**k-TUPLE TOTAL DOMINATION AND MYCIELESKIAN GRAPHS**

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Abstract. Let $k$ be a positive integer. A subset $S$ of $V(G)$ in a graph $G$ is a $k$-tuple total dominating set of $G$ if every vertex of $G$ has at least $k$ neighbors in $S$. The $k$-tuple total domination number $\gamma_{\times k,t}(G)$ of $G$ is the minimum cardinality of a $k$-tuple total dominating set of $G$. In this paper for a given graph $G$ with minimum degree at least $k$, we find some sharp lower and upper bounds on the $k$-tuple total domination number of the $m$-Mycieleskian graph $\mu_m(G)$ of $G$ in terms on $k$ and $\gamma_{\times k,t}(G)$. Specially we give the sharp bounds $\gamma_{\times k,t}(G) + 1$ and $\gamma_{\times k,t}(G) + k$ for $\gamma_{\times k,t}(\mu_1(G))$, and characterize graphs with $\gamma_{\times k,t}(\mu_1(G)) = \gamma_{\times k,t}(G) + 1$.

1. Introduction

In this paper, $G = (V, E)$ is a simple graph with the vertex set $V$ and the edge set $E$. The order $|V|$ of $G$ is denoted by $n = n(G)$. The open neighborhood and the closed neighborhood of a vertex $v \in V$ are $N_G(v) = \{u \in V(G) \mid uv \in E(G)\}$ and $N_G[v] = N_G(v) \cup \{v\}$, respectively. Also the degree of $v$ is $\deg_G(v) = |N_G(v)|$. The minimum and maximum degree of $G$ are denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively. We write $K_n$ and $C_n$ for the complete graph and the cycle of order $n$, respectively, while $G[S]$ and $K_{n_1, n_2, \ldots, n_p}$ denote the subgraph induced on $G$ by a vertex set $S$, and the complete $p$-partite graph, respectively.

Let $S \subseteq V$ and let $k$ be a positive integer. For each $k$-element subset $S' \subseteq S$ the $(S,k)$-private neighborhood $pn_k(S',S)$ of $S'$ is the set of all vertices $v \in V$ such that $N(v) \cap S = S'$. Further, the open $k$-boundary $OB_k(S)$ of $S$ is the set of all vertices $v$ in $G$ such that $v \in pn_k(S',S)$ for some $k$-element subset $S' \subseteq S$. Obviously, $OB_k(S) = \bigcup_{S'} pn_k(S',S)$, where $S'$ is a $k$-element subset of $S$.

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As we will see, the generalized Mycielskian graphs, which are also called cones over graphs [7], are natural generalization of Mycielski graphs. If $V(G) = V^0 = \{v^0_1, v^0_2, \ldots, v^0_n\}$ and $E(G) = E_0$, then for any integer $m \geq 1$ the $m$-Mycielskian $\mu_m(G)$ of $G$ is the graph with vertex set $V^0 \cup V^1 \cup V^2 \cup \cdots \cup V^m \cup \{u\}$, where $V^i = \{v^i_j \mid v^0_j \in V^0\}$ is the $i$-th distinct copy of $V^0$, for $i = 1, 2, \ldots, m$, and edge set $E_0 \cup \left( \bigcup_{i=0}^{m-1} \{v^i_j v^{i+1}_j \mid v^0_j v^0_j \in E_0\} \right) \cup \{v^m_j u \mid v^m_j \in V^m\}$. The 1-Mycielskian $\mu_1(G)$ of $G$ is the well-studied Mycielskian of $G$, and denoted simply by $\mu(G)$ or $M(G)$.

For positive integer $k$, the $k$-join of a graph $G$ to a graph $H$ of order at least $k$ is the graph obtained from the disjoint union of $G$ and $H$ by joining each vertex of $G$ to at least $k$ vertices of $H$. We denote the $k$-join of $G$ to $H$ by $G \circ_k H$.

Domination in graphs is now well-studied in graph theory and the literature on this subject has been surveyed and detailed in the two books by Haynes, Hedetniemi, and Slater [2, 3].

In [4], Henning and Kazemi introduced the $k$-tuple total domination number of a graph. Let $k$ be a positive integer. A subset $S$ of $V$ is a $k$-tuple total dominating set of $G$, abbreviated kTDS, if for every vertex $v \in V$, $|N(v) \cap S| \geq k$, that is, $S$ is a kTDS of $G$ if every vertex of $V$ has at least $k$ neighbors in $S$. The $k$-tuple total domination number $\gamma_{\times k,t}(G)$ of $G$ is the minimum cardinality of a kTDS of $G$. We remark that a 1-tuple total domination is the well-studied total domination number. Thus, $\gamma_k(G) = \gamma_{\times 1,t}(G)$. For a graph to have a $k$-tuple total dominating set, its minimum degree is at least $k$. Since every $(k+1)$-tuple total dominating set is also a $k$-tuple total dominating set, we note that $\gamma_{\times k,t}(G) \leq \gamma_{\times (k+1),t}(G)$ for all graphs with minimum degree at least $k+1$. A kTDS in a graph $G$ is a minimal kTDS if no proper subset of it is a kTDS in $G$. A kTDS of cardinality $\gamma_{\times k,t}(G)$ is called a $\gamma_{\times k,t}(G)$-set. A 2-tuple total dominating set is called a double total dominating set, abbreviated DTDS, and the 2-tuple total domination number is called the double total domination number. The redundancy involved in $k$-tuple total domination makes it useful in many applications. The references [5, 6] give more information about the $k$-tuple total domination number of a graph.

In this paper, we study the $k$-tuple total domination number of the $m$-Mycielskian graph of a graph $G$. We prove that for every positive integers $m$ and $k$ and every graph $G$ with $\delta(G) \geq k$, if $m - 1 \equiv r \pmod{4}$, where $0 \leq r \leq 3$, and $r' \equiv r + 1 \pmod{2}$, then

$$\gamma_{\times k,t}(G) + 1 \leq \gamma_{\times k,t}(\mu_m(G)) \leq \begin{cases} (1 + 2\lceil(m - 1)/4\rceil)\gamma_{\times k,t}(G) + kr' & \text{if } r = 0, 3, \\ 2\lceil(m - 1)/4\rceil\gamma_{\times k,t}(G) + kr' & \text{otherwise}. \end{cases}$$

Hence $\gamma_{\times k,t}(G) + 1 \leq \gamma_{\times k,t}(M(G)) \leq \gamma_{\times k,t}(G) + k$. We also prove that the bounds $\gamma_{\times k,t}(G) + 1$ and $\gamma_{\times k,t}(G) + k$ are sharp and characterize graphs with $\gamma_{\times k,t}(M(G)) = \gamma_{\times k,t}(G) + 1$.

Through of this paper, $k$ is a positive integer. The next results are useful for our investigations.

**Proposition 1.1. (Henning, Kazemi [4] 2010)** Let $G$ be a graph of order $n$ with $\delta(G) \geq k \geq 1$, and let $S$ be a kTDS in $G$. Then

1. $k + 1 \leq \gamma_{\times k,t}(G) \leq n$,
2. for every spanning subgraph $H$ of $G$, $\gamma_{\times k,t}(G) \leq \gamma_{\times k,t}(H)$,
3. for every vertex $v$ of degree $k$, $N_G(v) \subseteq S$. 

Proposition 1.2. (Henning, Kazemi [4] 2010) Let $G$ be a graph of order $n$ with $\delta(G) \geq k \geq 1$, and let $S$ be a kTDS in $G$. Then $S$ is a minimal kTDS of $G$ if and only if for each vertex $v \in S$, there exists a $k$-element subset $S_v \subseteq S$ such that $v \in S_v$ and $|\text{pt}_k(S_v, S)| \geq 1$.

Proposition 1.3. (Henning, Kazemi [4] 2010) Let $G$ be a graph with $\delta(G) \geq k \geq 1$. Then, $\gamma_{x,k,t}(G) = k + 1$ if and only if if $G = K_{k+1}$ or $G = F \circ_k K_{k+1}$ for some graph $F$.

Proposition 1.4. (Henning, Kazemi [5] 2010) Let $G$ be a graph of order $n$ with $\delta(G) \geq k \geq 1$. Then $\gamma_{x,k,t}(G) \geq \lceil kn/\Delta(G) \rceil$.

2. $m$-mycieleskian graphs

In the next theorem we give a lower bound and an upper bound on the $k$-tuple domination number of the $m$-Mycieleskian graph $\mu_m(G)$ in terms $k$ and the $k$-tuple domination number of $G$. First we state the following lemma which has an easy proof that is left to the reader.

Lemma 2.1. Let $G$ be a graph with $\delta(G) \geq k \geq 1$. Let $V(\mu_m(G)) = V^0 \cup V^1 \cup V^2 \cup \cdots \cup V^m \cup \{u\}$. If $\gamma_{x,k,t}(\mu_m(G)) = \gamma_{x,k,t}(G)$, then for every $\gamma_{x,k,t}(\mu_m(G))$-set $S$, $u \notin S$, and so $m = 1$.

Theorem 2.2. Let $m$ and $k$ be two positive integers, and let $G$ be a graph with $\delta(G) \geq k \geq 1$. Then

$$
\gamma_{x,k,t}(G) + 1 \leq \gamma_{x,k,t}(\mu_m(G)) \leq \begin{cases} 
(1 + 2[(m - 1)/4])\gamma_{x,k,t}(G) & \text{if } m \equiv 0 \pmod{4}, \\
(1 + 2[(m - 1)/4])\gamma_{x,k,t}(G) + k & \text{if } m \equiv 1 \pmod{4}, \\
2[(m - 1)/4]\gamma_{x,k,t}(G) & \text{if } m \equiv 2 \pmod{4}, \\
2[(m - 1)/4]\gamma_{x,k,t}(G) + k & \text{if } m \equiv 3 \pmod{4}.
\end{cases}
$$

Proof. Let $V(\mu_m(G)) = V^0 \cup V^1 \cup V^2 \cup \cdots \cup V^m \cup \{u\}$. Since $G$ is an induced subgraph of $\mu_m(G)$, $\gamma_{x,k,t}(G) \leq \gamma_{x,k,t}(\mu_m(G))$. If $\gamma_{x,k,t}(\mu_m(G)) = \gamma_{x,k,t}(G)$, then Lemma 2.1 implies $m = 1$ and for every $\gamma_{x,k,t}(M(G))$-set $S$, $u \notin S$. Since every vertex of $V^1$ is adjacent to at least $k$ vertices of $S \cap V^0$, we conclude that every vertex of $V^0$ is adjacent to at least $k$ vertices of $S \cap V^0$. Hence

$$
\gamma_{x,k,t}(G) \leq |S \cap V^0| = |S| - |S \cap V^1| \leq \gamma_{x,k,t}(G) - k < \gamma_{x,k,t}(G),
$$

a contradiction. Therefore $\gamma_{x,k,t}(\mu_m(G)) \geq \gamma_{x,k,t}(G) + 1$.

Now we prove the other inequality. For an arbitrary $\gamma_{x,k,t}(G)$-set $S$, let $S_i = \{v^i \mid v \in S\} \subseteq V^i$ be the $i$-th distinct copy of $S$ when $0 \leq i \leq m$. Let also $S_k$ be an arbitrary subset of $V^m$ of cardinality $k$. We continue our proof in the following four cases.

Case 0. $m \equiv 0 \pmod{4}$.

The set $S' = S^0 \cup (\bigcup_{t=1}^{[(m-1)/4]}(S^{4t-1} \cup S^{4t})) \cup (S^{m-1} \cup S^m)$ is a kTDS of $\mu_m(G)$ of cardinality

$$(3 + 2[(m - 1)/4])\gamma_{x,k,t}(G) = (1 + 2[(m - 1)/4])\gamma_{x,k,t}(G).$$

Case 1. $m \equiv 1 \pmod{4}$.

The set $S' = S^0 \cup (\bigcup_{t=1}^{[(m-1)/4]}(S^{4t-1} \cup S^{4t})) \cup S_k$ is a kTDS of $\mu_m(G)$ of cardinality

$$(1 + 2[(m - 1)/4])\gamma_{x,k,t}(G) + k = (1 + 2[(m - 1)/4])\gamma_{x,k,t}(G) + k.$$
Case 2. $m \cong 2 \pmod{4}$.

The set $S' = \bigcup_{t=1}^{[m-1/4]} (S^{4t-3} \cup S^{4t-2}) \cup (S^{m-1} \cup S^m)$ is a kTDS of $\mu_m(G)$ of cardinality $2 + 2\lceil (m - 1)/4 \rceil \gamma_{x,t}(G) = 2\lceil (m - 1)/4 \rceil \gamma_{x,t}(G)$.

Case 3. $m \cong 3 \pmod{4}$.

The set $S' = \bigcup_{t=1}^{[m-1/4]} (S^{4t-3} \cup S^{4t-2}) \cup S_k$ is a kTDS of $\mu_m(G)$ of cardinality $2\lceil (m - 1)/4 \rceil \gamma_{x,t}(G) + k$.

Therefore we have proved

$$\gamma_{x,t}(\mu_m(G)) \leq |S'| = \begin{cases} 
(1 + 2\lceil (m - 1)/4 \rceil) \gamma_{x,t}(G) & \text{if } m \cong 0 \pmod{4}, \\
(1 + 2\lceil (m - 1)/4 \rceil) \gamma_{x,t}(G) + 1 & \text{if } m \cong 1 \pmod{4}, \\
2\lceil (m - 1)/4 \rceil \gamma_{x,t}(G) & \text{if } m \cong 2 \pmod{4}, \\
2\lceil (m - 1)/4 \rceil \gamma_{x,t}(G) + 1 & \text{if } m \cong 3 \pmod{4},
\end{cases}$$

and this completes our proof.

\[ \square \]

**Corollary 2.3.** Let $m$ be a positive integer, and let $G$ be a graph without isolated vertex. Then

$$\gamma_t(G) + 1 \leq \gamma_t(\mu_m(G)) \leq \begin{cases} 
(1 + 2\lceil (m - 1)/4 \rceil) \gamma_t(G) & \text{if } m \cong 0 \pmod{4}, \\
(1 + 2\lceil (m - 1)/4 \rceil) \gamma_t(G) + 1 & \text{if } m \cong 1 \pmod{4}, \\
2\lceil (m - 1)/4 \rceil \gamma_t(G) & \text{if } m \cong 2 \pmod{4}, \\
2\lceil (m - 1)/4 \rceil \gamma_t(G) + 1 & \text{if } m \cong 3 \pmod{4}.
\end{cases}$$

3. Mycielski graph

Theorem 2.2 implies the next two theorems when $m = 1$.

**Theorem 3.1.** If $G$ is a graph with no isolated vertices, then $\gamma_t(M(G)) = \gamma_t(G) + 1$.

**Theorem 3.2.** If $G$ is a graph with $\delta(G) \geq k \geq 2$, then $\gamma_{x,t}(G) + 1 \leq \gamma_{x,t}(M(G)) \leq \gamma_{x,t}(G) + k$.

In the next theorem we give other lower bound for $\gamma_{x,t}(M(G))$.

**Theorem 3.3.** If $G$ is a graph with $\delta(G) \geq k \geq 2$, then

$$\gamma_{x,t}(M(G)) \geq \min \{ \gamma_{x,t}(G) + k, \gamma_{x(k-1),t}(G) + k + 1 \}.$$

**Proof.** Let $V(M(G)) = V^0 \cup V^1 \cup \{u\}$ and let $S$ be an arbitrary kTDS of $M(G)$. Then $|S \cap V^1| \geq k$. If $u \notin S$, then $|S \cap V^0| \geq \gamma_{x(k-1),t}(G)$. Since $S \cap V^0$ must be a $(k-1)$TDS of $V^1$. If $u \notin S$, then $|S \cap V^0| \geq \gamma_{x,k,t}(G)$. Since $S \cap V^0$ must be a kTDS of $V^1$. Therefore

$$\gamma_{x,t}(M(G)) \geq \min \{ \gamma_{x,t}(G) + k, \gamma_{x(k-1),t}(G) + k + 1 \}.$$

\[ \square \]

As an immediately result of Theorems 3.2 and 3.3 we have the following two corollaries.
**Corollary 3.4.** Let $G$ be a graph with $\delta(G) \geq k \geq 2$. If $\gamma_{x,k,t}(G) = \gamma_{x,(k-1),t}(G) + 1$, then

$$\gamma_{x,k,t}(M(G)) = \gamma_{x,k,t}(G) + 2.$$ 

**Corollary 3.5.** If $n \geq k + 1 \geq 3$, then $\gamma_{x,k,t}(M(K_n)) = \gamma_{x,k,t}(K_n) + k = 2k + 1$.

Also the following two results show that the upper bound $\gamma_{x,k,t}(G) + k$ in Theorem 3.2 is sharp for some of the complete multipartite graphs, the complete graph $K_{k+1}$ and the $k$-join $F \circ_k K_{k+1}$, for every graph $F$.

**Proposition 3.6.** Let $G = K_{n_1,...,n_p}$ be a complete $p$-partite graph. If $p \geq k + 1 \geq 3$, then

$$\gamma_{x,k,t}(M(G)) = \gamma_{x,k,t}(G) + k.$$ 

**Proof.** We suppose that $V(G)$ has partition $X_1 \cup X_2 \cup ... \cup X_p$ such that $|X_j| = n_j$ for $j = 1,2,...,p$. Let $V(M(G)) = V^0 \cup V^1 \cup \{u\}$, where $V^i = X^i_1 \cup X^i_2 \cup ... \cup X^i_p$ and $X^i_j = \{v^i_j \mid v^i_j \in X^i_j\}$, for $i = 0, 1$ and $j = 1,2,...,p$. Let also $S$ be an arbitrary $k$TDs of $M(G)$. Obviously $|S \cap V^1| \geq k$, and, without less of generality, we may assume that $|S \cap V^1| = k$. Let $S \cap V^1$ be a set which contains only one vertex of every $X^i_1$, for $1 \leq i \leq k$. Thus each vertex of $V^0 - \{v^0_i \mid v^0_i \in S \cap V^1\}$ is adjacent to all vertices in $S \cap V^1$. Since each vertex of $S \cap V^1$ must be adjacent to at least $k$ vertices of $S$, we have $|S \cap (V^0 \cup \{u\})| \geq k$. The assumptions $k \geq 2$ and $|S \cap V^0| \geq k - 1$ imply $S \cap V^0 \neq \emptyset$. We see that there exists an unique index $1 \leq j \leq k$ such that each vertex of $X^1_j$ is adjacent to $k - 1$ vertices of $S \cap (V^0 \cup \{u\})$. Hence $|S \cap (V^0 \cup \{u\})| \geq k + 1$, and so

$$\gamma_{x,k,t}(M(G)) = \min \{|S| \mid S \text{ is a kTDs of } M(G)\} \geq 2k + 1 = \gamma_{x,k,t}(G) + k.$$ 

Now Theorem 3.2 implies $\gamma_{x,k,t}(M(G)) = \gamma_{x,k,t}(G) + k$.

**Theorem 3.7.** Let $G$ be a graph of order $n \geq k + 1$ with $\delta(G) \geq k \geq 2$. If $G$ is the $k$-join $F \circ_k K_{k+1}$, for some graph $F$, then

$$\gamma_{x,k,t}(M(G)) = \gamma_{x,k,t}(G) + k.$$ 

**Proof.** Let $G$ be the $k$-join $F \circ_k K_{k+1}$, for some graph $F$. Then Proposition 1.3 implies $\gamma_{x,k,t}(G) = k + 1$. Since $G$ is a spanning subgraph of $K_n$, and hence $M(G)$ is a spanning subgraph of $M(K_n)$, we have

$$\gamma_{x,k,t}(M(G)) \geq \gamma_{x,k,t}(M(K_n)) = \gamma_{x,k,t}(K_n) + k = 2k + 1 = \gamma_{x,k,t}(G) + k.$$ 

Now Theorem 3.2 implies $\gamma_{x,k,t}(M(G)) = \gamma_{x,k,t}(G) + k = 2k + 1$. 

Theorem 3.1 shows that $\gamma_{x,k,t}(M(G)) = \gamma_{x,k,t}(G) + 1$, where $G$ is a graph with no isolated vertices and $k = 1$. Here, we give an equivalent condition for $\gamma_{x,k,t}(M(G)) = \gamma_{x,k,t}(G) + 1$, when $k \geq 2$. We recall that a kTDS $S$ is a minimal kTDS if and only if for each vertex $v \in S$, there exists a $k$-element subset $S_v \subseteq S$ such that $v \in S_v$ and $|p_k(S_v,S)| \geq 1$. 

Theorem 3.8. Let $G$ be a graph with $\delta(G) \geq k \geq 1$. Then $\gamma_{x_k,t}(M(G)) = \gamma_{x_k,t}(G) + 1$ if and only if $k = 1$ or $k \geq 2$ and $G$ has a $\gamma_{x_k,t}$-set $S$ with a $k$-subset $S' \subseteq S$ such that $S - S'$ is a $(k - 1)$TDS of $G$ and for every vertex $v$, $|S_v \cap S'| \leq 1$.

Proof. Let $V(M(G)) = V^0 \cup V^1 \cup \{u\}$, where $V^i = \{v_j \mid 1 \leq j \leq n\}$ for $i = 0, 1$. If $k = 1$, then $\gamma_{x_k,t}(M(G)) = \gamma_{x_k,t}(G) + 1$, by Theorem 3.1. Now, let $k \geq 2$. Let $S$ be a $\gamma_{x_k,t}(G)$-set which contains a $k$-subset $S' \subseteq S$ with this condition that $S - S'$ is a $(k - 1)$TDS of $G$ and for every vertex $v$, $|S_v \cap S'| \leq 1$. Since

$$D = \{v_j^0 \mid v_j \in S - S' \} \cup \{v_j^1 \mid v_j \in S' \} \cup \{u\}$$

is a $k$TDS of $M(G)$ of cardinality $\gamma_{x_k,t}(G) + 1$, Theorem 3.2 implies $\gamma_{x_k,t}(M(G)) = \gamma_{x_k,t}(G) + 1$.

Conversely, let $\gamma_{x_k,t}(M(G)) = \gamma_{x_k,t}(G) + 1$ and let $k \geq 2$. If $u$ belongs to a $\gamma_{x_k,t}$-set of $M(G)$, we have nothing to prove. Thus we assume that $u$ belongs to no $\gamma_{x_k,t}(M(G))$-set $D$, and so $|D \cap V^1| \geq k$ and $|D \cap V^0| \geq \gamma_{x_k,t}(G)$. Therefore

$$\gamma_{x_k,t}(G) + 1 = \gamma_{x_k,t}(M(G)) = |D| \geq \gamma_{x_k,t}(G) + k \geq \gamma_{x_k,t}(G) + 2,$$

a contradiction. □

The next proposition gives graphs that satisfy in the condition of Theorem 3.8. First, we present the definition of the Harary graph $H_m,n$.

Given $m \leq n$, place $n$ vertices $1, 2, \ldots, n$ around a circle, equally spaced. If $m$ is even, form $H_{m,n}$ by making each vertex adjacent to the nearest $m/2$ vertices in each direction around the circle. If $m$ is odd and $n$ is even, form $H_{m,n}$ by making each vertex adjacent to the nearest $(m - 1)/2$ vertices in each direction and to the diametrically opposite vertex. In each case, $H_{m,n}$ is $m$-regular. When $m$ and $n$ are both odd, index the vertices by the integers modulo $n$. Construct $H_{m,n}$ from $H_{m-1,n}$ by adding the edges $i \leftrightarrow i + (n - 1)/2$ for $0 \leq i \leq (n - 1)/2$.

Proposition 3.9. If $G$ is a cycle of order at least 3 or the Harary graph $H_{2m,\ell m+1}$, where $\ell \geq 3$ and $m \geq 1$, then

$$\gamma_{x_2,t}(M(G)) = \gamma_{x_2,t}(G) + 1.$$

Proof. Let $G = C_n$ be a cycle of order at least 3 with the vertex set $V(C_n) = V^0 = \{v_j \mid 1 \leq j \leq n\}$ and the edge set $E(C_n) = \{(v_j, v_{j+1}) \mid 1 \leq j \leq n\}$. Let also $V(M(G)) = V^0 \cup V^1 \cup \{u\}$. Proposition 1.1 implies $\gamma_{x_2,t}(G) = n$. Since $S = (V^0 - \{v_1, v_n\}) \cup \{v_{n-1}^1, v_n^1, u\}$ is a DTDS of $M(G)$ of cardinality $\gamma_{x_2,t}(C_n) + 1 = n + 1$, Theorem 2.2 implies $\gamma_{x_2,t}(M(C_n)) = \gamma_{x_2,t}(C_n) + 1 = n + 1$.

Now let $G = H_{2m,\ell m+1}$ and let $V(M(G)) = V^0 \cup V^1 \cup \{u\}$, where $V^i = \{v_j^i \mid 1 \leq j \leq \ell m + 1\}$ for $i = 0, 1$, $V^0 = V(G)$ and $\ell \geq 3$. Since $\{v_j^{0,m+1} \mid 0 \leq j \leq \lfloor(\ell m + 1)/m\rfloor - 1 = \ell\}$ is a DTDS of $G$, we have $\gamma_{x_2,t}(G) = \lfloor(\ell m + 1)/m\rfloor - 1 = \ell$ by Proposition 1.4. Since also $S = \{v_i^{0,m+1} \mid 1 \leq i \leq \ell - 1\} \cup \{v_1^1, v_{\ell m+1}^1, u\}$ is a DTDS of $M(G)$ of cardinality $\ell + 2 = \gamma_{x_2,t}(G) + 1$, Theorem 2.2 implies $\gamma_{x_2,t}(M(G)) = \gamma_{x_2,t}(G) + 1$. □

In the end of paper, the author states the following problem.
**Problem:** For integers $k, m \geq 1$, characterize graphs $G$ with $\delta(G) \geq k$ satisfy $\gamma_{\times k,t}(\mu_m(G)) = \gamma_{\times k,t}(G) + 1$ or

$$\gamma_{\times k,t}(\mu_m(G)) = \begin{cases} 
(1 + 2\lceil (m - 1)/4 \rceil)\gamma_{\times k,t}(G) & \text{if } m \equiv 0 \pmod{4}, \\
(1 + 2\lceil (m - 1)/4 \rceil)\gamma_{\times k,t}(G) + k & \text{if } m \equiv 1 \pmod{4}, \\
2\lceil (m - 1)/4 \rceil\gamma_{\times k,t}(G) & \text{if } m \equiv 2 \pmod{4}, \\
2\lceil (m - 1)/4 \rceil\gamma_{\times k,t}(G) + k & \text{if } m \equiv 3 \pmod{4}.
\end{cases}$$

**REFERENCES**


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