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ON THE UNIMODALITY OF INDEPENDENCE POLYNOMIAL OF CERTAIN CLASSES OF GRAPHS

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Abstract. The independence polynomial of a graph G is the polynomial $\sum i_k x^k$, where i_k denote the number of independent sets of cardinality k in G. In this paper we study unimodality problem for the independence polynomial of certain classes of graphs.

1. Introduction

Throughout this paper $G = (V, E)$ is a simple (i.e., finite, undirected, loopless and without multiple edges) graph with vertex set $V = V(G)$ and edge set $E = E(G)$. If $X \subset V$, then $G[X]$ is the subgraph of G spanned by X. By $G\backslash W$ we mean the subgraph $G[V \backslash W]$, if $W \subset V(G)$. We also denote by $G-F$ the partial subgraph of G obtained by deleting the edges of F, for $F \subset E(G)$, and we write shortly $G - e$, whenever $F = \{e\}$. The neighborhood of a vertex $v \in V$ is the set $N(v) = \{w \in V|\ vw \in E\}$, and $N[v] = N(v) \cup \{v\}$. A vertex v is pendant if its neighborhood contains only one vertex; an edge $e = uv$ is pendant if one of its endpoints is a pendant vertex. K_n, P_n and C_n denote the complete graph, the path, and the cycle on n vertices, respectively. The disjoint union of the graphs G_1 and G_2 is the graph $G = G_1 \cup G_2$ having as a vertex set the disjoint union of $V(G_1)$ and $V(G_2)$, and as an edge set the disjoint union of $E(G_1)$ and $E(G_2)$. In particular, nG denotes the disjoint union of $n > 1$ copies of the graph G. The corona of the graphs G and H is the graph $G \circ H$ obtained from G and $|V(G)|$ copies of H, such that each vertex of G is joined to all vertices of a copy of H.

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An independent set of a graph G is a set of vertices where no two vertices are adjacent. The independence number is the size of a maximum independent set in the graph. For a graph G with independence number α, let i_k denote the number of independent sets of cardinality k in G ($k = 0, 1, \ldots, \alpha$). The independence polynomial of G,

$$I(G; x) = \sum_{k=0}^{\alpha} i_k x^k,$$

is the generating polynomial for the independent sequence $(i_0, i_1, i_2, \ldots, i_\alpha)$. We say that the polynomial $P(x) = a_0 + a_1 x + \cdots + a_n x^n$ is unimodal if there exist $k \in \{0, \ldots, n\}$, called a mode of the sequence such that

$$a_0 \leq a_1 \leq \cdots \leq a_{k-1} \leq a_k \geq a_{k+1} \geq \cdots \geq a_n.$$

A polynomial, $P(x)$, as above is logarithmically concave (or simply log-concave) if for all $k = 1, \ldots, n-1$, we have

$$a_k^2 \geq a_{k-1} a_{k+1}.$$

It is trivial to show that if $P(x)$ is log-concave, then it is unimodal. Unimodality problems arise naturally in many branches of mathematics and have been extensively investigated. See survey article [6] and [18] for details.

Unimodality problems of graph polynomials have always been of great interest to researchers in graph theory. For example, it is conjectured that the chromatic polynomial of a graph is unimodal [17]. Also recently, it is conjectured that the domination polynomial of a graph is unimodal (see [1] or [2]). There has been an extensive literature in recent years on the unimodality problems of independence polynomials (see [10, 11, 12, 13, 14, 15, 16] for instance). Wang and Zhu in [19] established recurrence relations and gave factorizations of independence polynomials for certain classes of graphs, and then studied their unimodalities. Similar to their method we study the unimodality of certain graphs.

A cactus graph is a connected graph in which no edge lies in more than one cycle. So, each block of a cactus graph is either an edge or a cycle. If all blocks of a cactus G are cycles of the same length m, the cactus is m-uniform. A hexagonal cactus is a 6-uniform cactus, i.e., a cactus in which every block is a hexagon. A vertex shared by two or more hexagons is called a cut-vertex. If each hexagon of a hexagonal cactus G has at most two cut-vertices, and each cut-vertex is shared by exactly two hexagons, we say that G is a chain hexagonal cactus. The number of hexagons in G is called the length of the chain. We call G a polyphenyl hexagonal chain if each hexagon of G has at most two cut-vertices and each cut vertex is shared by exactly one hexagon and one cut-edge . Obviously, a polyphenyl hexagonal chain of length n has $6n$ vertices and $7n-1$ edges. Furthermore, any polyphenyl hexagonal chain of length greater than one has exactly two hexagons with only one cut-vertex. Such hexagons are called terminal hexagons. Any remaining hexagons are called internal hexagons. The polyphenyl ortho-chains, polyphenyl meta-chains, and polyphenyl para-chains of length n is denoted by $\overline{O}_n$;$\overline{M}_n$ and $\overline{P}_n$, respectively. For more information on chain hexagonal cacti and polyphenyl chains, see [4] and [8]. Examples of polyphenyl ortho-chains, polyphenyl meta-chains, and polyphenyl para-chains are shown in Figure 1.

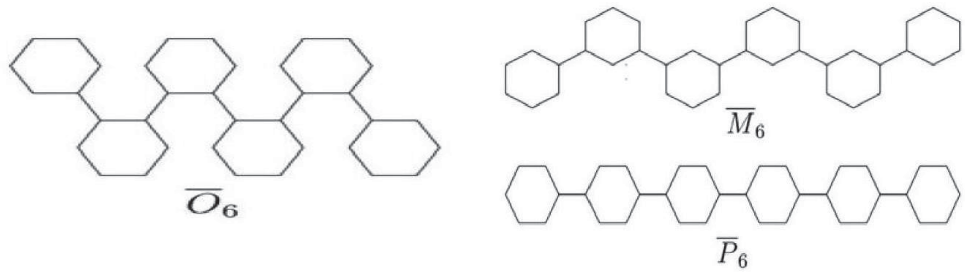

FIGURE 1. Example of polyphenyl ortho-chains, polyphenyl meta-chains, and polyphenyl para-chains of length 6.

In Section 2 we prove that the independence polynomial of $\overline{O}_n$ is unimodal. In Section 3 we study the unimodality of the independence polynomials of certain graphs.

2. Unimodality of independence polynomial of polyphenyl ortho-chains

In this section we prove that the independence polynomial of polyphenyl ortho-chains $\overline{O}_n$ is unimodal.

Hoede and Li [9] obtained the following recursive formula for the independence polynomial of a graph.

Theorem 2.1. *For any vertex v of a graph G, $I(G,x) = I(G-v,x) + xI(G-[v],x)$ where $[v]$ is the closed neighborhood of v, contains of v, together with all vertices incident with v.*

Using Theorem 2.1 we can obtain the recurrence relations for the independence polynomials of $\overline{P}_n, \overline{O}_n$ and $\overline{M}_n$ ([3]).

Theorem 2.2. ([3]) *If $n \geq 2$, then*

(i) $I(\overline{O}_n,x) = (1+6x+9x^2+2x^3)I(\overline{O}_{n-1},x) - (x^2+6x^3+11x^4+6x^5+x^6)I(\overline{O}_{n-2},x)$

(ii) $I(\overline{P}_n,x) = (1+5x+6x^2+2x^3)I(\overline{P}_{n-1},x) + (2x^2+9x^3+9x^4+4x^5-x^6)I(\overline{P}_{n-2},x)$

(iii) $I(\overline{M}_n,x) = (1+6x+8x^2+x^3)I(\overline{M}_{n-1},x) + (-x-4x^2-x^3-9x^4+6x^5+x^6)I(\overline{M}_{n-2},x)$.

We shall give factorizations of independence polynomial of $\overline{O_n}$. To explain our approach and obtain our results, we recall some lemmas (see [19]).

Lemma 2.3. ([7]) *Let $\{z_n\}_{n\geq 0}$ be a sequence satisfying the linear recurrence relation*

$$z_n = az_{n-1} + bz_{n-2}, \qquad n = 2,3,\ldots$$

If $a^2+4b > 0$, then the closed form for the sequence is

$$z_n = \frac{(z_1 - z_0\lambda_2)\lambda_1^n + (z_0\lambda_1 - z_1)\lambda_2^n}{\lambda_1 - \lambda_2}, \qquad n = 2, 3, \ldots$$

where

$$\lambda_1 = \frac{a + \sqrt{a^2 + 4b}}{2}, \qquad \lambda_2 = \frac{a - \sqrt{a^2 + 4b}}{2}$$

are the roots of quadratic equation $\lambda^2 - a\lambda - b = 0$.

Lemma 2.4. ([5]) *Let* $\lambda_1, \lambda_2 \in \mathbb{R}$ *and* $n \in \mathbb{N}$.

(i) *If* n *is odd, then* $\lambda_1^n - \lambda_2^n = (\lambda_1 - \lambda_2) \prod_{s=1}^{\frac{n-1}{2}} \left[(\lambda_1 + \lambda_n)^2 - 4\lambda_1\lambda_2 \cos^2 \frac{s\pi}{n}\right]$.

(ii) *If* n *is even, then* $\lambda_1^n - \lambda_2^n = (\lambda_1 - \lambda_2)(\lambda_1 + \lambda_2) \prod_{s=1}^{\frac{n-2}{2}} \left[(\lambda_1 + \lambda_n)^2 - 4\lambda_1\lambda_2 \cos^2 \frac{s\pi}{n}\right]$.

(iii) *If* n *is odd, then* $\lambda_1^n + \lambda_2^n = (\lambda_1 + \lambda_2) \prod_{s=1}^{\frac{n-1}{2}} \left[(\lambda_1 + \lambda_2)^2 - 4\lambda_1\lambda_2 \cos^2 \frac{(2s-1)\pi}{2n}\right]$.

(iv) *If* n *is even, then* $\lambda_1^n + \lambda_2^n = \prod_{s=1}^{\frac{n}{2}} \left[(\lambda_1 + \lambda_2)^2 - 4\lambda_1\lambda_2 \cos^2 \frac{(2s-1)\pi}{2n}\right]$.

Lemma 2.5. ([18]) *Let* $f(x)$ *and* $g(x)$ *be polynomials with positive coefficients.*

(i) *If both* $f(x)$ *and* $g(x)$ *are log-concave, then so is their product* $f(x)g(x)$.

(ii) *If* $f(x)$ *is log-concave and* $g(x)$ *is unimodal, then their product* $f(x)g(x)$ *is unimodal.*

(iii) *If both* $f(x)$ *and* $g(x)$ *are symmetric and unimodal, then so is their product* $f(x)g(x)$.

Now we are ready to obtain formula for the independence polynomial of $\overline{O_n}$:

Theorem 2.6. *For any even integer* n,

$$I(\overline{O_n}; x) = \prod_{s=1}^{\lfloor \frac{n}{2} \rfloor} \left[I^2(C_6; x) - 4(x^2 + 6x^3 + 11x^4 + 6x^5 + x^6) \cos^2 \frac{s\pi}{n+1}\right],$$

and for any odd integer n,

$$I(\overline{O_n}; x) = I(C_6; x) \prod_{s=1}^{\lfloor \frac{n}{2} \rfloor} \left[I^2(C_6; x) - 4(x^2 + 6x^3 + 11x^4 + 6x^5 + x^6) \cos^2 \frac{s\pi}{n+1}\right].$$

Proof. By Theorem 2.2

$$I(\overline{O_n}; x) = (1 + 6x + 9x^2 + 2x^3) I(\overline{O}_{n-1}; x) - (x^2 + 6x^3 + 11x^4 + 6x^5 + x^6) I(\overline{O}_{n-2}; x).$$

Put $h_n = I(\overline{O_n}; x)$, then $h_0 = 1, h_1 = 1 + 6x + 9x^2 + 2x^3 = a, b = -(x^2 + 6x^3 + 11x^4 + 6x^5 + x^6)$. By Lemma 2.3

$$h_n = \frac{(a - \lambda_2)\lambda_1^n + (\lambda_1 - a)\lambda_2^n}{(\lambda_1 - \lambda_2)} = \frac{\lambda_1^{n+1} - \lambda_2^{n+1}}{(\lambda_1 - \lambda_2)}.$$

Thus by Lemma 2.4 for even n we have

$$I(\overline{O_n};x)=\prod_{s=1}^{\lfloor\frac{n}{2}\rfloor}\left[I^2(C_6;x)-4(x^2+6x^3+11x^4+6x^5+x^6)\cos^2\frac{s\pi}{n+1}\right],$$

and for odd n,

$$I(\overline{O_n};x)=I(C_6;x)\prod_{s=1}^{\lfloor\frac{n}{2}\rfloor}\left[I^2(C_6;x)-4(x^2+6x^3+11x^4+6x^5+x^6)\cos^2\frac{s\pi}{n+1}\right]. \quad \square$$

We next give a result about the unimodality of the independence polynomials of $\overline{O_n}$.

Theorem 2.7. *$I(\overline{O_n};x)$ is log-concave and therefore unimodal.*

Proof. First suppose that n is even. We have

$$I(\overline{O_n};x)=\prod_{s=1}^{\lfloor\frac{n}{2}\rfloor}\left[I^2(C_6;x)-4(x^2+6x^3+11x^4+6x^5+x^6)\cos^2\frac{s\pi}{n+1}\right].$$

By substituting $a=\cos^2\frac{s\pi}{n+1}$, we have $0\le a<1$ and

$$\begin{aligned}I(\overline{O_n};x)=\prod_{s=1}^{\lfloor\frac{n}{2}\rfloor}\big(&1+12x+(54-4a)x^2+(112-24a)x^3+(105-44a)x^4\\&+(36-6a)x^5+(4-4a)x^6\big).\end{aligned}\tag{2.1}$$

Since all coefficients of each factor of the above equality is positive, to show the log-concavity of $I(\overline{O_n};x)$, by Lemma 2.5(i) it suffices to show that each factor on the right of (2.1) is log-concave. Now, simple calculations lead us to

$$(12)^2=144>1\cdot(54-4a),$$

$$1572-144a+8a^2>0\rightarrow(54-4a)^2>12(112-24a),$$

$$6874-2580a+400a^2>0\rightarrow(112-24a)^2>(54-4a)(105-44a),$$

$$6993-7568a+1792a^2>0\rightarrow(105-44a)^2>(112-24a)(36-6a),$$

$$876+164a-140a^2>0\rightarrow(36-6a)^2>(105-44a)(4-4a).$$

Thus $I(\overline{O_n};x)$ is log-concave and therefore unimodal. The proof for case odd n is similar to case even n. $\square$

3. Unimodality of independence polynomial of certain graphs

In this section we consider graphs of the form $P_n \circ (tK_m)$ and $C_n \circ (tK_m)$, where $n, m, t \in \mathbb{N}$. We study the unimodality of the independence polynomials of these kind of graphs. Firstly we obtain the independence polynomial of $P_n \circ (tK_m)$:

Theorem 3.1. *Let $n, m, t \in \mathbb{N}$. Then*

(i)

$$I(P_n \circ (tK_m); x) = (1+mx)^{t\lfloor \frac{n}{2} \rfloor} \prod_{s=1}^{\lfloor \frac{n+1}{2} \rfloor} \left[(1+mx)^t + 4x \cos^2 \frac{s\pi}{n+2} \right]. \tag{3.1}$$

(ii) *$I(P_n \circ (tK_m); x)$ is log-concave and therefore unimodal.*

Proof.

(i) Let $h_n = I(P_n \circ (tK_m); x)$. Then by Lemma 2.3 $h_n = (1+mx)^t \cdot h_{n-1} + x(1+mx)^t \cdot h_{n-2}$. Clearly, $h_0 = 1$ and $h_1 = (1+mx)^t + x$. By Lemma 2.3 we have

$$h_n = \frac{((1+mx)^t + x - \lambda_2)\lambda_1^n + (\lambda_2 - (1+mx)^t - x)\lambda_2^n}{\lambda_1 - \lambda_2}.$$

Therefore

$$h_n = \frac{(a + \frac{b}{a} - \lambda_2)\lambda_1^n + (\lambda_2 - a - \frac{b}{a})\lambda_2^n}{\lambda_1 - \lambda_2},$$

where $a = (1+mx)^t$ and $b = x(1+mx)^t$. Note that $a + \frac{b}{a} - \lambda_2 = \frac{\lambda_1^2}{a}$ and $\lambda_2 - a - \frac{b}{a} = -\frac{\lambda_2^2}{a}$. Hence

$$h_n = \frac{\lambda_1^{n+2} - \lambda_2^{n+2}}{a(\lambda_1 - \lambda_2)}.$$

Thus by Lemma 2.4 for odd n we have

$$h_n = \frac{(\lambda_1 - \lambda_2) \prod_{s=1}^{\frac{n+1}{2}} \left[(\lambda_1 + \lambda_2)^2 - 4\lambda_1\lambda_2 \cos^2 \frac{s\pi}{n+2} \right]}{a(\lambda_1 - \lambda_2)} = a^{\frac{n-1}{2}} \prod_{s=1}^{\frac{n+1}{2}} \left[a + 4\frac{b}{a} \cos^2 \frac{s\pi}{n+2} \right]. \tag{3.2}$$

and for even n,

$$h_n = \prod_{s=1}^{\frac{n}{2}} \left[(\lambda_1 + \lambda_2)^2 - 4\lambda_1\lambda_2 \cos^2 \frac{s\pi}{n+2} \right] = a^{\frac{n}{2}} \prod_{s=1}^{\frac{n}{2}} \left[a + 4\frac{b}{a} \cos^2 \frac{s\pi}{n+2} \right]. \tag{3.3}$$

Combining (3.2) and (3.3) we obtain:

$$I(P_n \circ (tK_m); x) = (1+mx)^{t\lfloor \frac{n}{2} \rfloor} \prod_{s=1}^{\lfloor \frac{n+1}{2} \rfloor} \left[(1+mx)^t + 4x \cos^2 \frac{s\pi}{n+2} \right].$$

(ii) To show log-concavity of $I(P_n \circ (tK_m); x)$, it suffices to show that each factor on the right of (3.2) is log-concave. Let $0 \leq a < 1$. We claim that the polynomial

$$(1+mx)^t + 4xa = 1 + (tm+4a)x + \frac{t(t-1)}{2}m^2x^2 + \frac{t(t-1)(t-2)}{6}m^3x^3 + ... + m^tx^t$$

is log-concave. Actually, since $(1+mx)^t$ is log-concave, it suffices to prove the inequality

$$\left[\frac{t(t-1)}{2}m^2\right]^2 \geq (mt+4a)\left[\frac{t(t-1)(t-2)}{6}m^3\right].$$

Clearly, it suffices to prove the inequality for $a = 1$. In this case, the inequality is equivalent to $mt^2 + (m-4)t + 16 \geq 0$, which is obviously true for $m \geq 4$. For $m = 1, 2$ and 3, we have $t^2 - 3t + 16 \geq 0$, $2t^2 - 2t + 16 \geq 0$ and $3t^2 - t + 16 \geq 0$, respectively which are obviously true. Thus $I(P_n \circ (tK_m); x)$ is log-concave and therefore is unimodal. □

We now consider the independence polynomial of $C_n \circ (tK_m)$.

Theorem 3.2. *For $n \geq 3$*

(i)

$$I(C_n \circ (tK_m); x) = (1+mx)^{t\lfloor\frac{n+1}{2}\rfloor} \prod_{s=1}^{\lfloor\frac{n}{2}\rfloor}\left[(1+mx)^t + 4x\cos^2\frac{(2s-1)\pi}{2n}\right].$$

(ii) $I(C_n \circ (tK_m); x)$ *is log-concave and therefore unimodal.*

Proof.

(i) Let h_n and f_n be the independence polynomials of $I(P_n \circ (tK_m); x)$ and $I(C_n \circ (tK_m); x)$ respectively. By Lemma 2.3, we have

$$f_n = (1+mx)^t \cdot h_{n-1} + x(1+mx)^{2t} \cdot h_{n-3},$$

Recall that

$$h_n = \frac{\lambda_1^{n+2} - \lambda_2^{n+2}}{(1+mx)^t(\lambda_1 - \lambda_2)}.$$

Therefore using Lemma 2.4 we have

$$\begin{aligned} f_n &= \frac{\lambda_1^{n+1} - \lambda_2^{n+1}}{(\lambda_1 - \lambda_2)} + x(1+mx)^t\frac{\lambda_1^{n-1} - \lambda_2^{n-1}}{(\lambda_1 - \lambda_2)} \\ &= \frac{\lambda_1^{n+1} - \lambda_2^{n+1}}{(\lambda_1 - \lambda_2)} - \lambda_1\lambda_2\frac{\lambda_1^{n-1} - \lambda_2^{n-1}}{(\lambda_1 - \lambda_2)} \\ &= \lambda_1^n + \lambda_2^n \\ &= (1+mx)^{t\lfloor\frac{n+1}{2}\rfloor} \prod_{s=1}^{\lfloor\frac{n}{2}\rfloor}\left[(1+mx)^t + 4x\cos^2\frac{(2s-1)\pi}{2n}\right]. \end{aligned}$$

(ii) Proof is similar to the proof of Theorem 3.1(ii).

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