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ENERGY OF BINARY LABELED GRAPHS

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ABSTRACT. Let G be a graph with vertex set $V(G)$ and edge set $X(G)$ and consider the set $A = \{0, 1\}$. A mapping $l : V(G) \longrightarrow A$ is called binary vertex labeling of G and $l(v)$ is called the label of the vertex v under l. In this paper we introduce a new kind of graph energy for the binary labeled graph, the labeled graph energy $E_l(G)$. It depends on the underlying graph G and on its binary labeling, upper and lower bounds for $E_l(G)$ are established. The labeled energies of a number of well known and much studied families of graphs are computed.

1. Introduction

Let G be a graph of order n. The energy of the graph G was first defined by Gutman [7] in 1978 as the sum of the absolute eigenvalues of G. It represents a proper generalization of a formula valid for the total π-electron energy of a conjugated hydrocarbon as calculated by the Huckel molecular orbital (HMO) method in quantum chemistry. For recent mathematical work on the energy of a graph see [1, 2, 3, 9, 13, 14, 15]. In connection with graph energy, energy-like quantities were also considered for other matrices: Laplacian [8], distance [10], minimum covering [4], maximum degree [5] etc.

All graphs considered in this paper are finite, simple and undirected.

Let $G = (V, X)$ be a simple binary labeled graph [11] with n vertices $v_1, v_2, \ldots, v_n$. We define

$$l_{ij} = \begin{cases} a, & \text{if } v_iv_j \in X(G) \text{ and } l(v_i) = l(v_j) = 0, \\ b, & \text{if } v_iv_j \in X(G) \text{ and } l(v_i) = l(v_j) = 1, \\ c, & \text{if } v_iv_j \in X(G) \text{ and } l(v_i) = 0, l(v_j) = 1 \text{ or vice-versa}, \\ 0, & \text{otherwise.} \end{cases}$$

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where a, b and c are distinct non zero real numbers.

Then the $n \times n$ matrix $A_l(G) = [l_{ij}]$ is called the adjacency matrix of labeled graph G or just label matrix of G. The adjacency matrix explains graph and spectral properties of a graph. In the same way, the label matrix $A_l(G)$ explains features of a binary labeled graph and leads to study the spectral properties of the underlying labeled graph. The characteristic polynomial of the label matrix $A_l(G)$ is defined by

$$\begin{aligned}\phi(A_l(G), \lambda) &= \det(\lambda I - A_l(G)) \\ &= c_0\lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \cdots + c_n\end{aligned}$$

where I is the unit matrix of order n. The roots $\lambda_1, \lambda_2, \ldots, \lambda_n$ assumed in non-increasing order of $\phi(A_l(G), \lambda) = 0$ are called *label eigenvalues* of binary labeled graph G. The label energy of a graph G is defined as $E_l(G) = \sum_{i=1}^{n} |\lambda_i|$. Since $A_l(G)$ is a real symmetric matrix with zero trace, these eigenvalues of binary labeled graph are real with sum equal to zero.

Thus $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and $\sum_{i=1}^{n} \lambda_i = 0$.

There are situations when chemists use labeled graphs rather than graphs, such as vertices represent two distinct chemical species and the edges represent a particular reaction takes place between the two corresponding species. It is possible that the label energy that we are considering in this paper has similar applications in chemistry as well as in other areas. We mention that this paper deals only the mathematical aspects of label energy of a graph. It is worth mentioning that the energy of labeled graph means a totally new concept in the literature. This concept can be extended to other labeling of graphs too.

2. Some basic properties of label energy

It is easy to check that,

(1) if $V(G)$ is labeled only as '0', then $A_l(G)$ denoted as $A_{l_0}(G)$ is equal to $aA(G)$.
Therefore, $E_{l_0}(G) = aE(G)$.

(2) if $V(G)$ is labeled only as '1', then $A_l(G)$ denoted as $A_{l_1}(G)$ is equal to $bA(G)$.
Hence, $E_{l_1}(G) = b(E(G))$.

Example 1. *: We first compute the label energy of graph G for three different labelings of G in Fig.1.*

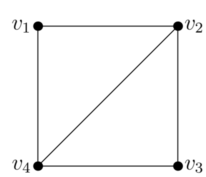

FIGURE 1. Graph G

Let $l(v_i) = 0, for \quad i = 1, 2, 3, 4.$ *Then*

$$A_{l_0}(G) = \begin{bmatrix} 0 & a & 0 & a \\ a & 0 & a & a \\ 0 & a & 0 & a \\ a & a & a & 0 \end{bmatrix}$$

The label energy of the respective graph is $E_{l_a}(G) = 5.1231a.$
Let $l(v_i) = 1, \quad for \quad i = 1, 2, 3, 4.$ *Then*

$$A_{l_1}(G) = \begin{bmatrix} 0 & b & 0 & b \\ b & 0 & b & b \\ 0 & b & 0 & b \\ b & b & b & 0 \end{bmatrix}$$

The label energy of the corresponding graph is $E_{l_1}(G) = 5.1231b.$
Let $l(v_1) = l(v_3) = 0$ *and* $l(v_2) = l(v_4) = 1.$ *Then*

$$A_{l_{0,1}}(G) = \begin{bmatrix} 0 & c & 0 & c \\ c & 0 & c & b \\ 0 & c & 0 & c \\ c & b & c & 0 \end{bmatrix}$$

In this case, the characteristic polynomial of $A_{l_{0,1}}$ *is* $\lambda^4 - (b^2 + 4c^2)\lambda - 4bc^2$.

Example 1 illustrates the fact that the label energy of a graph G depends on the assignment of vertex labels. i.e., that the label energy is not a graph invariant. The following observations can be made about a label matrix $A_l(G)$ of graph G.

(1) The degree of the vertex equals the number of non zero elements in the corresponding row (column).
(2) The occurrence of 'a' s and 'b' s above the principal diagonal of $A_l(G)$ give the number of labels 0 and 1 respectively in graph G.
(3) Given any label matrix $A_l(G)$ of order n, one can always construct a graph G of n vertices and identify the labeling of G.

We now give the explicit expression for the coefficient c_i of $\lambda^{n-i}, i = 0, 1, 2, 3$ in the *characteristic polynomial* of $A_l(G)$.

Theorem 2.1. *Let G be a labeled graph with vertex set V and edge set X.*
Let $\phi(A_l(G), \lambda) = c_0\lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + c_3\lambda^{n-3} + \cdots + c_n$ *be the characteristic polynomial of* $A_l(G)$. *Then,*

(i) $c_0 = 1,$
(ii) $c_1 = 0,$
(iii) $c_2 = -[n_1(a)^2 + n_2(b)^2 + n_3(c)^2]$, *where* n_1, n_2, n_3 *denote the number of edges of the graph G with end vertices are labeled as* $(0, 0), (1, 1)$ *and* $(0, 1)$ *respectively.*

(iv) $c_3 = -2 \sum_{\substack{\triangle v_i v_j v_k \\ i<j<k}} S(i)$, *where,*

$$S(i) = \begin{cases} a^3, & \text{if } l_i = l_j = l_k = 0 \\ b^3, & \text{if } l_i = l_j = l_k = 1 \\ ac^2, & \text{if } l_i = 0 \text{ is the repeated label} \\ bc^2, & \text{if } l_i = 1 \text{ is the repeated label.} \end{cases}$$

and $l_i = l(v_i)$.

Proof. (i) Directly from the definition of $\phi(A_l(G), \lambda)$, it follows that $c_0 = 1$

(ii) $c_1 = trace(A_l(G)) = 0$.

(iii) We have

$$c_2 = \sum_{1 \le j < k \le n} \begin{vmatrix} 0 & l_{jk} \\ l_{kj} & 0 \end{vmatrix}$$

But

$$\begin{vmatrix} 0 & l_{jk} \\ l_{kj} & 0 \end{vmatrix} = \begin{cases} -l_{jk}^2, & \text{if } v_j \text{ and } v_k \text{ are adjacent} \\ 0, & \text{otherwise} \end{cases}$$

Therefore, $c_2 = -[n_1(a)^2 + n_2(b)^2 + n_3(c)^2]$

(iv) We have

$$\begin{aligned} c_3 &= (-1)^3 \sum_{1 \le i < j < k \le n} \begin{vmatrix} l_{ii} & l_{ij} & l_{ik} \\ l_{ji} & l_{jj} & l_{jk} \\ l_{ki} & l_{kj} & l_{kk} \end{vmatrix} \\ &= -2 \sum_{1 \le i < j < k \le n} l_{ij} l_{jk} l_{ki} \\ &= -2 \sum_{\substack{\triangle v_i v_j v_k \\ i < j < k}} S(i) \end{aligned}$$

Consider a triangle of vertices v_i, v_j, v_k, when $i < j < k$. There are only four possible non-isomorphic labelings of a triangle i.e. (0,0,0), (1,1,1), (0,1,0), (1,0,1). Then,

$$S(i) = \begin{cases} a^3, & \text{if } l_i = l_j = l_k = 0 \\ b^3, & \text{if } l_i = l_j = l_k = 1 \\ ac^2, & \text{if } l_i = 0 \text{ is the repeated label} \\ bc^2, & \text{if } l_i = 1 \text{ is the repeated label.} \end{cases}$$

□

Theorem 2.2. *If* $\lambda_1, \lambda_2, \ldots, \lambda_n$ *are the eigenvalues of* $A_l(G)$*, then* $\sum_{i=1}^n \lambda_i^2 = -2c_2$

Proof. We have

$$\begin{aligned}\sum_{i=1}^{n}\lambda_i^2 &= trace \quad (A_l(G))^2\\ &= \sum_{i=1}^{n}\sum_{k=1}^{n} l_{ik}l_{ki}\\ &= \sum_{i=1}^{n}\sum_{k=1}^{n} l_{ik}^2\\ &= 2[n_1(a)^2+n_2(b)^2+n_3(c)^2]\\ &= -2c_2\end{aligned}$$

where n_1, n_2, n_3 denote the number of edges of the graph G with end vertices are labeled as $(0,0),(1,1)$ and $(0,1)$ respectively. □

Bounds for $E_l(G)$, similar to McClelland's inequalities [12] for graph energy are given in the following theorem.

Theorem 2.3. *Let G be a labeled graph with n vertices and m edges. Then*
$\sqrt{2[n_1(a)^2+n_2(b)^2+n_3(c)^2]+n(n-1)p^{\frac{2}{n}}} \leq E_l(G) \leq \sqrt{2n(n_1(a)^2+n_2(b)^2+n_3(c)^2)}$

Proof. We have

$$\begin{aligned}E_l^2(G) &= (\sum_{i=1}^{n}|\lambda_i|)^2\\ &= \sum_{i=1}^{n}|\lambda_i|^2+\sum_{i\neq j}|\lambda_i||\lambda_j|\\ &\geq 2[(n_1(a)^2+n_2(b)^2+n_3(c)^2)]+n(n-1)[\prod_{i=1}^{n}|\lambda_i|]^{\frac{2}{n}}\end{aligned}$$

by using Arithmetic and Geometric mean inequality and Theorem 2.2 .
Hence,

$$E_l(G) \geq \sqrt{(2(n_1(a)^2+n_2(b)^2+n_3(c)^2)+n(n-1)p^{\frac{2}{n}})}, \quad \text{where,} \quad p=\prod_{i=1}^{n}\lambda_i$$

We have

$$\begin{aligned}E_l(G) &= \sum_{i=1}^{n}|\lambda_i|\\ &\leq \sqrt{\sum_{i=1}^{n}|\lambda_i|^2}\ \sqrt{n} \text{ ,on employing Holder's inequality.}\\ &= \sqrt{2n(n_1(a)^2+n_2(b)^2+n_3(c)^2)}\end{aligned}$$

Hence, $\sqrt{2[n_1(a)^2+n_2(b)^2+n_3(c)^2]+n(n-1)p^{\frac{2}{n}}} \leq E_l(G) \leq \sqrt{2n(n_1(a)^2+n_2(b)^2+n_3(c)^2)}$. □

Theorem 2.4. *Let G be a binary labeled graph. If the energy $E_l(G)$ is rational, then $E_l(G) \equiv \text{trace}(A_l(G))(\mod 2)$.*

Proof. Let $\lambda_1, \lambda_2, \dots, \lambda_r$ $(r \in \{1, 2, \dots, n\})$ be positive, and the remaining eigenvalues be non-positive. Then,

$$\begin{aligned} E_l(G) &= \sum_{i=1}^{n} |\lambda_i| \\ &= (\lambda_1 + \lambda_2 + \cdots + \lambda_r) - (\lambda_{r+1} + \lambda_{r+2} + \cdots + \lambda_n) \\ &= 2(\lambda_1 + \lambda_2 + \cdots + \lambda_r) - trace(A_l(G)) \end{aligned}$$

since $\lambda_1, \lambda_2, \dots, \lambda_n$ are algebraic integers, so is their sum.
Hence $(\lambda_1 + \lambda_2 + \cdots + \lambda_r)$ must be an integer if $E_l(G)$ is rational. Therefore, $E_l(G) \equiv trace(A_l(G))(\mod 2)$. ☐

Theorem 2.5. *Let G be a binary labeled graph of order n. Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be the label eigenvalues of G. Then $\lambda_1 \leq [\frac{2[n_1(a^2)+n_2(b^2)+n_3(c^2)](n-1)}{n}]^{\frac{1}{2}}$.*

Proof. For a labeled graph G, we have

$$\sum_{i=1}^{n} \lambda_i = 0 \tag{2.1}$$

$$\sum_{i=1}^{n} \lambda_i^2 = 2[n_1(a^2) + n_2(b^2) + n_3(c^2)], \tag{2.2}$$

where n_1, n_2 and n_3 denote number of edges incident to vertices in G whose labels are $(0,0)$, $(1,1)$ and $(0,1)$ respectively.
From equation(2.1),

$$\lambda_1 = -\sum_{i=2}^{n} \lambda_i$$

Therefore,

$$\lambda_1 \leq \sum_{i=2}^{n} |\lambda_i| \tag{2.3}$$

$$\text{From equation(2.2)}, \sum_{i=2}^{n} \lambda_i^2 = 2[n_1(a^2) + n_2(b^2) + n_3(c^2)] - \lambda_1^2 \tag{2.4}$$

$$\begin{aligned}
\text{By Cauchy Schwartz inequality, } [\sum_{i=2}^{n}|\lambda_i|]^2 &\leq (n-1)\sum_{i=2}^{n}\lambda_i^2, \\
\text{By employing equation (2.3) we get, } \sum_{i=2}^{n}\lambda_i^2 \geq \frac{1}{n-1}(\sum_{i=2}^{n}|\lambda_i|)^2 &\geq \frac{\lambda_1^2}{n-1} \\
\text{From equation(2.4), } \quad 2[n_1(a^2)+n_2(b^2)+n_3(c^2)] - \lambda_1^2 &\geq \frac{\lambda_1^2}{n-1} \\
\text{Therefore, } \quad 2[n_1(a^2)+n_2(b^2)+n_3(c^2)] &\geq \lambda_1^2(1+\frac{1}{n-1}) \\
\lambda_1^2 &\leq \frac{2(n_1(a^2)+n_2(b^2)+n_3(c^2))(n-1)}{n} \\
\text{Thus, } \quad \lambda_1 &\leq (\frac{2(n_1(a^2)+n_2(b^2)+n_3(c^2))(n-1)}{n})^{\frac{1}{2}}
\end{aligned}$$

□

3. **Label Energies of Some families of graphs**

Theorem 3.1. *For $n \geq 2$, the characteristic polynomial of labeled complete graph K_n is* $\phi(A_l(K_n),\lambda) = (\lambda+a)^{m-1}(\lambda+b)^{n-(m+1)}[\lambda^2 - (a(m-1)+b(n-m-1))\lambda + ab(mn-m^2+1) - (n-m)(ab+mc^2)]$ *and label energy of complete graph is* $E_l(K_n) = a(m-1)+b(n-m-1)+\sqrt{(a(m-1)+b(n-m-1))^2+4(ab(mn-m^2+1)-(n-m)(ab+mc^2))}$, *where m vertices are labeled zero, n-m vertices are labeled one and $0 \leq m \leq n$.*

Proof. Let $v_1, v_2, \ldots, v_m$ vertices of K_n be labeled zero and $v_{m+1}, v_{m+2}, \ldots, v_n$ be labeled 1. Then

$$A_l(K_n) = \begin{bmatrix}
0 & a & a & \cdots & a & a & c & c & c & \cdots & c & c \\
a & 0 & a & \cdots & a & a & c & c & c & \cdots & c & c \\
\vdots & & & \ddots & & & & & & \ddots & & \vdots \\
a & a & a & \cdots & a & 0 & c & c & c & \ldots & c & c \\
c & c & c & \cdots & c & c & 0 & b & b & \cdots & b & b \\
c & c & c & \cdots & c & c & b & 0 & b & \cdots & b & b \\
\vdots & & & \ddots & & & & & & \ddots & & \vdots \\
c & c & c & \cdots & c & c & b & b & b & \cdots & b & 0
\end{bmatrix}$$

The *characteristic polynomial* of $A_l(K_n)$ is

$$\phi(A_l(K_n),\lambda) = \begin{vmatrix} \lambda & -a & -a & \cdots & -a & -a & -c & -c & -c & \cdots & -c & -c \\ -a & \lambda & -a & \cdots & -a & -a & -c & -c & -c & \cdots & -c & -c \\ \vdots & & & \ddots & & & & & & \ddots & & \vdots \\ -a & -a & -a & \cdots & -a & \lambda & -c & -c & -c & \ldots & -c & -c \\ -c & -c & -c & \cdots & -c & -c & \lambda & -b & -b & \cdots & -b & -b \\ -c & -c & -c & \cdots & -c & -c & -b & \lambda & -b & \cdots & -b & -b \\ \vdots & & & \ddots & & & & & & \ddots & & \vdots \\ -c & -c & -c & \cdots & -c & -c & -b & -b & -b & \cdots & -b & \lambda \end{vmatrix}$$

$$= \begin{vmatrix} [(\lambda+a)I - aJ]_m & -cJ_{m\times n-m} \\ -cJ^T_{n-m\times m} & [(\lambda+b)I - bJ]_{n-m} \end{vmatrix}$$

where J denotes the matrix with all entries equal to unity. If A is a non-singular square matrix, then

$$\begin{vmatrix} A & B \\ B^T & C \end{vmatrix} = |A||C - B^TA^{-1}B|$$

Thus,

$$\phi(A_l(K_n),\lambda) = |((\lambda+a)I - J)_m||((\lambda+b)I - bJ)_{n-m} - (-cJ)\frac{I_m}{(\lambda+a)^{m-1}[\lambda - a(m-1)]}(-cJ^T)|$$

$$= (\lambda+a)^{m-1}(\lambda - a(m-1)\mathrm{diag}((\lambda - b(n-m-1) - \frac{c^2m(n-m)}{\lambda - a(m-1}), \lambda+b, \lambda+b, \cdots, \lambda+b)$$

$$= (\lambda+a)^{m-1}(\lambda+b)^{n-(m+1)}[\lambda^2 - (a(m-1)+b(n-m-1))\lambda + ab(m-1)(n-m-1) - c^2$$

$$m(n-m)]$$

Spectrum of $A_l(K_n)$ is given by

$$\begin{pmatrix} -a & m-1 \\ -b & n-(m+1) \\ \frac{a(m-1)+b(n-m-1)+\sqrt{(a(m-1)+b(n-m-1))^2+4(ab(m-1)(n-m-1)-c^2m(n-m))}}{2} & 1 \\ \frac{a(m-1)+b(n-m-1)-\sqrt{(a(m-1)+b(n-m-1))^2+4(ab(m-1)(n-m-1)-c^2m(n-m))}}{2} & 1 \end{pmatrix}$$

Hence,

$E_l(K_n) = a(m-1)+b(n-m-1)+\sqrt{(a(m-1)+b(n-m-1))^2 + 4(ab(mn-m^2+1)-(n-m)(ab+mc^2))}$. □

Corollary 3.1.1. *Suppose $a = 1$, $b = 2$ and $c = 3$, then $E_l(K_n) = 14m - 8$ is an integer when $n = 3m - 1$.*

Proof. By substituting $n = 3m - 1$ in the expression of $E_l(K_n)$ for $a = 1$, $b = 2$ and $c = 3$ in Theorem 3.1, we obtain $E_l(K_n) = 5m - 5 + \sqrt{81m^2 - 54m + 9} = 5m - 5 + 9m - 3 = 14m - 8$. □

Remark 3.2. *For $a = 1$, $b = 2$ and $c = 3$, $E_l(K_n)$ increases monotonically and linearly with m, if $m \leq \lfloor \frac{n+1}{2} \rfloor$. The maximum value of $E_l(K_n)$ is attained for $\lfloor \frac{n+1}{2} \rfloor$ and thereafter it decreases monotonically.*

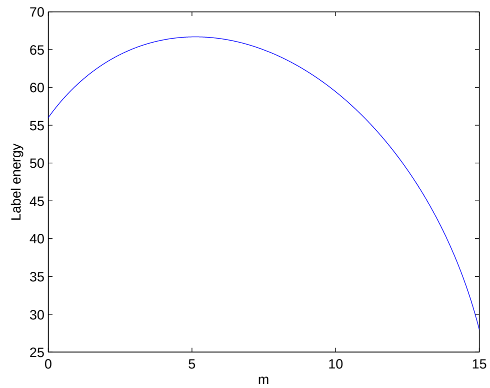

FIGURE 2. The dependence of $E_l(K_n)$ on m for $n = 15$ and $0 \leq m \leq n$

Theorem 3.3. *Let m vertices of star graph S_n including the central vertex be labeled zero and the remaining vertices $n-m$ be labeled one. Then $E_l(S_n) = 2\sqrt{a^2(m-1) + c^2(n-m)}$,
$0 \leq m \leq n$.*

Proof. Let v_1 be the central vertex of S_n. Let $v_1, v_2, \ldots, v_m$ vertices be labeled zero and $v_{m+1}, v_{m+2}, \ldots, v_n$ vertices be labeled one. Then

$$A_l(S_n) = \begin{bmatrix} 0 & a & a & \cdots & a & a & c & c & \cdots & c & c \\ a & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ a & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & & & \ddots & & & & & \ddots & & \vdots \\ a & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \ldots & 0 & 0 \\ a & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ c & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ c & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & & & \ddots & & & & & \ddots & & \vdots \\ c & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

The *characteristic* polynomial is

$$|\lambda I_n - A_l(S_n)| = \begin{vmatrix} \lambda & -a & -a & \cdots & -a & -a & -c & -c & \cdots & -c & -c \\ -a & \lambda & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ -a & 0 & \lambda & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & & & \ddots & & & & & \ddots & & \vdots \\ -a & 0 & 0 & \cdots & \lambda & 0 & 0 & 0 & \ldots & 0 & 0 \\ -a & 0 & 0 & \cdots & 0 & \lambda & 0 & 0 & \cdots & 0 & 0 \\ -c & 0 & 0 & \cdots & 0 & 0 & \lambda & 0 & \cdots & 0 & 0 \\ -c & 0 & 0 & \cdots & 0 & 0 & 0 & \lambda & \cdots & 0 & 0 \\ \vdots & & & \ddots & & & & & \ddots & & \vdots \\ -c & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & \lambda \end{vmatrix}$$

Thus,

$$\begin{aligned} |\lambda I_n - A_l(S_n)| &= \lambda^{m-2}(\lambda^2 - a^2(m-1))|\lambda I_{n-m} - Diag(\frac{c^2\lambda(n-m)}{\lambda^2 - a^2(m-1)}, 0, 0, \cdots, 0)| \\ &= \lambda^{m-2}(\lambda(\lambda^2 - a^2(m-1)) - c^2\lambda(n-m))\lambda^{n-m-1} \\ &= \lambda^{n-2}(\lambda^2 - (a^2(m-1) + c^2(n-m)) \end{aligned}$$

Spectrum of $A_l(S_n)$ is

$$\begin{pmatrix} 0 & \sqrt{a^2(m-1) + c^2(n-m)} & -\sqrt{a^2(m-1) + c^2(n-m)} \\ n-2 & 1 & 1 \end{pmatrix}$$

and $E_l(S_n) = 2\sqrt{a^2(m-1) + c^2(n-m)}$ □

Corollary 3.3.1. *Let m vertices of star graph S_n including the central vertex be labeled one and the remaining vertices $n-m$ be labeled zero. Then $E_l(S_n) = 2\sqrt{b^2(m-1) + c^2(n-m)}$, $0 \leq m \leq n$.*

Proof. By replacing a by b in the matrix of $A_l(S_n)$ in Theorem 3.3, the proof follows. □

Corollary 3.3.2. *The label energy of star graph $E(A_l(S_n)) = 2c\sqrt{n-1}$ is maximum when central vertex has label zero(one)and the remaining vertices have label one(zero) and $c > max\{a, b\}$.*

Proof. By substituting $m = 1$ in the Theorem 3.3 (Corollary 3.3.1), the proof follows. □

Corollary 3.3.3. *The label energy of path graph $E(A_l(P_n)) = cE(P_n)$ is maximum when the vertices are labeled $0, 1$ alternatively and $c > max\{a, b\}$.*

Corollary 3.3.4. *The label energy of cycle graph of even order $E(A_l(C_n)) = cE(C_n)$ is maximum when the vertices are labeled $0, 1$ alternatively and $c > max\{a, b\}$.*

Theorem 3.4. *The multiplicity of the label eigenvalues $-a$ and $-b$ of a labeled complete graph K_n is the degree of the vertex of the induced subgraph K_m whose vertices are labeled zero and K_{n-m} whose vertices are labeled one respectively.*

Proof. We have $\phi(A_l(K_n),\lambda) = (\lambda+a)^{m-1}(\lambda+b)^{n-(m+1)}[\lambda^2 - (a(m-1)+b(n-m-1))\lambda + ab(mn - m^2+1) - (n-m)(ab+mc^2)]$, where m denotes the number of vertices with zero labels.

$$A_l(K_m) = (l_{ij}) = \begin{cases} a, & if \quad (l(v_i), l(v_j)) = (0,0) \\ 0, & \qquad if \quad i = j \end{cases}$$

Therefore, $\phi(A_l(K_m),\lambda) = (\lambda+a)^{m-1}(\lambda - a(m-1))$ which implies $-a$ is the label eigenvalue with multiplicity $m-1$, the degree of the vertex in K_m. Also

$$A_l(K_{n-m}) = (l_{ij}) = \begin{cases} b, & if \quad (l(v_i), l(v_j)) = (1,1) \\ 0, & \qquad if \quad i = j \end{cases}$$

$Hence$, $\phi(A_l(K_{n-m}),\lambda) = (\lambda+b)^{n-m-1}(\lambda - b(n-m-1))$ which implies $-b$ is the label eigenvalue with multiplicity $n-m-1$,the degree of the vertex in K_{n-m}. □

Theorem 3.5. *The characteristic polynomial of binary labeled complete bipartite graph $K(r,s)$ with $m_1 \le r, m_2 \le s$, the number of zeros in the vertex set of order r, s respectively, is given by* $\phi(A_l(K(r,s)),\lambda) = \lambda^{r+s-4}[\lambda^4 - \{((r-m_1)b^2 + m_1c^2)(s-m_2) + m_2((r-m_1)c^2 + m_1a^2)\}\lambda^2 + m_1m_2(r-m_1)(s-m_2)(c^2-ab)^2]$

Proof. Let the labels of $r+s$ vertices be $\underbrace{000\ldots0}_{m_1}\underbrace{111\ldots1}_{r-m_1}$ and $\underbrace{000\ldots0}_{m_2}\underbrace{111\ldots1}_{s-m_2}$.

$$A_l(K(r,s)) = \left[\begin{array}{c|c} 0_r & B_{r\times s} \\ \hline B^T_{s\times r} & 0_s \end{array}\right]_{(r+s)\text{x}(r+s)}$$

where

$$B = \begin{bmatrix} a & a & a & \cdots & a & c & c & \cdots & c & c \\ a & a & a & \cdots & a & c & c & \cdots & c & c \\ a & a & a & \cdots & a & c & c & \cdots & c & c \\ \vdots & & & \ddots & & & & \ddots & & \vdots \\ a & a & a & \cdots & a & c & c & \cdots & c & c \\ c & c & c & \cdots & c & b & b & \cdots & b & b \\ c & c & c & \cdots & c & b & b & \cdots & b & b \\ \vdots & & & \ddots & & & & \ddots & & \vdots \\ c & c & c & \cdots & c & b & b & \cdots & b & b \end{bmatrix}$$

The characteristic polynomial of $A_l(K(r,s))$ is

$$|\lambda I - A_l(K(r,s))| = \begin{vmatrix} \lambda I_r & -B \\ -B^T & \lambda I_s \end{vmatrix}$$

where $B^TB = \left[\begin{array}{c|c} [(r-m_1)c^2+m_1a^2]J_{m_2\times m_2} & [(r-m_1)bc+acm_1]J_{m_2\times(s-m_2)} \\ \hline [(r-m_1)bc+acm_1]J_{(s-m_2)\times m_2} & [(r-m_1)b^2+m_1c^2]J_{(s-m_2)\times(s-m_2)} \end{array}\right]$

$$\begin{aligned}\text{Hence, } |\lambda I - A_l(K(r,s))| &= |\lambda I_r| \left|\lambda I_s - (-B^T)\times \frac{I_r}{\lambda} \times (-B)\right| \\ &= \lambda^r \left|\frac{\lambda^2 I_s - B^TB}{\lambda}\right| \\ &= \lambda^{r-s}\left|\lambda^2 I_s - B^TB\right|\end{aligned}$$

Using elementary row and column operations successively, we get

$$B^TB = \begin{bmatrix} m_2[(r-m_1)c^2+m_1a^2] & 0 & \cdots & 0 & (s-m_2)[(r-m_1)bc+acm_1] & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & & \ddots & & & & \ddots & \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ m_2([(r-m_1)bc+acm_1] & 0 & \cdots & 0 & (s-m_2)[(r-m_1)b^2+m_1c^2] & 0 & \cdots & 0 \\ \vdots & & \ddots & & & & \ddots & \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

Thus, $\left|\lambda^2 I_s - B^TB\right|$

$$= \left|\begin{array}{cccc|cccc} \lambda^2 - m_2[(r-m_1)c^2+m_1a^2] & 0 & \cdots & 0 & -(s-m_2)[(r-m_1)bc+acm_1] & 0 & \cdots & 0 \\ 0 & \lambda^2 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & & & \ddots & & & & \ddots \\ 0 & 0 & \cdots & \lambda^2 & 0 & 0 & \cdots & 0 \\ \hline -m_2[(r-m_1)bc+acm_1] & 0 & \cdots & 0 & \lambda^2-(s-m_2)[(r-m_1)b^2+m_1c^2] & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \lambda^2 & \cdots & 0 \\ \vdots & & & \ddots & & & & \ddots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & \lambda^2 \end{array}\right|$$

$$\begin{aligned} &= \lambda^{2m_2-2}[\lambda^2 - m_2[(r-m_1)c^2+m_1a^2]] \\ &\qquad \left|\mathrm{diag}(\lambda^2-(s-m_2)[(r-m_1)b^2+m_1c^2] - \frac{m_2[(r-m_1)bc+acm_1]^2(s-m_2)}{\lambda^2-m_2[(r-m_1)c^2+m_1a^2]}, \lambda^2, \cdots, \lambda^2)\right| \\ &= \lambda^{2m_2-2}[\lambda^2 - m_2[(r-m_1)c^2+m_1a^2])]\lambda^{2(s-m_2-1)} \\ &\qquad [\lambda^2-(s-m_2)((r-m_1)b^2+m_1c^2) - \frac{m_2((r-m_1)bc+acm_1)^2(s-m_2)}{\lambda^2-m_2((r-m_1)c^2+m_1a^2)}] \\ &= \lambda^{2s-4}\{[\lambda^2-(s-m_2)((r-m_1)b^2+m_1c^2)] \\ &\qquad [\lambda^2-m_2((r-m_1)c^2+m_1a^2)] - m_2(s-m_2)((r-m_1)bc+acm_1)^2\} \end{aligned}$$

$$\begin{aligned}\text{Thus, } |\lambda I - A_l(K(r,s))| = \lambda^{r+s-4}[&\lambda^4 - \{(s-m_2)((r-m_1)b^2+m_1c^2)+m_2((r-m_1)c^2+m_1a^2)\}\lambda^2 \\ &+ m_1m_2(r-m_1)(s-m_2)(c^2-ab)^2]\end{aligned}$$

□

Theorem 3.6. *For $n \geq 4$, the characteristic polynomial of binary labeled double star graph $S(m,n)$ is, $\phi(A_l(S(m,n)),\lambda) = \lambda^{m+n-4}[\lambda^4 - (c^2[(m+n)-(m_1+m_2)] + a^2(m_1+m_2-2)+1)\lambda^2 + (c^2(m-m_1) + a^2(m_1-1))(c^2(n-m_2)+a^2(m_2-1))]$, where m_1 , m_2 ($m_1 \leq m$ and $m_2 \leq n$) are the number of vertices with zero label including the central vertices.*

Proof. Let $(v_1, v_2, \ldots, v_{m_1}, v_{m_1+1}, \ldots, v_{m-1}, v_m)$ be labeled as $(0, 0, \ldots, 0, 1, \ldots, 1, 0)$ and $(v_{m+1}, v_{m+2}, \ldots, v_{m_2}, v_{m_2+1}, \ldots, v_{m+n-1}, v_{m+n})$ be labeled as $(0, 0, \ldots, 0, 1, \ldots, 1, 1)$,where v_m and v_{m+1} are the central vertices.

Then, the label matrix of $S(m,n)$ is

$$A_l(S(m,n)) = \left[\begin{array}{c|c} A_l(S_m) & B \\ \hline B^T & A_l(S_n) \end{array}\right]$$

Where, $B = [b_{ij}]$ is the $m \times n$ matrix with zero elements except $b_{m,m+1} = a$. The characteristic polynomial of $A_l(S(m,n))$ is

$$\phi(A_l(S(m,n)),\lambda) = \left|\begin{array}{ccccccccccccccccc}
\lambda & 0 & 0 & \cdots & 0 & 0 & \cdots & -a & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
0 & \lambda & 0 & \cdots & 0 & 0 & \cdots & -a & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \lambda & \cdots & 0 & 0 & \cdots & -a & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
\vdots & & & \ddots & & & \ddots & & & & \ddots & & & \ddots & & \vdots \\
0 & 0 & 0 & \cdots & \lambda & 0 & \cdots & -a & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & \lambda & \cdots & -c & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & \cdots & -c & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
\vdots & & & \ddots & & & \ddots & & & & \ddots & & & \ddots & & \vdots \\
0 & 0 & 0 & \cdots & 0 & 0 & \cdots & -c & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
-a & -a & -a & \cdots & -a & -c & \cdots & \lambda & -a & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & \cdots & -a & \lambda & -a & \cdots & -a & -c & \cdots & -c & -c \\
0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & -a & \lambda & \cdots & 0 & 0 & \cdots & 0 & 0 \\
\vdots & & & \ddots & & & \ddots & & & & \ddots & & & \ddots & & \vdots \\
0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & -a & 0 & \cdots & \lambda & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & -c & 0 & \cdots & 0 & \lambda & \cdots & 0 & 0 \\
\vdots & & & \ddots & & & \ddots & & & & \ddots & & & \ddots & & \vdots \\
0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & -c & 0 & \cdots & 0 & 0 & \cdots & \lambda & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & -c & 0 & \cdots & 0 & 0 & \cdots & 0 & \lambda
\end{array}\right|$$

$$\begin{aligned}
\phi(A_l(S(m,n)),\lambda) &= \begin{vmatrix} A_{m\times m} & B_{m\times n} \\ B^T_{n\times m} & C_{n\times n} \end{vmatrix} \\
&= |A||C - B^T A^{-1} B|
\end{aligned}$$

Where

$$|A| = \begin{vmatrix} \lambda & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & -a \\ 0 & \lambda & 0 & \cdots & 0 & 0 & \cdots & 0 & -a \\ \vdots & & & \ddots & & & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & \lambda & 0 & \dots & 0 & -a \\ 0 & 0 & 0 & \cdots & 0 & \lambda & \cdots & 0 & -c \\ \vdots & & & \ddots & & \ddots & & \vdots & \\ -a & -a & -a & \cdots & -a & -c & \cdots & -c & \lambda \end{vmatrix}$$

$$= |Diag(\lambda + \sqrt{c^2(m-m_1)+a^2(m_1-1)}, \lambda, \lambda, \dots, \lambda, \lambda - \sqrt{c^2(m-m_1)+a^2(m_1-1)})$$

$$= \lambda^{m-2}[\lambda^2 - (c^2(m-m_1)+a^2(m_1-1))]$$

Similarly, $|C| = \lambda^{n-2}(\lambda^2 - (c^2(n-m_2)+a^2(m_2-1)))$

$$\begin{aligned} \phi(A_l(S(m,n)),\lambda) &= \lambda^{m-2}(\lambda^2 - (c^2(m-m_1)+a^2(m_1-1)))|Diag(\lambda + \sqrt{c^2(n-m_2)+a^2(m_2-1)}, \lambda, \dots, \lambda, \lambda \\ &- \sqrt{c^2(n-m_2)+a^2(m_2-1)}) - \frac{1}{\lambda^{m-2}[\lambda^2 - (c^2(m-m_1)+a^2(m_1-1))]} Diag(a^2, 0, 0, \dots, 0)| \\ &= \lambda^{m-2}(\lambda^2 - (c^2(m-m_1)+a^2(m_1-1)))|Diag(\lambda^2 - (c^2(n-m_2)+a^2(m_2-1)), \lambda, \lambda, \dots, \lambda, 1) \\ &- Diag(\frac{a^2\lambda^2}{\lambda^2 - (c^2(m-m_1)+a^2(m_1-1))}, 0, 0, \dots, 0)| \\ &= \lambda^{m+n-4}[(\lambda^2 - (c^2(m-m_1)+a^2(m_1-1)))(\lambda^2 - (c^2(n-m_2)+a^2(m_2-1))) - a^2\lambda^2] \\ &= \lambda^{m+n-4}[\lambda^4 - (c^2[(m+n)-(m_1+m_2)] + a^2(m_1+m_2-2)+a^2)\lambda^2 \\ &+ (c^2(m-m_1)+a^2(m_1-1))(c^2(n-m_2)+a^2(m_2-1))] \end{aligned}$$

□

Corollary 3.6.1. *If the central vertices of $S(m,n)$ are labeled one, then* $\phi(S(m,n),\lambda) = \lambda^{m+n-4}[\lambda^4 - (c^2[(m+n)-(m_1+m_2]) + b^2(m_1+m_2-2)+b^2)\lambda^2 + (c^2(m-m_1) + b^2(m_1-1))(c^2(n-m_2)+b^2(m_2-1))$,*where m_1, m_2 denote the number of vertices including the central vertices which are labeled one in the vertex sets of order m, n respectively.*

Corollary 3.6.2. *If the central vertices are labeled 0,1 or vice-versa, then* $\phi(S(m,n),\lambda) = \lambda^{m+n-4}[\lambda^4 - (c^2[(m+n)-(m_1+m_2)] + a^2(m_1-1)+b^2(m_2-1)+c^2)\lambda^2 + (c^2(m-m_1)+a^2(m_1-1))(c^2(n-m_2)+b^2(m_2-1))]$,*where m_1, m_2 denote the number of vertices including the central vertices which are labeled zero, one in the vertex sets of order m, n respectively.*

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