Transactions on Combinatorics
ISSN (print): 2251-8657, ISSN (on-line): 2251-8665
Vol. 2 No. 4 (2013), pp. 13-24.

www.combinatorics.ir

www.ui.ac.ir

RECIPROCAL DEGREE DISTANCE OF SOME GRAPH OPERATIONS

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Communicated by Ali Reza Ashrafi

Abstract. The reciprocal degree distance (RDD), defined for a connected graph G as vertex-degree-weighted sum of the reciprocal distances, that is, $RDD(G) = \sum\limits_{u,v \in V(G)} \frac{d_G(u)+d_G(v)}{d_G(u,v)}$. The reciprocal degree distance is a weight version of the Harary index, just as the degree distance is a weight version of the Wiener index. In this paper, we present exact formulae for the reciprocal degree distance of join, tensor product, strong product and wreath product of graphs in terms of other graph invariants including the degree distance, Harary index, the first Zagreb index and first Zagreb coindex. Finally, we apply some of our results to compute the reciprocal degree distance of fan graph, wheel graph, open fence and closed fence graphs.

1. Introduction

All the graphs considered in this paper are simple and connected. For vertices $u, v \in V(G)$, the distance between u and v in G, denoted by $d_G(u, v)$, is the length of a shortest (u, v)-path in G and let $d_G(v)$ be the degree of a vertex $v \in V(G)$. For two simple graphs G and H their *tensor product,* denoted by $G \times H$, has vertex set $V(G) \times V(H)$ in which (g_1, h_1) and (g_2, h_2) are adjacent whenever g_1g_2 is an edge in G and h_1h_2 is an edge in H. Note that if G and H are connected graphs, then $G \times H$ is connected only if at least one of the graph is nonbipartite. The *strong product* of graphs G and H, denoted by $G \boxtimes H$, is the graph with vertex set $V(G) \times V(H) = \{(u, v) : u \in V(G), v \in V(H)\}$ and $(u, x)(v, y)$ is an edge whenever (i) $u = v$ and $xy \in E(H)$, or (ii) $uv \in E(G)$ and $x = y$, or (iii) $uv \in E(G)$ and $xy \in E(H)$. Similarly, the *wreath product* of the graphs G and H, denoted by $G \circ H$, has vertex set $V(G) \times V(H)$ in which $(g_1, h_1)(g_2, h_2)$ is an edge whenever g_1g_2 is an edge in G or, $g_1 = g_2$ and h_1h_2 is an edge in H, see Fig.1. The tensor product of graphs has been extensively studied in relation

MSC(2010): Primary: 05C12; Secondary: 05C76.

Keywords: Reciprocal degree distance, Harary index, Graph operations.

Received: 20 August 2013 , Accepted: 15 September 2013.

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to the areas such as graph colorings, graph recognition, decompositions of graphs, design theory, see [1, 2, 3, 12, 16].

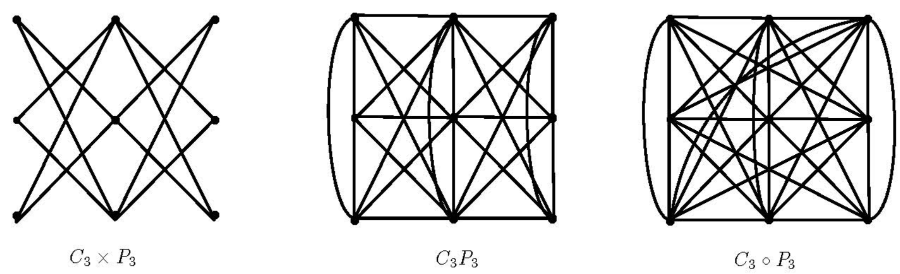

Fig.1. Tensor, strong and wreath product of C_3 and P_3

A *sum* $G+H$ of two graphs G and H with disjoint vertex sets $V(G)$ and $V(H)$ is the graph on the vertex set $V(G) \cup V(H)$ and the edge set $E(G) \cup E(H) \cup \{uv \mid u \in V(G),\ v \in V(H)\}$. Hence, the sum of two graphs is obtained by connecting each vertex of one graph to each vertex of the other graph, while keeping all edges of both graphs. The sum of two graphs is sometimes also called a *join,* and is denoted by $G\nabla H$.

A *topological index* of a graph is a real number related to the graph; it does not depend on labeling or pictorial representation of a graph. In theoretical chemistry, molecular structure descriptors (also called topological indices) are used for modeling physicochemical, pharmacologic, toxicologic, biological and other properties of chemical compounds [9]. There exist several types of such indices, especially those based on vertex and edge distances. One of the most intensively studied topological indices is the Wiener index; for other related topological indices see [22].

Let G be a connected graph. Then $Wiener\ index$ of G is defined as $W(G) = \frac{1}{2} \sum\limits_{u,\,v\,\in V(G)} d_G(u,v)$ with the summation going over all pairs of distinct vertices of G. Similarly, the $Harary\ index$ of G is defined as $H(G) = \frac{1}{2} \sum\limits_{u,\,v\,\in V(G)} \frac{1}{d_G(u,v)}$.

Dobrynin and Kochetova [5] and Gutman [8] independently proposed a vertex-degree-weighted version of Wiener index called degree distance or Schultz molecular topological index, which is defined for a connected graph G as $DD(G) = \frac{1}{2} \sum\limits_{u,v \in V(G)} (d_G(u) + d_G(v)) d_G(u,v)$, where $d_G(u)$ is the degree of the vertex u in G. Note that the degree distance is a degree-weight version of the Wiener index. Hua and Zhang [11] introduced a new graph invariant named reciprocal degree distance, which can be seen as a degree-weight version of Harary index, that is, $RDD(G) = \frac{1}{2} \sum\limits_{u,v \in V(G)} \frac{d_G(u)+d_G(v)}{d_G(u,v)}$.

The Harary index of a graph G was introduced independently by Plavsic et al. [20] and by Ivanciuc et al. [13] in 1993. Its applications and mathematical properties are well studied in [4, 7, 23, 15]. Zhou et al. [24] have obtained the lower and upper bounds of the Harary index of a connected graph. Very recently, Xu et al. [21] have obtained lower and upper bounds for the Harary index of a connected graph in relation to $\chi(G)$, chromatic number of G and $\omega(G)$, clique number of G. and characterized the extremal graphs that attain the lower and upper bounds of Harary index. Also, Feng et. al. [7]

have given a sharp upper bound for the Harary indices of graphs based on the matching number, that is, the size of a maximum matching. Various topological indices on tensor product, Cartesian product and strong product have been studied various authors, see [17, 18, 19, 14, 10]. Hua and Zhang [11] have obtained lower and upper bounds for the reciprocal degree distance of graph in terms of other graph invariants including the degree distance, Harary index, the first Zagreb index, the first Zagreb coindex, pendent vertices, independence number, chromatic number and vertex-, and edge-connectivity.

The *first Zagreb index* and *first Zagerb coindex* are defined as $M_1(G) = \sum\limits_{u\in V(G)} d_G(u)^2$ and $\overline{M}_1(G) = \sum\limits_{uv\notin E(G)} (d_G(u) + d_G(v))$. In fact, one can rewrite the first Zagreb index as $M_1(G) = \sum\limits_{uv\in E(G)} (d_G(u) + d_G(v))$. The Zagreb indices are found to have appilications in QSPR and QSAR studies as well, see [6].

Denoted by P_n, C_n and K_n the path, cycle and complete graphs on n vertices, respectively. We call C_3 a triangle. In this paper, we present exact formulae for the reciprocal degree distance of join, tensor product, strong product and wreath product of graphs in terms of other graph invariants including the degree distance, Harary index, the first Zagreb index and first Zagreb coindex. Finally, we apply some of our results to compute the reciprocal degree distance of fan graph, wheel graph, open fence and closed fence graphs.

2. Reciprocal degree distance of $G + H$

In this section, we compute the reciprocal degree distance of join of two connected graphs.

Theorem 2.1. *Let G and H be graphs with n and m vertices, respectively. Then* $RDD(G + H) = M_1(G) + M_1(H) + \frac{1}{2}(\overline{M}_1(G) + \overline{M}_1(H)) + 3(n\,|E(H)| + m\,|E(G)|) + \frac{nm}{2}(3n + 3m - 2)$.

Proof. Set $V(G) = \{u_1, u_2, \ldots, u_n\}$ and $V(H) = \{v_1, v_2, \ldots, v_m\}$. By definition of the join of two graphs, one can see that,

$$d_{G+H}(x) = \begin{cases} d_G(x) + |V(H)|\,, & \text{if } x \in V(G) \\ d_H(x) + |V(G)|\,, & \text{if } x \in V(H) \end{cases}$$

and
$$d_{G+H}(u,v) = \begin{cases} 0, & \text{if } u = v \\ 1, & \text{if } uv \in E(G) \text{ or } uv \in E(H) \text{ or } (u \in V(G) \text{ and } v \in V(H)) \\ 2, & \text{otherwise.} \end{cases}$$

Therefore,

$$\begin{aligned}
RDD(G+H) &= \frac{1}{2}\sum_{u,v\in V(G+H)} \frac{d_{G+H}(u)+d_{G+H}(v)}{d_{G+H}(u,v)} \\
&= \frac{1}{2}\Big(\sum_{uv\in E(G)} \big(d_G(u)+m+d_G(v)+m\big) + \frac{1}{2}\sum_{uv\notin E(G)} \big(d_G(u)+m+d_G(v)+m\big) \\
&\quad + \sum_{uv\in E(H)} \big(d_H(u)+n+d_H(v)+n\big) + \frac{1}{2}\sum_{uv\in E(H)} \big(d_H(u)+n+d_H(v)+n\big) \\
&\quad + \sum_{u\in V(G),\ v\in V(H)} \big(d_G(u)+m+d_H(v)+n\big)\Big) \\
&= M_1(G)+M_1(H)+\frac{1}{2}(\overline{M}_1(G)+\overline{M}_1(H))+3(n\,|E(H)|+m\,|E(G)|) \\
&\quad +\frac{nm}{2}(3n+3m-2).
\end{aligned}$$

□

Using Theorem 2.1, we have the following corollary.

Corollary 2.2. *Let G be graph on n vertices. Then* $RDD(G+K_m) = M_1(G)+\frac{1}{2}\overline{M}_1(G)+3m\,|E(G)|+\frac{m}{2}\Big(2(m-1)^2+3n(m-1)+n(3n+3m-2)\Big)$.

One can observe that $M_1(C_n)=4n,\ n\geq 3,\ M_1(P_1)=0,\ M_1(P_n)=4n-6,\ n>1$ and $M_1(K_n)=n(n-1)^2$. Similarly, $\overline{M_1}(K_n)=0$, $\overline{M_1}(P_n)=2(n-2)^2$ and $\overline{M_1}(C_n)=2n(n-3)$.

Using $M_1(C_n)$,$M_1(P_n)$,$\overline{M_1}(P_n)$ and $\overline{M_1}(C_n)$ and Corollary 2.2, we compute the formulae for reciprocal degree distance of fan and wheel graphs, P_n+K_1 and C_n+K_1, see Figs. 2a and 2b.

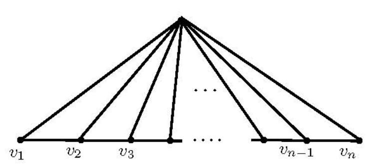

Fig. 2a. Fan graph P_n+K_1

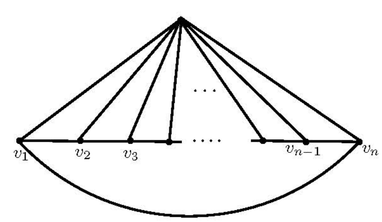

Fig.2b. Wheel graph C_n+K_1

Example 1.
(i) $RDD(P_n+K_1)=\frac{1}{2}(5n^2+7n-10)$.
(ii) $RDD(C_n+K_1)=\frac{1}{2}(5n^2+9n)$.

3. Reciprocal degree distance of tensor product of graphs

In this section, we compute the reciprocal degree distance of $G\times K_r$.

The proof of the following lemma follows easily from the properties and structure of $G\times K_r$. The lemma is used in the proof of the main theorem of this section.

Lemma 3.1. *Let G be a connected graph on $n \geq 2$ vertices. For any pair of vertices $x_{ij}, x_{kp} \in V(G \times K_r)$, $r \geq 3$*
(i) If $v_iv_k \in E(G)$, then

$$d_{G\times K_r}(x_{ij}, x_{kp}) = \begin{cases} 1, & \text{if } j \neq p, \\ 2, & \text{if } j = p \text{ and } v_iv_k \text{ is on a triangle of } G, \\ 3, & \text{if } j = p \text{ and } v_iv_k \text{ is not on a triangle of } G. \end{cases}$$

(ii) If $v_iv_k \notin E(G)$, then $d_{G\times K_r}(x_{ij}, x_{kp}) = d_G(v_i, v_k)$.
(iii) $d_{G\times K_r}(x_{ij}, x_{ip}) = 2$.

Theorem 3.2. *Let G be a connected graph with $n \geq 2$ vertices and m edges. Then $RDD(G \times K_r) = r(r-1)\left(rRDD(G) - \frac{M_1(G)}{2} - \frac{1}{6}\sum\limits_{u_iu_k\in E_2}(d(u_i) + d(u_k)) + (r-1)m\right)$, where $r \geq 3$.*

Proof. Set $V(G) = \{u_1, u_2, \ldots, u_n\}$ and $V(K_r) = \{v_1, v_2, \ldots, v_r\}$. Let x_{ij} denote the vertex (u_i, v_j) of $G \times K_r$. The degree of the vertex x_{ij} in $G \times K_r$ is $d_G(u_i)d_{K_r}(v_j)$, that is, $d_{G\times K_r}(x_{ij}) = (r-1)d_G(u_i)$. By the definition of reciprocal degree distance

$$\begin{aligned} RDD(G \times K_r) &= \frac{1}{2}\sum_{x_{ij},\, x_{kp}\in V(G\times K_r)} \frac{d_{G\times K_r}(x_{ij}) + d_{G\times K_r}(x_{kp})}{d_{G\times K_r}(x_{ij}, x_{kp})} \\ &= \frac{1}{2}\Bigg(\sum_{i=0}^{n-1}\sum_{\substack{j,p=0\\ j\neq p}}^{r-1} \frac{d_{G\times K_r}(x_{ij}) + d_{G\times K_r}(x_{ip})}{d_{G\times K_r}(x_{ij}, x_{ip})} + \sum_{\substack{i,k=0\\ i\neq k}}^{n-1}\sum_{j=0}^{r-1} \frac{d_{G\times K_r}(x_{ij}) + d_{G\times K_r}(x_{kj})}{d_{G\times K_r}(x_{ij}, x_{kj})} \\ &\quad + \sum_{\substack{i,k=0\\ i\neq k}}^{n-1}\sum_{\substack{j,p=0\\ j\neq p}}^{r-1} \frac{d_{G\times K_r}(x_{ij}) + d_{G\times K_r}(x_{kp})}{d_{G\times K_r}(x_{ij}, x_{kp})}\Bigg) \\ &= \frac{1}{2}\{A_1 + A_2 + A_3\}, \qquad (3.1) \end{aligned}$$

where A_1 to A_3 are the sums of the above terms, in order.

We shall calculate A_1 to A_3 of (3.1) separately.

$(\mathbf{A_1})$ First we compute $\sum\limits_{i=0}^{n-1}\sum\limits_{\substack{j,p=0\\ j\neq p}}^{r-1} \frac{d_{G\times K_r}(x_{ij})+d_{G\times K_r}(x_{ip})}{d_{G\times K_r}(x_{ij},x_{ip})}$.

$$\begin{aligned} \sum_{i=0}^{n-1}\sum_{\substack{j,p=0\\ j\neq p}}^{r-1} \frac{d_{G\times K_r}(x_{ij}) + d_{G\times K_r}(x_{ip})}{d_{G\times K_r}(x_{ij}, x_{ip})} &= \sum_{i=0}^{n-1}\sum_{\substack{j,p=0\\ j\neq p}}^{r-1} \frac{d_G(u_i)(r-1) + d_G(u_i)(r-1)}{2}, \text{ by Lemma 3.1} \\ &= 2r(r-1)^2m. \qquad (3.2) \end{aligned}$$

$(\mathbf{A_2})$ Next we compute $\sum\limits_{j=0}^{r-1}\sum\limits_{\substack{i,k=0\\ i\neq k}}^{n-1} \frac{d_{G\times K_r}(x_{ij})+d_{G\times K_r}(x_{kj})}{d_{G\times K_r}(x_{ij},x_{kj})}$.

Let $E_1 = \{uv \in E(G) \mid uv \text{ is on a } C_3 \text{ in } G\}$ and $E_2 = E(G) - E_1$.

$$
\begin{aligned}
&\sum_{\substack{i,k=0\\ i\neq k}}^{n-1} \frac{d_{G\times K_r}(x_{ij})+d_{G\times K_r}(x_{kj})}{d_{G\times K_r}(x_{ij},x_{kj})}\\
&= \Big(\sum_{\substack{i,k=0\\ i\neq k\\ u_iu_k\notin E(G)}}^{n-1} + \sum_{\substack{i,k=0\\ i\neq k\\ u_iu_k\in E_1}}^{n-1} + \sum_{\substack{i,k=0\\ i\neq k\\ u_iu_k\in E_2}}^{n-1}\Big)\Big(\frac{d_{G\times K_r}(x_{ij})+d_{G\times K_r}(x_{kj})}{d_{G\times K_r}(x_{ij},x_{kj})}\Big)\\
&= \Bigg(\sum_{\substack{i,k=0\\ i\neq k\\ u_iu_k\notin E(G)}}^{n-1} \frac{d_G(u_i)(r-1)+d_G(u_k)(r-1)}{d_G(u_i,u_k)} + \sum_{\substack{i,k=0\\ i\neq k\\ u_iu_k\in E_1}}^{n-1} \frac{d_G(u_i)(r-1)+d_G(u_k)(r-1)}{2}\\
&\quad + \sum_{\substack{i,k=0\\ i\neq k\\ u_iu_k\in E_2}}^{n-1} \frac{d_G(u_i)(r-1)+d_G(u_k)(r-1)}{3}\Bigg), \text{ by Lemma 3.1}\\
&= (r-1)\Bigg\{\Bigg(\sum_{\substack{i,k=0\\ i\neq k\\ u_iu_k\notin E(G)}}^{n-1} \frac{d_G(u_i)+d_G(u_k)}{d_G(u_i,u_k)} + \sum_{\substack{i,k=0\\ i\neq k\\ u_iu_k\in E_1}}^{n-1} \frac{d_G(u_i)+d_G(u_k)}{d_G(u_i,u_k)} + \sum_{\substack{i,k=0\\ i\neq k\\ u_iu_k\in E_2}}^{n-1} \frac{d_G(u_i)+d_G(u_k)}{d_G(u_i,u_k)}\Bigg)\\
&\quad - \sum_{\substack{i,k=0\\ i\neq k\\ u_iu_k\in E_1}}^{n-1} \frac{d_G(u_i)+d_G(u_k)}{2} - 2\sum_{\substack{i,k=0\\ i\neq k\\ u_iu_k\in E_2}}^{n-1} \frac{d_G(u_i)+d_G(u_k)}{3}\Bigg\}\\
&= (r-1)\Bigg\{2RDD(G) - \sum_{\substack{i,k=0\\ i\neq k\\ u_iu_k\in E(G)}}^{n-1} \frac{d_G(u_i)+d_G(u_k)}{2} - \sum_{\substack{i,k=0\\ i\neq k\\ u_iu_k\in E_2}}^{n-1} \frac{d_G(u_i)+d_G(u_k)}{6}\Bigg\}
\end{aligned}
$$

$$
(3.3)\quad = (r-1)\Bigg\{2RDD(G) - M_1(G) - \sum_{\substack{i,k=0\\ i\neq k\\ u_iu_k\in E_2}}^{n-1} \frac{d_G(u_i)+d_G(u_k)}{3}\Bigg\},
$$

Now summing (3.3) over $j = 0, 1, \ldots, r-1$, we get,

$$
(3.4)\quad \sum_{j=0}^{r-1}\left(\sum_{\substack{i,k=0\\ i\neq k}}^{n-1} \frac{d_{G\times K_r}(x_{ij})+d_{G\times K_r}(x_{kj})}{d_{G\times K_r}(x_{ij},x_{kj})}\right) = r(r-1)\Bigg\{2RDD(G) - M_1(G) - \sum_{\substack{i,k=0\\ i\neq k\\ u_iu_k\in E_2}}^{n-1} \frac{d_G(u_i)+d_G(u_k)}{3}\Bigg\}.
$$

($\mathbf{A_3}$) Next we compute $\sum_{\substack{i,k=0\\ i\neq k}}^{n-1}\left(\sum_{\substack{j,p=0\\ j\neq p}}^{r-1} \frac{d_{G\times K_r}(x_{ij})+d_{G\times K_r}(x_{kp})}{d_{G\times K_r}(x_{ij},x_{kp})}\right)$.

$$\sum_{\substack{i,k=0\\ i\neq k}}^{n-1}\sum_{\substack{j,p=0,\\ j\neq p}}^{r-1}\frac{d_{G\times K_r}(x_{ij})+d_{G\times K_r}(x_{kp}))}{d_{G\times K_r}(x_{ij},x_{kp}} = \sum_{\substack{i,k=0\\ i\neq k}}^{n-1}\sum_{\substack{j,p=0,\\ j\neq p}}^{r-1}\frac{d_G(u_i)(r-1)+d_G(u_k)(r-1)}{d_G(u_i,u_k)}, \text{ by Lemma 3.1}$$

$$= r(r-1)^2\sum_{\substack{i,k=0\\ i\neq k}}^{n-1}\frac{d_G(u_i)+d_G(u_k)}{d_G(u_i,u_k)}$$

$$(3.5)\qquad = 2r(r-1)^2RDD(G).$$

Using (3.1) and the sums $\mathbf{A_1}$,$\mathbf{A_2}$ and $\mathbf{A_3}$ in (3.2),(3.4) and (3.5), respectively, we have,

$$RDD(G\times K_r) = r(r-1)\Bigg(rRDD(G)-\frac{1}{2}M_1(G)-\frac{1}{6}\sum_{u_iu_k\in E_2}(d_G(u_i)+d_G(u_k))+(r-1)m\Bigg).$$

□

Using Theorem 3.2, we have the following corollaries.

Corollary 3.3. *Let G be a connected graph on $n\geq 2$ vertices with m edges. If each edge of G is on a C_3, then $RDD(G\times K_r)=r(r-1)\Big(rRDD(G)-\frac{1}{2}M_1(G)+(r-1)m\Big)$, where $r\geq 3$.*

Corollary 3.4. *If G is a connected triangle free graph on $n\geq 2$ vertices and m edges, then $RDD(G\times K_r)=r(r-1)\Big(rRDD(G)-\frac{2}{3}M_1(G)+(r-1)m\Big)$.*

4. Reciprocal degree distance of strong product of graphs

In this section, we obtain the reciprocal degree distance of $G\boxtimes K_r$.

Theorem 4.1. *Let G be a connected graph with n vertices and m edges. Then $RDD(G\boxtimes K_r)=r\Big(r^2RDD(G)+2r(r-1)H(G)+2r(r-1)m+n(r-1)^2\Big)$.*

Proof. Set $V(G)=\{u_1,u_2,\ldots,u_n\}$ and $V(K_r)=\{v_1,v_2,\ldots,v_r\}$. Let x_{ij} denote the vertex (u_i,v_j) of $G\boxtimes K_r$. The degree of the vertex x_{ij} in $G\boxtimes K_r$ is $d_G(u_i)+d_{K_r}(v_j)+d_G(u_i)d_{K_r}(v_j)$, that is $d_{G\boxtimes K_r}(x_{ij})=rd_G(u_i)+(r-1)$. One can see that for any pair of vertices x_{ij}, $x_{kp}\in V(G\boxtimes K_r)$, $d_{G\boxtimes K_r}(x_{ij},x_{ip})=1$ and $d_{G\boxtimes K_r}(x_{ij},x_{kp})=d_G(u_i,u_k)$.

By the definition of reciprocal degree distance

$$\begin{aligned} RDD(G \boxtimes K_r) &= \frac{1}{2} \sum_{x_{ij},\, x_{kp} \in V(G\boxtimes K_r)} \frac{d_{G\boxtimes K_r}(x_{ij}) + d_{G\boxtimes K_r}(x_{kp})}{d_{G\boxtimes K_r}(x_{ij}, x_{kp})} \\ &= \frac{1}{2}\Bigg(\sum_{i=0}^{n-1} \sum_{\substack{j,\,p=0 \\ j\neq p}}^{r-1} \frac{d_{G\boxtimes K_r}(x_{ij}) + d_{G\boxtimes K_r}(x_{ip})}{d_{G\boxtimes K_r}(x_{ij}, x_{ip})} + \sum_{\substack{i,\,k=0 \\ i\neq k}}^{n-1} \sum_{j=0}^{r-1} \frac{d_{G\boxtimes K_r}(x_{ij}) + d_{G\boxtimes K_r}(x_{kj})}{d_{G\boxtimes K_r}(x_{ij}, x_{kj})} \\ &\quad + \sum_{\substack{i,\,k=0 \\ i\neq k}}^{n-1} \sum_{\substack{j,\,p=0 \\ j\neq p}}^{r-1} \frac{d_{G\boxtimes K_r}(x_{ij}) + d_{G\boxtimes K_r}(x_{kp})}{d_{G\boxtimes K_r}(x_{ij}, x_{kp})} \Bigg) \\ &= \frac{1}{2}\{A_1 + A_2 + A_3\}, \qquad (4.1) \end{aligned}$$

where A_1, A_2 and A_3 are the sums of the terms of the above expression, in order.
We shall obtain A_1 to A_3 of (4.1), separately.

$$\begin{aligned} A_1 &= \sum_{i=0}^{n-1} \sum_{\substack{j,\,p=0 \\ j\neq p}}^{r-1} \frac{d_{G\boxtimes K_r}(x_{ij}) + d_{G\boxtimes K_r}(x_{ip})}{d_{G\boxtimes K_r}(x_{ij}, x_{ip})} \\ &= \sum_{i=0}^{n-1} \sum_{\substack{j,\,p=0 \\ j\neq p}}^{r-1} \Big(2d_G(u_i) + 2(r-1) + 2(r-1)d_G(u_i) \Big) \\ &= 4r^2(r-1)m + 2nr(r-1)^2 \qquad (4.2) \end{aligned}$$

$$\begin{aligned} A_2 &= \sum_{j=0}^{r-1} \sum_{\substack{i,\,k=0 \\ i\neq k}}^{n-1} \frac{d_{G\boxtimes K_r}(x_{ij}) + d_{G\boxtimes K_r}(x_{kj})}{d_{G\boxtimes K_r}(x_{ij}, x_{kj})} \\ &= \sum_{j=0}^{r-1} \sum_{\substack{i,\,k=0 \\ i\neq k}}^{n-1} \frac{d_G(u_i) + (r-1)d_G(u_i) + d_G(u_k) + (r-1)d_G(u_k) + 2(r-1)}{d_G(u_i, u_k)} \\ &= r \sum_{j=0}^{r-1} \sum_{\substack{i,\,k=0 \\ i\neq k}}^{n-1} \frac{d_G(u_i) + d_G(u_k)}{d_G(u_i, u_k)} + \sum_{j=0}^{r-1} \sum_{\substack{i,\,k=0 \\ i\neq k}}^{n-1} \frac{2(r-1)}{d_G(u_i, u_k)} \\ &= 2r^2 RDD(G) + 4r(r-1)H(G). \qquad (4.3) \end{aligned}$$

$$\begin{aligned} A_3 &= \sum_{\substack{i,\,k=0 \\ i\neq k}}^{n-1} \sum_{\substack{j,\,p=0, \\ j\neq p}}^{r-1} \frac{d_{G\boxtimes K_r}(x_{ij}) + d_{G\boxtimes K_r}(x_{kp})}{d_{G\boxtimes K_r}(x_{ij}, x_{kp})} \\ &= r^2(r-1) \sum_{\substack{i,\,k=0 \\ i\neq k}}^{n-1} \frac{d_G(u_i) + d_G(u_k)}{d_G(u_i, u_k)} + 2r(r-1)^2 \sum_{\substack{i,\,k=0 \\ i\neq k}}^{n-1} \frac{1}{d_G(u_i, u_k)} \\ &= 2r^2(r-1)RDD(G) + 4r(r-1)^2 H(G). \qquad (4.4) \end{aligned}$$

Using (4.2), (4.3) and (4.4) in (4.1), we have

$$RDD(G \boxtimes K_r) = r\Big(r^2 RDD(G) + 2r(r-1)H(G) + 2r(r-1)m + n(r-1)^2\Big).$$

□

By direct calculations we obtain expressions for the values of the Harary indices of P_n and C_n.

$$H(P_n) = n\Big(\sum_{i=1}^{n} \frac{1}{i}\Big) - n \text{ and } H(C_n) = \begin{cases} n\Big(\sum_{i=1}^{\frac{n}{2}} \frac{1}{i}\Big) - 1 & \text{n is even} \\ n\Big(\sum_{i=1}^{\frac{n-1}{2}} \frac{1}{i}\Big) & \text{n is odd.} \end{cases}$$

The following are the reciprocal degree distance for complete graph, path and cycle on n vertices by direct calculations:

$RDD(K_n) = n(n-1)^2$, $RDD(P_n) = H(P_n) + 4\Big(\sum_{i=1}^{n-1} \frac{1}{i}\Big) - \frac{3}{n-1}$ and $RDD(C_n) = 4H(C_n)$.

As an application we present formulae for reciprocal degree distance of open and closed fences, $P_n \boxtimes K_2$ and $C_n \boxtimes K_2$, see Fig. 3.

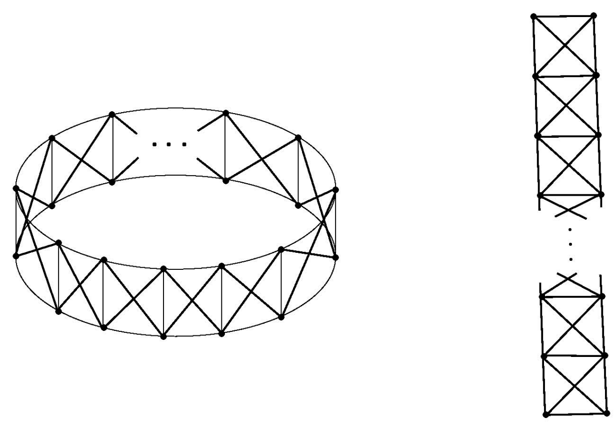

Fig. 3. Closed and open Fence graphs

By using Theorem 4.1, $RDD(C_n)$ and $H(C_n)$, we obtain the exact reciprocal degree distance of the following graphs.

Example 2.

(*i*) $RDD(P_n \boxtimes K_2) = 16\Big(\sum_{i=1}^{n} \frac{1}{i}\Big) + 32\Big(\sum_{i=1}^{n-1} \frac{1}{i}\Big) - 6n - \frac{24}{n-1} - 8.$

(*ii*) $$RDD(C_n \boxtimes K_2) = \begin{cases} 10n\Big(1 + 4\sum_{i=1}^{\frac{n}{2}} \frac{1}{i}\Big) - 40 & \text{n is even} \\ 10n\Big(1 + 4\sum_{i=1}^{\frac{n-1}{2}} \frac{1}{i}\Big) & \text{n is odd.} \end{cases}$$

5. Reciprocal degree distance of $G \circ G'$

In this section, we obtain the reciprocal degree distance of the wreath product of graphs.

Theorem 5.1. *Let G and G' be two connected graphs with n_1 and n_2 vertices, respectively. Then* $RDD(G \circ G') = n_2^3 RDD(G) + 2H(G)\Big(2\,|E(G')| + M_1(G') + \overline{M}_1(G')\Big) + n_2\,|E(G)|\,\Big(n_2^2 + 2\,|E(G')| - n_2\Big) + \frac{n_1}{2}\Big(2M_1(G') + \overline{M}_1(G')\Big)$.

Proof. Let $V(G) = \{u_1, u_2, \ldots, u_{n_1}\}$ and let $V(G') = \{v_1, v_2, \ldots, v_{n_2}\}$. Let x_{ij} denote the vertex (u_i, v_j) of $G \circ G'$. The degree of the vertex x_{ij} in $G \circ G'$ is $n_2 d_G(u_i) + d_{G'}(v_j)$. By the definition of reciprocal degree distance

$$
\begin{aligned}
RDD(G \circ G') &= \frac{1}{2} \sum_{x_{ij}, x_{k\ell} \in V(G\circ G')} \frac{d_{G\circ G'}(x_{ij}) + d_{G\circ G'}(x_{k\ell})}{d_{G\circ G'}(x_{ij}, x_{k\ell})} \\
&= \frac{1}{2}\Bigg(\sum_{i=0}^{n_1-1} \sum_{\substack{j,\,\ell=0 \\ j\neq \ell}}^{n_2-1} \frac{d_{G\circ G'}(x_{ij}) + d_{G\circ G'}(x_{i\ell})}{d_{G\circ G'}(x_{ij}, x_{i\ell})} + \sum_{\substack{i,\,k=0 \\ i\neq k}}^{n_1-1} \sum_{j=0}^{n_2-1} \frac{d_{G\circ G'}(x_{ij}) + d_{G\circ G'}(x_{kj})}{d_{G\circ G'}(x_{ij}, x_{kj})} \\
&\quad + \sum_{\substack{i,\,k=0 \\ i\neq k}}^{n_1-1} \sum_{\substack{j,\,\ell=0 \\ j\neq \ell}}^{n_2-1} \frac{d_{G\circ G'}(x_{ij}) + d_{G\circ G'}(x_{k\ell})}{d_{G\circ G'}(x_{ij}, x_{k\ell})} \Bigg) \\
&= \frac{1}{2}\{A_1 + A_2 + A_3\}, \qquad (5.1)
\end{aligned}
$$

where A_1 to A_3 are the sums of the above terms, in order

We shall calculate the terms A_1 to A_3 of above expression separately.

$$
\begin{aligned}
A_1 &= \sum_{i=0}^{n_1-1} \sum_{\substack{j,\,\ell=0 \\ j\neq \ell}}^{n_2-1} \frac{d_{G\circ G'}(x_{ij}) + d_{G\circ G'}(x_{i\ell})}{d_{G\circ G'}(x_{ij}, x_{i\ell})} \\
&= \sum_{i=0}^{n_1-1} \sum_{\substack{j,\,\ell=0 \\ j\neq \ell}}^{n_2-1} \frac{2n_2 d_G(u_i) + d_{G'}(v_j) + d_{G'}(v_\ell)}{d_{G'}(v_j, v_\ell)} \\
&= \sum_{i=0}^{n_1-1} \sum_{\substack{j,\,\ell=0 \\ j\neq \ell}}^{n_2-1} \frac{2n_2 d_G(u_i)}{d_{G'}(v_j, v_\ell)} + \sum_{i=0}^{n_1-1} \sum_{\substack{j,\,\ell=0 \\ j\neq \ell}}^{n_2-1} \frac{d_{G'}(v_j) + d_{G'}(v_\ell)}{d_{G'}(v_j, v_\ell)} \\
&= 2n_2 \sum_{i=0}^{n_1-1} d_G(u_i) \Bigg(\sum_{v_j v_\ell \in E(G')} \frac{1}{d_{G'}(v_j, v_\ell)} + \sum_{v_j v_\ell \notin E(G')} \frac{1}{d_{G'}(v_j, v_\ell)} \Bigg) \\
&\quad + \sum_{i=0}^{n_1-1} \Bigg(\sum_{v_j v_\ell \in E(G')} \frac{d_{G'}(v_j) + d_{G'}(v_\ell)}{d_{G'}(v_j, v_\ell)} + \sum_{v_j v_\ell \notin E(G')} \frac{d_{G'}(v_j) + d_{G'}(v_\ell)}{d_{G'}(v_j, v_\ell)} \Bigg) \\
&= 4n_2\,|E(G)| \Bigg(\sum_{v_j \in V(G')} d_{G'}(v_j) + \sum_{v_j \in V(G')} \frac{1}{2}(|E(G')| - d_{G'}(v_j) - 1) \Bigg) \\
&\quad + \sum_{i=0}^{n_1-1} \Bigg(\sum_{v_j v_\ell \in E(G')} (d_{G'}(v_j) + d_{G'}(v_\ell)) + \sum_{v_j v_\ell \notin E(G')} \frac{d_{G'}(v_j) + d_{G'}(v_\ell)}{2} \Bigg),
\end{aligned}
$$

since each row

induces a copy of G' and $d_{G\circ G'}(x_{ij}, x_{i\ell}) = \begin{cases} 1, & \text{if } v_j v_\ell \in E(G') \\ 2, & \text{if } v_j v_\ell \notin E(G'). \end{cases}$

$$
= 2n_2\,|E(G)|\,(n_2^2 + 2\,|E(G')| - n_2) + n_1(2M_1(G') + \overline{M}_1(G')). \qquad (5.2)
$$

$$
\begin{aligned}
A_2 &= \sum_{\substack{i,\,k=0\\ i\neq k}}^{n_1-1}\sum_{j=0}^{n_2-1}\frac{d(x_{ij})+d(x_{kj})}{d_{G\circ G'}(x_{ij},x_{kj})}\\
&= \sum_{\substack{i,\,k=0\\ i\neq k}}^{n_1-1}\sum_{j=0}^{n_2-1}\frac{n_2(d(u_i)+d(u_k))+2d(v_j)}{d_G(u_i,u_k)},\\
&\quad \text{since the distance between a pair of vertices}\\
&\quad \text{in a column is same as the distance between the corresponding}\\
&\quad \text{vertices of other column}\\
&= \sum_{\substack{i,\,k=0\\ i\neq k}}^{n_1-1}\sum_{j=0}^{n_2-1}\frac{n_2(d(u_i)+d(u_k))}{d_G(u_i,u_k)}+\sum_{\substack{i,\,k=0\\ i\neq k}}^{n_1-1}\sum_{j=0}^{n_2-1}\frac{2d(v_j)}{d_G(u_i,u_k)}\\
&= 2n_2^2\,RDD(G)+8\,|E(G')|\,H(G). \qquad (5.3)
\end{aligned}
$$

$$
\begin{aligned}
A_3 &= \sum_{\substack{i,\,k=0\\ i\neq k}}^{n_1-1}\sum_{\substack{j,\,\ell=0\\ j\neq \ell}}^{n_2-1}\frac{d(x_{ij})+d(x_{k\ell})}{d_{G\circ G'}(x_{ij},x_{k\ell})}\\
&= \sum_{\substack{i,\,k=0\\ i\neq k}}^{n_1-1}\sum_{\substack{j,\,\ell=0\\ j\neq \ell}}^{n_2-1}\frac{n_2d(u_i)+d(v_j)+n_2d(u_k)+d(v_\ell)}{d_G(u_i,u_k)},\\
&\quad \text{since } d_{G\circ G'}(x_{ij},x_{k\ell})=d_G(u_i,u_k) \text{ for all } j \text{ and } k \text{ and further the distance}\\
&\quad \text{between the corresponding vertices of the layers is counted in } A_2\\
&= \sum_{\substack{i,\,k=0\\ i\neq k}}^{n_1-1}\sum_{\substack{j,\,\ell=0\\ j\neq \ell}}^{n_2-1}\frac{n_2(d(u_i)+d(u_k))}{d_G(u_i,u_k)}+\sum_{\substack{i,\,k=0\\ i\neq k}}^{n_1-1}\sum_{\substack{j,\,\ell=0\\ j\neq \ell}}^{n_2-1}\frac{d(v_j)+d(v_\ell)}{d_G(u_i,u_k)},\\
&= 2n_2^2(n_2-1)RDD(G)+4\,H(G)(M_1(G')+\overline{M}_1(G')). \qquad (5.4)
\end{aligned}
$$

Using (5.2),(5.3) and (5.4) in (5.2), we have,

$$
\begin{aligned}
RDD(G\circ G') &= n_2^3RDD(G)+2H(G)\Big(2\left|E(G')\right|+M_1(G')+\overline{M}_1(G')\Big)\\
&\quad +n_2\,|E(G)|\,\Big(n_2^2+2\left|E(G')\right|-n_2\Big)+\frac{n_1}{2}\Big(2M_1(G')+\overline{M}_1(G')\Big).
\end{aligned}
$$

□

Acknowledgments

The author would like to thank reviewer for careful reading and suggestions, which improved the manuscript.

References

[1] N. Alon and E. Lubetzky, Independent set in tensor graph powers, *J. Graph Theory*, **54** (2007) 73-87.

[2] A. M. Assaf, Modified group divisible designs, *Ars Combin.*, **29** (1990) 13-20.

[3] B. Bresar, W. Imrich, S. Klavžar and B. Zmazek, Hypercubes as direct products, *SIAM J. Discrete Math.*, **18** (2005) 778-786.

[4] K. C. Das, B. Zhou and N. Trinajstić, Bounds on Harary index, *J. Math. Chem.*, **46** (2009) 1377-1393.

[5] A. A. Dobrynin and A. A. Kochetova, Degree distance of a graph: a degree analogue of the Wiener index, *J. Chem. Inf. Comput. Sci.*, **34** (1994) 1082-1086.

[6] J. Devillers, A. T. Balaban and Eds., *Topological indices and related descriptors in QSAR and QSPR*, Gordon and Breach, Amsterdam, The Netherlands, 1999.

[7] L. Feng and A. Ilić, Zagreb, Harary and hyper-Wiener indices of graphs with a given matching number, *Appl. Math. Lett.*, **23** (2010) 943-948.

[8] I. Gutman, Selected properties of the Schultz molecular topological index, *J. Chem. Inf. Comput. Sci.*, **34** (1994) 1087-1089.

[9] I. Gutman and O. E. Polansky, *Mathematical Concepts in Organic Chemistry*, Springer-Verlag, Berlin, 1986.

[10] M. Hoji, Z. Luo and E. Vumar, Wiener and vertex PI indices of kronecker products of graphs, *Discrete Appl. Math.*, **158** (2010) 1848-1855.

[11] H. Hua and S. Zhang, On the reciprocal degree distance of graphs, *Discrete Appl. Math.*, **160** (2012) 1152-1163.

[12] W. Imrich and S. Klavžar, Product graphs: Structure and Recognition, John Wiley, New York, 2000.

[13] O. Ivanciuc and T. S. Balaban, Reciprocal distance matrix, related local vertex invariants and topological indices, *J. Math. Chem.*, **12** (1993) 309-318.

[14] M. H. Khalifeh, H. Youseri-Azari and A. R. Ashrafi, Vertex and edge PI indices of Cartesian product of graphs, *Discrete Appl. Math.*, **156** (2008) 1780-1789.

[15] B. Lučić, A. Milicević, S. Nikolić and N. Trinajstić, Harary index-twelve years later, *Croat. Chem. Acta*, **75** (2002) 847-868.

[16] A. Mamut and E. Vumar, Vertex vulnerability parameters of Kronecker products of complete graphs, *Inform. Process. Lett.*, **106** (2008) 258-262.

[17] K. Pattabiraman and P. Paulraja, On some topological indices of the tensor product of graphs, *Discrete Appl. Math.*, **160** (2012) 267-279.

[18] K. Pattabiraman and P. Paulraja, Wiener and vertex PI indices of the strong product of graphs, *Discuss. Math. Graph Thoery*, **32** (2012) 749-769.

[19] K. Pattabiraman and P. Paulraja, Wiener index of the tensor product of a path and a cycle, *Discuss. Math. Graph Thoery*, **31** (2011) 737-751.

[20] D. Plavšić, S. Nikolić, N. Trinajstić and Z. Mihalić, On the Harary index for the characterization of chemical graphs, *J. Math. Chem.*, **12** (1993) 235-250.

[21] K. Xu and K. C. Das, On Harary index of graphs, *Discrete. Appl. Math.*, **159** (2011) 1631-1640.

[22] H. Yousefi-Azari, M. H. Khalifeh and A. R. Ashrafi, Calculating the edge Wiener and edge Szeged indices of graphs, *J. Comput. Appl. Math.*, **235** (2011) 4866-4870.

[23] B. Zhou, Z. Du and N. Trinajstić, Harary index of landscape graphs, *Int. J. Chem. Model.*, **1** (2008) 35-44.

[24] B. Zhou, X. Cai and N. Trinajstić, On the Harary index, *J. Math. Chem.*, **44** (2008) 611-618.

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