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NOTE ON DEGREE KIRCHHOFF INDEX OF GRAPHS

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Abstract. The degree Kirchhoff index of a connected graph G is defined as the sum of the terms $d_i\, d_j\, r_{ij}$ over all pairs of vertices, where d_i is the degree of the i-th vertex, and r_{ij} the resistance distance between the i-th and j-th vertex of G. Bounds for the degree Kirchhoff index of the line and para-line graphs are determined. The special case of regular graphs is analyzed.

1. Introduction

Throughout this paper all graphs are assumed to be finite and simple. Let $G = (V, E)$ be such a graph of order n, having m edges. In other words, $|V(G)| = n$ and $|E(G)| = m$. By $\deg(v)$ is denoted the degree (= number of first neighbors) of the vertex $v \in V(G)$.

The eigenvalues of the adjacency matrix $\mathcal{A}(G)$ of G are called the eigenvalues of G and will be denoted by $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$.

The Laplacian matrix of G is defined as $\mathcal{L}(G) = \mathcal{D}(G) - \mathcal{A}(G)$ where $\mathcal{D}(G) = [d_{ij}]$ is the diagonal matrix with $d_{ii} = \deg(v_i)$, and $d_{ij} = 0$ for $i \neq j$. The eigenvalues of $\mathcal{L}(G)$ are called the Laplacian eigenvalues of G and will be denoted by $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$, where $\mu_n = 0$ for all graphs.

Provided the graph G has no isolated vertices, the normalized Laplacian matrix $\tilde{\mathcal{L}}(G)$ is defined as

$$\tilde{\mathcal{L}} = \mathcal{D}^{-1/2}\, \mathcal{L}\, \mathcal{D}^{-1/2}$$

which implies that its (i, j)-entry is equal to 1 if $i = j$, equal to $-1/\sqrt{\deg(v_i)\, \deg(v_j)}$ if $i \neq j$ and the vertices v_i, v_j are adjacent, and zero otherwise. The eigenvalues of $\tilde{\mathcal{L}}$ are called the normalized

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Laplacian eigenvalues of G [5,6], and will be denoted by $\delta_1 \geq \delta_2 \geq \cdots \geq \delta_n$, where $\delta_n = 0$ for all graphs.

In [13], Klein and Randić introduced the notion of resistance-distance as a second distance function on the vertex set of a graph. If r_{ij} denotes the resistance–distance between vertices v_i and v_j in a graph G then the Kirchhoff index of G is defined as $Kf(G) = \sum\limits_{i<j} r_{ij}$ [13].

Originally, the r_{ij} was conceived as the resistance between the nodes i and j in an electrical network corresponding to the graph G, in which all edges are replaced by resistors of unit resistance. Eventually, it was shown [18,19,1,20] that the resistance distance can be expressed in terms of Laplacian matrix and its spectrum.

In [11], it was proven that $Kf(G) = n \sum\limits_{i=1}^{n-1} 1/\mu_i$. We refer to [10,23,24,25,14,15] for more information about the mathematical properties of the Kirchhoff index. Recently, Chen and Zhang [8], introduced the degree Kirchhoff index of G as $Kf^*(G) = \sum\limits_{i<j} d_i\, d_j\, r_{ij}$ and proved that $Kf^*(G) = 2m \sum\limits_{i=1}^{n-1} 1/\delta_i$, see [4,9,25,16,17] for details.

The line graph $L(G)$ of a graph G is the graph whose vertices correspond to the edges of G. Two vertices of $L(G)$ are adjacent if and only if the corresponding edges of G have a common vertex. Define $L^0(G) = G$, $L^k(G) = L(L^{k-1}(G))$, $k \geq 1$. A well–known result in graph theory states that the line graph $L^k(G)$ of an r-regular G of order n is a $(2^k\, r - 2^{k+1} + 2)$-regular graph of order $n \prod\limits_{i=0}^{k-1} (2^{i-1}\, r - 2^i + 1)$, having $n \prod\limits_{i=0}^{k} (2^{i-1}\, r - 2^i + 1)$ edges. Following Yan et al. [21], a para-line graph of G, denoted by $C(G)$, is defined as a line graph of the subdivision graph $S(G)$ of G. Here, $S(G)$ is the graph obtained from G by inserting a vertex to every edge of G. This graph is also called the clique-inserted graph [21,22]. Let $C^0(G) = G$ and $C^k(G) = C(C^{k-1}(G))$, $k \geq 1$. Notice that for $k \geq 0$, $C^k(G)$ is r-regular with $n'_k = n\, r^k$ vertices and $m'_k = \frac{1}{2}\, n\, r^{k+1}$ edges.

It is well known that if G is connected, then $L(G)$ and $S(G)$ are also connected. For two graphs G_1 and G_2, $G_1 \cup G_2$ is the disjoint union of G_1 and G_2. The join $G_1 + G_2$ is the graph obtained from $G_1 \cup G_2$ by connecting all vertices of $V(G_1)$ with all vertices of $V(G_2)$. If $G_1, G_2, \ldots, G_k$ are graphs with mutually disjoint vertex sets, we denote $G_1 + G_2 + \cdots + G_k$ by $\sum\limits_{j=1}^{k} G_j$. In the case that $G_1 = G_2 = \cdots = G_k = G$, we denote $\sum\limits_{j=1}^{k} G_j$ by $k\,G$. The following results are crucial throughout this paper.

Lemma 1.1 [7]. Let G be a graph of order $n \geq 2$ without isolated vertices. Then 0 is a simple eigenvalue of $\tilde{\mathcal{L}}(G)$ if and only if G is connected. Moreover, $\sum_{i=1}^{n} \delta_i = n$.

Lemma 1.2 [22]. Let G be a simple r-regular graph with n vertices, $m = \frac{1}{2}\, nr$ edges, and eigenvalues $r = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Then the eigenvalues of the para-line graph $C(G)$ are: 0 with multiplicity $m - n$, -2 with multiplicity $m - n$ and simple eigenvalues $\frac{1}{2}\left[r + 2 \pm \sqrt{r^2 + 4(\lambda_i + 1)}\right]$, where $1 \leq i \leq n$.

As well known [7], if G is a connected r-regular graph, then $\delta(G) \leq 2$ with equality if and only if G is bipartite.

Lemma 1.3 [5]. Let G_1 be an r_1-regular graph of order n_1, and G_2 an r_2-regular graph of order n_2. Let $2 \geq \delta_1(G_1) \geq \delta_2(G_1) \geq \cdots \geq \delta_n(G_1) = 0$ be the normalized Laplacian eigenvalues of G_1 and $2 \geq \delta_1(G_2) \geq \delta_2(G_2) \geq \cdots \geq \delta_n(G_2) = 0$ the normalized Laplacian eigenvalues of G_2. Then the normalized Laplacian eigenvalues of the join graph $G_1 + G_2$ is computed as follows:

$$\begin{pmatrix} \frac{n_2}{n_2+r_1} + \frac{n_1}{n_1+r_2} & \frac{n_2 + r_1\,\delta_i(G_1)}{n_2+r_1} & \frac{n_1 + r_2\,\delta_i(G_2)}{n_1+r_2} & 0 \\ 1 & 1 \leq i \leq n_1 - 1 & 1 \leq i \leq n_2 - 1 & 1 \end{pmatrix}.$$

Throughout this paper our notation is standard. The complement of a graph G is denoted by $\overline{G}$. K_n and $\overline{K}_n$ are the complete and empty graphs on n vertices, respectively. The complete bipartite graph with bipartitions of size n_1 and n_2 is denoted by K_{n_1,n_2}. The Kneser graph $KG_{m,n}$ is the graph whose vertices are the m-subsets of an n-set, two such subsets being adjacent if only if their intersection is empty.

2. Main results

The aim of this section is to find bounds for degree Kirchhoff index of graphs. Some formulas for the degree Kirchhoff index of $L^k(G)$ and $C^k(G)$, $k \geq 1$, are also presented.

Proposition 2.1. Let G be a connected r-regular graph of order $n \geq 3$, having t spanning trees. Then $Kf^*(G) \geq n\,r^2(n-1)(t\,n)^{-1/(n-1)}$, with equality if and only if $G \cong K_n$.

Proof. By the arithmetic–geometric inequality [12],

$$\frac{1}{n-1}\,Kf^*(G) = \frac{2m}{n-1} \sum_{i=1}^{n-1} \frac{1}{\delta_i} \quad \geq \quad 2m \left(\prod_{i=1}^{n-1} \frac{1}{\delta_i} \right)^{1/(n-1)} .$$

Since $\delta_i = \mu_i/r$ and $m = \frac{1}{2}\,n\,r$, the matrix–tree theorem [3] implies that $t\,n = \prod_{i=1}^{n-1} \mu_i$. Thus,

$$Kf^*(G) \quad \geq \quad n\,r(n-1) \left(\prod_{i=1}^{n-1} \frac{r}{\mu_i} \right)^{1/(n-1)} = n\,r^2(n-1) \left(\frac{1}{t\,n} \right)^{1/(n-1)} .$$

Equality holds if and only if all δ_i's for $i = 1, 2, \ldots, n-1$, are mutually equal. This happens only if $G \cong K_n$ [7]. □

Corollary 2.2. Let G be a connected r-regular graph of order $n \geq 3$ and let $\overline{G}$ be also connected, having $\bar{t}$ spanning trees. Then

$$Kf^*(\overline{G}) > n(n-r-1)^2\,(n-1) \left(\frac{1}{\bar{t}\,n} \right)^{1/(n-1)} .$$

Proposition 2.3. Suppose that G_1 and G_2 are graphs with n_1, n_2 vertices and m_1, m_2 edges, respectively. Then

$$Kf^*(G_1 \cup G_2) = Kf^*(G_1) + Kf^*(G_2) + 2\left(m_2 \sum_{i=1}^{n_1-1} \frac{1}{\delta_i(G_1)} + m_1 \sum_{j=1}^{n_2-1} \frac{1}{\delta_j(G_2)} \right)$$

i.e.,

$$Kf^*(G_1 \cup G_2) = (m_1 + m_2)\left(\frac{1}{m_1}\, Kf^*(G_1) + \frac{1}{m_2}\, Kf^*(G_2) \right).$$

Proof. By definition, the normalized Laplacian eigenvalues of $G_1 \cup G_2$ are: $\delta_i(G_1)$, $\delta_j(G_2)$, 0, 0, where $1 \le i \le n_1 - 1$ and $1 \le j \le n_2 - 1$. This yields the result. $\square$

Proposition 2.4. Suppose G_1 and G_2 are r_1- and r_2-regular graphs with n_1, n_2 vertices and m_1, m_2 edges, respectively. Then,

$$\begin{aligned} Kf^*(G_1 + G_2) &= Kf^*(G_1 + \overline{K}_{n_2}) + Kf^*(G_2 + \overline{K}_{n_1}) - Kf^*(K_{n_1,n_2}) \\ &+ \frac{n_2\, r_2(n_2 + r_1)}{n_1 + n_2}\, Kf(G_1 + \overline{K}_{n_2}) + \frac{n_1\, r_1(n_1 + r_2)}{n_1 + n_2}\, Kf(G_2 + \overline{K}_{n_1}) \\ &- n_1\, n_2 \left[r_1 + r_2 + 1 + \frac{r_1}{2n_2 + r_1} + \frac{r_2}{2n_1 + r_2} - 2B \right] \\ &- n_1\, r_1 \left[n_1 + r_2 \left(\frac{1}{n_1 + n_2} + \frac{n_1 - 1}{n_2} \right) - \frac{n_2}{2n_2 + r_1} - B \right] \\ &- n_2\, r_2 \left[n_2 + r_1 \left(\frac{1}{n_1 + n_2} + \frac{n_2 - 1}{n_1} \right) - \frac{n_1}{2n_1 + r_2} - B \right] \end{aligned}$$

where

$$B = \frac{(n_2 + r_1)(n_1 + r_2)}{n_2(n_1 + r_2) + n_1(n_2 + r_1)}\ .$$

Proof. Obviously, $m_1 = \frac{1}{2}\, n_1\, r_1$, $m_2 = \frac{1}{2}\, n_2\, r_2$, $r_1\, \delta_i(G_1) = \mu_i(G_1)$, and $r_2\, \delta_i(G_2) = \mu_i(G_2)$, where $\mu_i(G_j)$ is a Laplacian eigenvalue of G_j, $j = 1, 2$ and $1 \le i \le n_j$. So,

$$\begin{aligned} Kf^*(G_1 + G_2) &= (n_1\, r_1 + n_2\, r_2 + 2n_1\, n_2) \left[\sum_{i=1}^{n_1-1} \frac{n_2 + r_1}{n_2 + \mu_i(G_1)} + \sum_{i=1}^{n_2-1} \frac{n_1 + r_2}{n_1 + \mu_i(G_2)} \right. \\ &+ \left. \frac{(n_2 + r_1)(n_1 + r_2)}{n_2(n_1 + r_2) + n_1(n_2 + r_1)} \right]. \end{aligned} \tag{2.1}$$

Thus,

$$Kf^*(G_1 + \overline{K}_{n_2}) = (n_1\, r_1 + 2n_1\, n_2) \left[n_2 - 1 + \frac{n_2 + r_1}{2n_2 + r_1} + \sum_{i=1}^{n_1-1} \frac{n_2 + r_1}{n_2 + \mu_i(G_1)} \right] \tag{2.2}$$

and

$$Kf^*(G_2+\overline{K}_{n_1}) = (n_2\,r_2+2n_1\,n_2)\left[n_1-1+\frac{n_1+r_2}{2n_1+r_2}+\sum_{i=1}^{n_2-1}\frac{n_1+r_2}{n_2+\mu_i(G_2)}\right]. \tag{2.3}$$

We further have:

$$Kf(G_1+\overline{K}_{n_2}) = 1+(n_1+n_2)\left(\frac{n_2-1}{n_1}+\sum_{i=1}^{n_1-1}\frac{1}{n_2+\mu_i(G_1)}\right) \tag{2.4}$$

and

$$Kf(G_2+\overline{K}_{n_1}) = 1+(n_1+n_2)\left(\frac{n_1-1}{n_2}+\sum_{i=1}^{n_2-1}\frac{1}{n_1+\mu_i(G_2)}\right). \tag{2.5}$$

Then the proof is obtained by substituting Eqs. (2.2)–(2.5) into (2.1). □

Corollary 2.6. Let G be an r-regular graph of order n. Then

$$Kf^*(2G) = 4n(n+r)^2\left(\sum_{i=1}^{n-1}\frac{1}{n+\mu_i(G)}+\frac{1}{4n}\right).$$

Lemma 2.7. Let G be a connected r-regular graph with n vertices and m edges. Then,

$$Kf^*(L(G)) = (r-1)^2\left[2r\,Kf(G)+\frac{n^2}{2}(r-2)\right].$$

Proof. Suppose that the eigenvalues of G are $r=\lambda_1(G)\geq\lambda_2(G)\geq\cdots\geq\lambda_n(G)$. Then 0, $\frac{r-\lambda_i(G)}{2r-2}$, $2\leq i\leq n$, and $\frac{r-1}{r}$ with multiplicity $\frac{n(r-2)}{2}$ are the normalized Laplacian eigenvalues of $L(G)$. Since the number of edges of $L(G)$ is $\frac{n\,r(r-1)}{2}$,

$$\begin{aligned}
Kf^*(L(G)) &= nr(r-1)\left[\sum_{i=2}^{n}\frac{2r-2}{r-\lambda_i}+\frac{n(r-2)(r-1)}{2r}\right]\\
&= 2nr(r-1)^2\sum_{i=2}^{n}\frac{1}{\mu_i}+\frac{n^2(r-1)^2(r-2)}{2}\\
&= (r-1)^2\left[2r\,Kf(G)+\frac{n^2}{2}(r-2)\right]
\end{aligned}$$

proving the result. □

Proposition 2.8. Let G be an r-regular graph with n vertices. Then

$$\begin{aligned}
Kf^*(L^{k+1}(G)) &\sim \frac{n^2}{2}(2^k r-2^{k+1}+1)^2\prod_{j=0}^{k-2}r_j^2\\
&\times \left[\frac{1}{2}\prod_{j=k-2}^{k-1}(r_j-1)+\frac{r-2}{2^k}(2^{k-1}r-2^k+2)^2\right]
\end{aligned}$$

where, $r_j=2^j\,r-2^{j+1}+2$ and k is a sufficiently large integer.

Proof. Let r_k and n_k denote the degree and the number of vertices of $L^k(G)$, respectively. By Lemma 2.7, we get $Kf^*(L^{k+1}(G)) = (r_k-1)^2\left[2r_k\, Kf(L^k(G)) + \frac{n_k^2}{2}(r_k-2)\right]$. From [21, Theorem 3.3], we have

$$Kf(L^k(G)) \sim \frac{n^2}{4}(2^{k-2}r - 2^{k-1}+1)\prod_{j=0}^{k-2}(2^{j-1}-2^j+1)^2, \text{ when } k\to\infty\ .$$

Therefore,

$$\begin{aligned} Kf^*(L^{k+1}(G)) &= \left[\frac{1}{2}(2^{k-1}r-2^k+1)(2^{k-2}-2^{k-1}+1)\prod_{j=0}^{k-2}r_j^2 + \frac{r-2}{2^k}\left(\prod_{j=0}^{k-1}r_j\right)^2\right] \\ &\times \frac{n^2}{2}(2^kr-2^{k+1}+1)^2 = \left[\frac{1}{2}(2^{k-1}r-2^k+1)(2^{k-2}-2^{k-1}+1)\right. \\ &+ \left.\frac{r-2}{2^k}(2^{k-1}r-2^k+2)^2\right]\frac{n^2}{2}(2^kr-2^{k+1}+1)^2\prod_{j=0}^{k-2}r_j^2 \\ &= \frac{n^2}{2}(2^kr-2^{k+1}+1)^2\prod_{j=0}^{k-2}r_j^2\left[\frac{1}{2}\prod_{j=k-2}^{k-1}(r_j-1) + \frac{r-2}{2^k}(2^{k-1}r-2^k+2)^2\right] \end{aligned}$$

where $r_j = 2^jr - 2^{j+1}+2$ and k is an enough large integer. $\square$

Lemma 2.9. Let G be an r-regular graph of order n. Then

$$Kf^*(C(G)) = n\,r^2\left[\frac{r(r+2)}{n}\,Kf(G) + \frac{n(r-2)(r+1)+r}{r+2}\right].$$

Proof. By Lemma 1.2, since $C(G)$ is r-regular, the normalized Laplacian eigenvalues of $C(G)$ are:

$$\begin{pmatrix} \dfrac{r+2\pm\sqrt{r^2+4(\lambda_i+1)}}{2r} & \dfrac{r+2}{r} & 1 & 0 \\ \\ 2\le i\le n & \dfrac{n(r-2)}{2}+1 & \dfrac{n(r-2)}{2} & 1 \end{pmatrix}$$

Applying the definition of the degree Kirchhoff index we then have:

$$\begin{aligned} Kf^*(C(G)) &= 2n\,r^3\sum_{i=2}^{n}\left(\frac{1}{r+2+\sqrt{r^2+4(\lambda_i+1)}} + \frac{1}{r+2-\sqrt{r^2+4(\lambda_i+1)}}\right) \\ &+ n\,r^2\left[\frac{(n(r-2)+2)r}{2(r+2)} + \frac{n(r-2)}{2}\right] \\ &= n\,r^2\left[\sum_{i=2}^{n}\frac{r(r+2)}{r-\lambda_i} + \frac{n(r-2)(r+1)+r}{r+2}\right] \\ &= n\,r^2\left[\frac{r(r+2)}{n}\,Kf(G) + \frac{n(r-2)(r+1)+r}{r+2}\right] \end{aligned}$$

as desired. □

Proposition 2.10. Let G be an r-regular graph of order n. Then

$$\begin{aligned} Kf^*(C^{k+1}(G)) &\sim r^{k+2}\,(r+2)^k \left[n^2\,(r-2)(r+1)\left(\frac{1}{2}+\frac{r^k}{(r+2)^{k+1}}\right)\right. \\ &+ \left.\frac{n\,r}{r+1}+r(r+2)Kf(G)\right]+\frac{n\,r^{k+3}}{r+2} \qquad \text{when } k\to\infty\ . \end{aligned}$$

Moreover, if $k\to\infty, r\to\infty$ then

$$Kf^*(C^{k+1}(G))\sim n\,r^{k+2}\left[(r+2)^{k+1}\left(\frac{n}{2}+\frac{Kf(G)}{n}\right)+n\,r^{k+1}+1\right].$$

Proof. Let $C^k(G)$ has exactly n'_k vertices. By Lemma 2.9,

$$\begin{aligned} Kf^*(C^{k+1}(G)) &= n'_k\,r^2\left[\frac{r(r+2)}{n'_k}\,Kf(C^k(G))+\frac{n'_k\,(r-2)(r+1)+r}{r+2}\right] \\ &= r^3\,(r+2)\,Kf(C^k(G))+\frac{n^2\,r^{2k+2}\,(r-2)(r+1)+n\,r^{k+3}}{r+2}\ . \end{aligned}$$

From [21, Theorem 3.6], we have

$$Kf(C^k(G))\sim\left[\frac{n^2\,(r-2)(r+1)}{2r(r+2)}+\frac{n}{(r+1)(r+2)}+Kf(G)\right]r^k\,(r+2)^k \qquad \text{for } k\to\infty\,.$$

Then,

$$\begin{aligned} Kf^*(C^{k+1}(G)) &\sim r^{k+2}\,(r+2)^k\left[\frac{n^2\,(r-2)(r+1)}{2}+\frac{n\,r}{r+1}+r(r+2)\,Kf(G)\right] \\ &+ \frac{n^2\,r^{2k+2}\,(r-2)(r+1)+n\,r^{k+3}}{r+2} \\ &= r^{k+2}\,(r+2)^k\left[n^2\,(r-2)(r+1)\left(\frac{1}{2}+\frac{r^k}{(r+2)^{k+1}}\right)\right. \\ &+ \left.\frac{n\,r}{r+1}+r(r+2)\,Kf(G)\right]+\frac{n\,r^{k+3}}{r+2} \qquad \text{for } k\to\infty\,. \end{aligned}$$

Also, from [21], we have

$$Kf(C^k(G))\sim r^k(r+2)^k\left(\frac{n^2}{2}+Kf(G)\right) \qquad \text{for } k\to\infty\ ,\ r\to\infty\,.$$

Hence, for $k\to\infty$ and $r\to\infty$, we have:

$$\begin{aligned} Kf^*(C^{k+1}(G)) &\sim r^{k+3}\,(r+2)^{k+1}\left(\frac{n^2}{2}+Kf(G)\right)+n^2\,r^{2k+3}+n\,r^{k+2} \\ &= n\,r^{k+2}\left[(r+2)^{k+1}\left(\frac{n}{2}+\frac{Kf(G)}{n}\right)+n\,r^{k+1}+1\right]. \end{aligned}$$

□

3. Applications

In this section, we apply our results given the Section 2 and gives formulas for the degree Kirchhoff index of some classes of graphs.

Example 3.1. In this example, degree Kirchhoff index of some graphs constructed from complete graph K_n, $n \geq 2$, are computed. From [2], we have that $KG_{1,n} \cong K_n$, $n \geq 3$. Thus, $Kf^*(K_n) = Kf^*(KG_{1,n}) = (n-1)^3$. Note that K_n is $(n-1)$-regular with n vertices and $Kf(K_n) = n-1$. In [3], a triangular graph Δ_n, is defined as a line graph of the complete graph K_n and in [2], it is proved that $\overline{KG_{2,n}} \cong L(K_n)$, $n \geq 5$. Now by Lemma 2.7, we have

$$Kf^*(\overline{KG_{2,n}}) = Kf^*(L(K_n)) = \frac{n^4}{2}(n-3) - 4(n^3 - 5\,n^2 + 6n - 2)\ .$$

By Lemma 2.9,

$$Kf^*(C(K_n)) = \frac{n}{n+1}(n^5 - 5n^4 + 8n^3 - 6n^2 + 3n - 1) + n^5 - 3n^4 + 2n^3 + 2n^2 - 3n + 1\ .$$

Since Δ_n, is an $(2n-4)$-regular graph with $\frac{n(n-1)}{2}$ vertices and $Kf(\Delta_n) = \frac{1}{8}(n-2)(n^2+3n-2)$, by Lemma 2.7

$$Kf^*(L(\Delta_n)) = n^7 - 8\,n^6 + \frac{1}{4}(105\,n^5 - 267\,n^4 + 745\,n^3 - 1439\,n^2) + 330\,n - 100\ .$$

Note that, $Kf^*(L(\Delta_n)) = Kf^*(L(\overline{KG_{2,n}})) = Kf^*(L^2(K_n))$. By Lemma 2.8,

$$\begin{aligned} Kf^*(C(\Delta_n)) &= \frac{n}{n-1}\big(2\,n^7 - 21\,n^6 + 89\,n^5 - 193\,n^4 + 219\,n^3 - 108\,n^2 - 4n + 16\big) \\ &+ 2\big(n^7 - 6\,n^6 + 3\,n^5 + 58\,n^4 - 184\,n^3 + 240\,n^2 - 144\,n + 32\big)\ . \end{aligned}$$

Example 3.2. Consider the cycle graph C_n. It is well know that $Kf(C_n) = \frac{n}{12}\,(n^2-1)$. Since C_n is 2-regular with n vertices, and $Kf^*(C_n) = Kf^*(L(C_n))$, by Lemma 2.7, $Kf^*(L(C_n)) = \frac{n}{3}\,(n^2-1)$. Apply Lemma 2.9, we have $Kf^*(C(C_n)) = \frac{2n}{3}\,(4n^2-1)$.

Example 3.3. Consider the complete bipartite graph $K_{n,n}$, for which $Kf = 4n-3$. Recall that the normalized Laplacian spectrum of $K_{n,n}$ is $\{2\,, \underbrace{1, \ldots, 1}_{2n-2}, 0\}$ and $K_{n,n}$ is n-regular with $2n$ vertices and n^2 edges. Then $Kf^*(K_{n,n}) = n^2\,(4n-3)$. The line graph of $K_{n,n}$, $n \geq 2$, is known as the lattice graph $L_2(n)$. By Lemma 2.7, we have $Kf^*(L(K_{n,n})) = 2n(n^4 - 6n^2 + 8n - 3)$. On the other hand, by Lemma 2.9,

$$Kf^*(C(K_{n,n})) = n^3\,(4\,n^2 + 5n - 6) + \frac{2n^4}{n+2}\,(2\,n^2 - 2n - 3)\ .$$

By continuing this reasoning, since $K_{n,n,n}$ is $2n$-regular of order $3n$ and $Kf(K_{n,n,n}) = \frac{1}{2}\,(9n-5)$, $Kf^*(L(K_{n,n,n})) = n(36\,n^4 - 67\,n^2 + 49\,n - 10)$, it follows that

$$Kf^*(C(K_{n,n,n})) = \frac{12\,n^4}{n+1}\,(6\,n^2 - 3n - 2) + 8\,n^3\,(9n^2 + 4n - 5)\ .$$

Example 3.4. Cocktail–party graph CP_n . Notice that CP_n is $(2n-2)$-regular with $2n$ vertices and $2n(n-1)$ edges. As well-known, $Kf(CP_n) = \frac{2n^2-2n+1}{n-1}$ and $Kf^*(CP_n) = 4(2n^3 - 4n^2 + 3n - 1)$. By Lemma 2.7,

$$Kf^*(L(CP_n)) = 4(4\,n^5 - 12\,n^4 + n^3 + 28\,n^2 - 30\,n + 9)$$

whereas by Lemma 2.9,

$$Kf^*(C(CP_n)) = 8(8\,n^5 - 22\,n^4 + 23\,n^3 - 13\,n^2 + 5\,n^5 - 1)\ .$$

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