearly $\sum(S',T) \leq \sum(S',T') + 1 \leq m' + 1 = m - 7$. Let $S = S' \cup \{v_1, v_2, w, w_1, w_2\}$. Then $\sum(S,T) = \sum(S',T) + \deg_T(v_1) + \deg_T(v_2) + \deg_T(w) + \deg_T(w_1) + \deg_T(w_2) \leq m$ which implies that $a(T) \geq |S| = |S'| + 5 = a(T') + 5$. Now the result follows by inductive hypothesis.

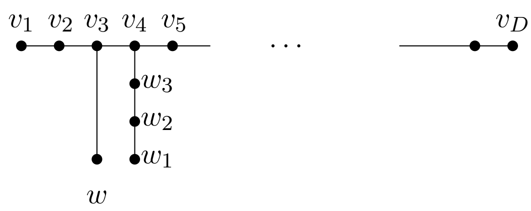

Figure 3. Case 3.3

Case 3.4. $\deg_T(v_4) = 3$ and there is a path $w_4w_3w_2w_1$ in T such that $v_4w_3 \in E(T)$, $\deg_T(w_3) = 3, \deg_T(w_2) = 2$, $\deg_T(w_1) = \deg_T(w_4) = 1$ and $w_3 \notin \{v_3, v_5\}$.
Let $T' = T - T_{v_4}$. Then every $\gamma_{r2}(T')$-function can be extended to a 2RDF of T by assigning $\{1\}$ to w_4, w, v_3, w_1, $\emptyset$ to v_2, v_4, w_2 and $\{2\}$ to v_1, w_3. Hence $\gamma_{r2}(T) \le \gamma_{r2}(T') + 6$. If $v_5 \notin S'$, then $\sum(S', T) = \sum(S', T')$ and if $v_5 \in S'$, then $\sum(S', T) = \sum(S', T')+1$. Thus, $\sum(S', T) \le \sum(S', T')+1 \le m' + 1 = m - 8$. Let $S = S' \cup \{v_1, v_2, w, w_1, w_2, w_4\}$. Then $\sum(S, T) = \sum(S', T) + 8 \le m$, implying that $a(T) \ge |S| = |S'| + 6 = a(T') + 6$. By inductive hypothesis, we have $\gamma_{r2}(T) \le \gamma_{r2}(T') + 6 \le (a(T') + 1) + 6 \le a(T) + 1$.

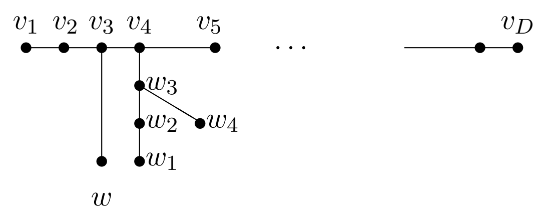

Figure 4. Case 3.4

Case 3.5. $\deg_T(v_4) = 3$ and v_4 is adjacent to a leaf, say w'.
Let $T' = T - T_{v_4}$. Then every $\gamma_{r2}(T')$-function can be extended to a 2RDF of T by assigning $\{1\}$ to v_1, w', w, $\emptyset$ to v_2, v_4 and $\{2\}$ to v_3. Hence $\gamma_{r2}(T) \le \gamma_{r2}(T') + 4$. As above, we have $\sum(S', T) \le \sum(S', T') + 1 \le m' + 1 = m - 5$. Let $S = S' \cup \{v_1, v_2, w, w'\}$. Then $\sum(S, T) = \sum(S', T) + \deg_T(v_1) + \deg_T(v_2) + \deg_T(w) + \deg_T(w') \le m$. Therefore, $a(T) \ge |S| = |S'| + 4 = a(T') + 4$ and the result follows by inductive hypothesis.

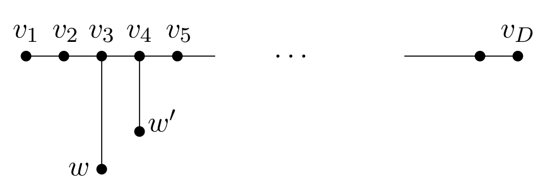

Figure 5. Case 3.5

Case 3.6. $\deg_T(v_4) = 3$ and v_4 is adjacent to a support vertex of degree 2, say w_2, other than v_5. Let w_1 be the leaf adjacent to w_2. We need to consider the following subcases.

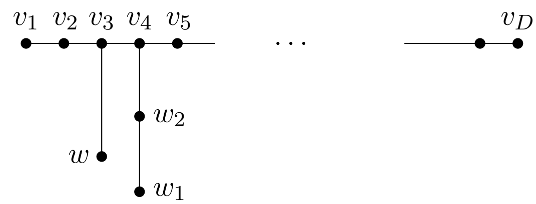

Figure 6. Case 3.6

Subcase 3.6.1 $\deg_T(v_5) \geq 4$.
Let $T' = T - T_{v_4}$. Then every $\gamma_{r2}(T')$-function can be extended to a 2RDF of T by assigning $\{1\}$ to w, v_1, w_1, w_2, $\emptyset$ to v_2, v_4 and $\{2\}$ to v_3 . Hence $\gamma_{r2}(T) \leq \gamma_{r2}(T') + 5$. Suppose that $v_5 \notin S'$. Then $\sum(S', T) = \sum(S', T')$. In this case, let $S = S' \cup \{w, v_1, v_2, w_1, w_2\}$. Then $\sum(S, T) = \sum(S', T) + \deg_T(w) + \deg_T(v_1) + \deg_T(v_2) + \deg_T(w_1) + \deg_T(w_2) = \sum(S', T') + 7 \leq m' + 7 = m$, implying that $a(T) \geq a(T') + 5$ and the result follows by inductive hypothesis.

Now let $v_5 \in S'$. Suppose $S = (S' - \{v_5\}) \cup \{w, v_1, v_2, v_3, w_1, w_2\}$. Then $\sum(S, T) = \sum(S', T') - \deg_{T'}(v_5) + \deg_T(w) + \deg_T(v_1) + \deg_T(v_2) + \deg_T(v_3) + \deg_T(w_1) + \deg_T(w_2) \leq m$. Therefore, $a(T) \geq |S| = |S'| + 5 = a(T') + 5$. By inductive hypothesis, we obtain $\gamma_{r2}(T) \leq \gamma_{r2}(T') + 5 \leq (a(T') + 1) + 5 \leq a(T) + 1$.

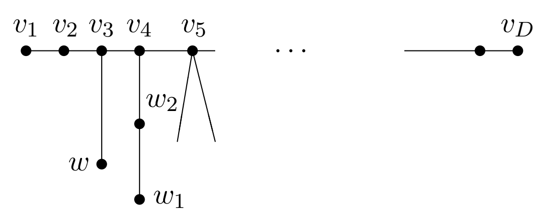

Figure 7. Subcase 3.6.1

Subcase 3.6.2 $\deg_T(v_5) = 2$.
Assume $T' = T - (T_{v_3} \cup T_{w_2})$. Since $\deg_T(v_5) = 2$, there is a $\gamma_{r2}(T')$-function, say f, so that $|f(v_4)| = 1$ by Observation 2.2. We may assume without loss of generality that $f(v_4) = \{1\}$. Now f can be extended to a 2RDF of T by assigning $\{1\}$ to v_1, v_2, $\emptyset$ to v_3, w_2 and $\{2\}$ to w, w_1. Hence $\gamma_{r2}(T) \leq \gamma_{r2}(T') + 4$. As above, we have $\sum(S', T) \leq \sum(S', T') + 1$. If $v_4 \notin S'$, then let $S = S' \cup \{v_1, w_1, w, v_2\}$ and if $v_5 \in S'$, then let $S = (S' - \{v_4\}) \cup \{v_1, v_2, w, w_1, w_2\}$. It is easy to see that $\sum(S, T) \leq m$ and hence $a(T) \geq |S| = |S'| + 4 = a(T') + 4$. Now the result follows by inductive hypothesis.

Subcase 3.6.3 $\deg_T(v_5) = 3$ and there is a path $v_5 z_4 z_3 z_2 z_1$ in T such that $\deg_T(z_4) = \deg_T(z_3) = \deg_T(z_2) = 2$, $\deg_T(z_1) = 1$ and $z_4 \neq v_6$.

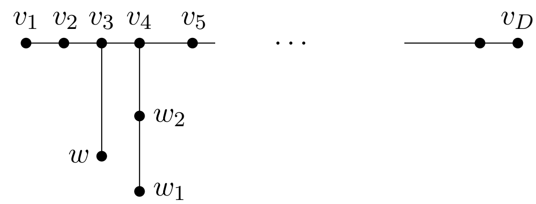

FIGURE 8. Subcase 3.6.2

Let $T' = T - \{z_1, z_2\}$. By Observation 2.2, T' has a $\gamma_{r2}(T')$-function such as f that $|f(z_3)| = 1$. We may assume without loss of generality that $f(z_3) = \{1\}$. Now f can be extended to a 2RDF of T by assigning $\{2\}$ to z_1 and $\emptyset$ to z_2. Hence $\gamma_{r2}(T) \leq \gamma_{r2}(T') + 1$. As above, we have $\sum(S', T) \leq \sum(S', T') + 1$. Let $S = S' \cup \{z_1\}$. Clearly $\sum(S, T) \leq m$ and hence $a(T) \geq |S| = |S'| + 1 = a(T') + 1$ and the result follows by inductive hypothesis.

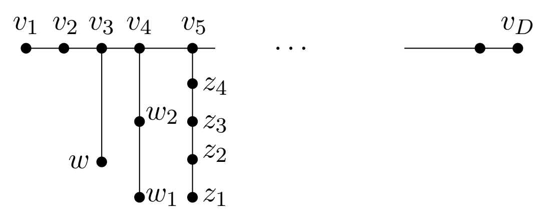

FIGURE 9. Subcase 3.6.3

Subcase 3.6.4 $\deg_T(v_5) = 3$ and there is a path $v_5v'_4v'_3v'_2v'_1$ in T such that $T_{v'_4} \simeq T_{v_4}$.
Let $T' = T - T_{v_5}$. It is easy to see that $\gamma_{r2}(T) \leq \gamma_{r2}(T') + 10$ and $a(T) \geq a(T') + 10$. Hence the result follows by inductive hypothesis.

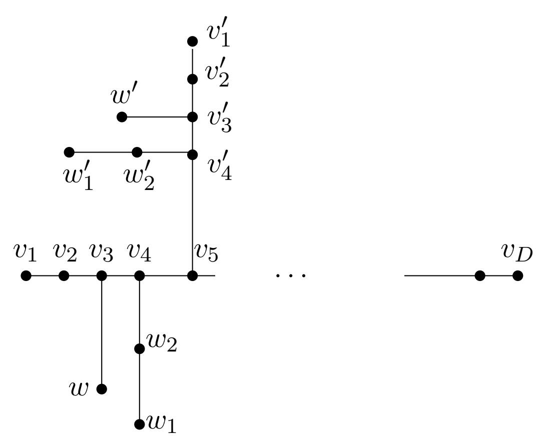

FIGURE 10. Subcase 3.6.4

Subcase 3.6.5 $\deg_T(v_5) = 3$ and there is a path $v_5z_3z_2z_1$ in T such that $\deg_T(z_3) = \deg_T(z_2) = 2$, $\deg_T(z_1) = 1$ and $z_3 \neq v_6$.

Let $T' = T - T_{v_5}$. Then every $\gamma_{r2}(T')$-function can be extended to a 2RDF of T by assigning $\{1\}$ to z_1, v_4, v_3, $\emptyset$ to v_2, v_5, z_2, w_2 and $\{2\}$ to v_1, w_1, w, z_3. Thus $\gamma_{r2}(T) \le \gamma_{r2}(T') + 7$. Clearly $\sum(S', T) \le \sum(S', T') + 1$. Let $S = S' \cup \{w, w_1, v_1, z_1, v_2, z_2, z_3\}$. Then $\sum(S, T) = \sum(S', T) + \deg_T(v_1) + \deg_T(v_2) + \deg_T(w) + \deg_T(w_1) + \deg_T(z_1) + \deg_T(z_2) + \deg_T(z_3) \le m$, implying that $a(T) \ge |S| = |S'| + 7 = a(T') + 7$. It follows from inductive hypothesis that $\gamma_{r2}(T) \le \gamma_{r2}(T') + 7 \le (a(T') + 1) + 7 \le a(T) + 1$.

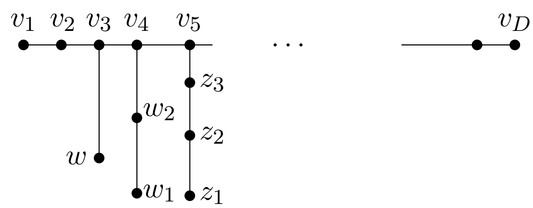

FIGURE 11. Subcase 3.6.5

Subcase 3.6.6 $\deg_T(v_5) = 3$ and there is a path $v_5z_3z_2z_1$ in T such that all neighbors of z_3, except v_5 and z_2, are leaves, $\deg_T(z_1) = 1$, $\deg_T(z_3) \ge 3$, $\deg_T(z_2) = 2$ and $z_3 \ne v_6$.
Assume $T' = T - T_{v_5}$. Then every $\gamma_{r2}(T')$-function can be extended to a 2RDF of T by assigning $\{1,2\}$ to z_3, $\{1\}$ to z_1, v_1, v_2, v_4, $\emptyset$ to v_3, v_5, z_2, w_2 and every leaves at z_3 and $\{2\}$ to w, w_1. Thus $\gamma_{r2}(T) \le \gamma_{r2}(T') + 8$. Clearly $\sum(S', T) \le \sum(S', T') + 1$. Let z_4 be a leaf adjacent to z_3 and let $S = S' \cup \{w, w_1, v_1, z_1, z_4, v_2, w_2, z_2\}$. Then $\sum(S, T) \le m$ and so $a(T) \ge |S| = |S'| + 8 = a(T') + 8$. It follows from inductive hypothesis that $\gamma_{r2}(T) \le \gamma_{r2}(T') + 8 \le (a(T') + 1) + 8 \le a(T) + 1$.

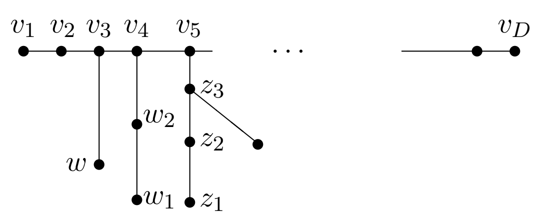

FIGURE 12. Subcase 3.6.6

Subcase 3.6.7 $\deg_T(v_5) = 3$ and v_5 is adjacent to the center, say $z \ne v_6$, of a spider different from P_4.
Let $T' = T - T_z$. Using an argument similar to that described in the proof of Lemma 2.1 and applying inductive hypothesis show that $\gamma_{r2}(T) \le a(T) + 1$.

Subcase 3.6.8 $\deg_T(v_5) = 3$ and v_5 is adjacent to a support vertex of degree 2, say z_2.
Let z_1 be the leaf adjacent to z_2 and let $T' = T - \{v_1, w, w_1\}$. Considering the paths $v_5z_2z_1$ and $v_4v_3v_2$ in T', it follows from Observation 2.2 that T' has a $\gamma_{r2}(T')$-function f such that $|f(v_5)| = |f(v_4)| \ge 1$.

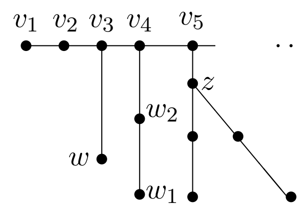

Figure 13. Subcase 3.6.7

Since also $|f(w_2)| + |f(v_4)| + |f(v_3)| + |f(v_2)| \geq 3$, we may assume without loss of generality that $f(v_4) = \{1\}, f(v_3) = \emptyset, f(w_2) = f(v_2) = \{2\}$. Now the function $g : V(T) \to \mathcal{P}(\{1,2\})$ defined by $g(w_1) = \{2\}, g(v_1) = g(w) = \{1\}, g(w_2) = \emptyset$ and $g(x) = f(x)$ if $x \in V(T) - \{v_1, w, w_1, w_2\}$ is a 2RDF of T, implying that $\gamma_{r2}(T) \leq \gamma_{r2}(T')+2$. If $|S' \cap \{v_2, v_3, w_2\}| \leq 1$, then $\sum(S', T) \leq \sum(S', T')+1$. In this case, let $S = S' \cup \{v_1, w\}$. Then $\sum(S, T) \leq m$ implying that $a(T) \geq a(T')+2$. Let $|S' \cap \{v_2, v_3, w_2\}| \geq 2$. If $v_3 \in S'$, then let $S = (S' - \{v_3\}) \cup \{v_1, w, w_1\}$. Obviously $\sum(S, T) \leq m$ and so $a(T) \geq a(T') + 2$. If $v_3 \notin S'$, then let $S = (S' - \{v_2\}) \cup \{v_1, w, w_1\}$. Then $\sum(S, T) \leq m$ and so $a(T) \geq a(T') + 2$. Thus, in all cases $a(T) \geq a(T') + 2$ and the result follows by inductive hypothesis.

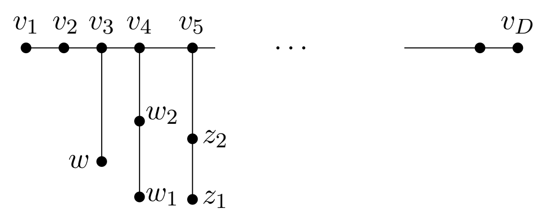

Figure 14. Subcase 3.6.8

Subcase 3.6.9 $\deg_T(v_5) = 3$ and v_5 is adjacent to a leaf z.
If $\mathrm{diam}(T) = 5$, then clearly $\gamma_{r2}(T) = a(T) = 7$ and the result is true. Let $\mathrm{diam}(T) \geq 6$. Let $T' = T - T_{v_5}$. Then every $\gamma_{r2}(T')$-function can be extended to a 2RDF of T by assigning $\{1\}$ to w, w_1, z, v_1, v_2 and $\emptyset$ to w_2, v_5, v_3 and $\{2\}$ to v_4. Thus $\gamma_{r2}(T) \leq \gamma_{r2}(T') + 6$. Clearly $\sum(S', T) \leq \sum(S', T') + 1$. Let $S = S' \cup \{v_1, v_2, w_1, w_2, w, z\}$. Then $\sum(S, T) = \sum(S', T) + \deg_T(v_1) + \deg_T(v_2) + \deg_T(w_1) + \deg_T(w_2) + \deg_T(w) + \deg_T(z) \leq m$, implying that $a(T) \geq a(T') + 6$. Now the result follows from inductive hypothesis. (■)

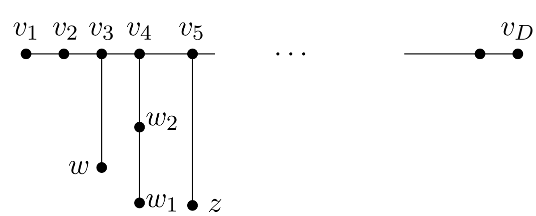

Figure 15. Subcase 3.6.9

We now return to the proof of Theorem. Note that by Claim 1 we may assume $\deg_T(v_2) = 2$ and by Claim 3 we have $\deg_T(v_3) = 2$. First let $\deg_T(v_4) = 2$. Assume $T' = T - T_{v_2}$. By Observation 2.2, there is a $\gamma_{r2}(T')$-function f such that $|f(v_3)| = 1$. Suppose without loss of generality that $f(v_3) = \{1\}$. Now f can be extended to a 2RDF of T by assigning $\emptyset$ to v_2 and $\{2\}$ to v_1. Thus $\gamma_{r2}(T) \leq \gamma_R(T') + 1$. Let $S = S' \cup \{v_1\}$. Then $\sum(S,T) \leq m$, implying that $a(T) \geq |S| = |S'| + 1 = a(T') + 1$. Applying inductive hypothesis we obtain the result.

Now let $\deg_T(v_4) \geq 3$. Assume $T' = T - T_{v_3}$. Then every $\gamma_{r2}(T')$-function can be extended to a 2RDF of T by assigning $\{1\}$ to v_1, $\emptyset$ to v_2 and $\{2\}$ to v_3. Thus $\gamma_{r2}(T) \leq \gamma_{r2}(T') + 2$. If $v_4 \notin S$, then let $S = S' \cup \{v_1, v_2\}$ and if $v_4 \in S'$, then let $S = (S' - \{v_4\} \cup \{v_1, v_2, v_3\}$. Then $\sum(S,T) \leq m$, implying that $a(T) \geq |S| = |S'| + 2 = a(T') + 2$. Now the result follows from inductive hypothesis and the proof is complete. □

We conclude this paper with an open problem.

Problem. Characterize all trees achieving the bound in Theorem 2.3.

Acknowledgments

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